EC9D3 Advanced Microeconomics, Part I: Lecture 5

Francesco Squintani

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Two Steps to Profit Maximization

The profit maximization can be obtained in two sequential steps:

1. Given y, find the choice of inputs that allows the producer to obtain y at the minimum cost;

   This generates *conditional factor demands* and the *cost function*;

2. Given the cost function, find *the profit maximizing output level*.

Step 1 is common to firms that behave *competitively* in the input market but *not necessarily* in the output market.

In step 2 we impose the competitive assumption on the output market.

Cost Minimization

We shall start from *cost minimization:*

$$\min_{x} \quad w\,x$$
$$s.t. \quad f(x) \geq y$$

The necessary *first order conditions* are:

$$y = f(x^*),$$
$$w \geq \lambda \nabla f(x^*)$$
$$[w - \lambda \nabla f(x^*)]\ x^* = 0$$

or for every input $l = 1, \ldots, h$: $w_l \geq \lambda \dfrac{\partial f(x^*)}{\partial x_l}$ with equality if $x_l^* > 0$.

Cost Minimization (2)

The first order conditions are also sufficient if $f(x)$ *is quasi-concave* (the input requirement set is convex).

Alternatively, a set of sufficient conditions for a local minimum are that $f(x)$ *is quasi-concave in a neighborhood of* x^*.

This can be checked by means of the bordered hessian matrix and its minors.

Cost Minimization (3)

In the case of only two inputs $f(x_1, x_2)$ we have:

$$w_l \geq \lambda \frac{\partial f(x^*)}{\partial x_l}, \qquad \forall l = 1, 2$$

with equality if $x_l^* > 0$

SOC:

$$\begin{vmatrix} f_{11}(x^*) & f_{12}(x^*) & f_1(x^*) \\ f_{21}(x^*) & f_{22}(x^*) & f_2(x^*) \\ f_1(x^*) & f_2(x^*) & 0 \end{vmatrix} > 0$$

Cost Minimization (4)

In the case the two first order conditions are satisfied with equality (no corner solutions) we can rewrite the necessary conditions as:

$$\text{MRTS} = \frac{\partial f(x^*)/\partial x_1}{\partial f(x^*)/\partial x_2} = \frac{w_1}{w_2}$$

and

$$y = f(x^*)$$

Notice a close formal analogy with consumption theory (expenditure minimization).

Conditional Factor Demands and Cost Function

This leads to define:

- the solution to the cost minimization problem:

$$x^* = z(w, y) = \begin{pmatrix} z_1(w, y) \\ \vdots \\ z_h(w, y) \end{pmatrix}$$

as the *conditional factor demands* (correspondence).

- the minimand function of the cost minimization problem:

$$c(w, y) = w\, z(w, y)$$

as the *cost function.*

Properties of the Cost Function and Cond. Factor Demand

1. $c(w, y)$ is non-decreasing in y.
2. $c(w, y)$ is homogeneous of degree 1 in w.
3. $c(w, y)$ is a concave function in w.
4. $z(w, y)$ is homogeneous of degree 0 in w.

Shephard's Lemma

5. *Shephard's Lemma:* if $z(w, y)$ is single valued with respect to w then $c(w, y)$ is differentiable with respect to w and

$$\frac{\partial c(w, y)}{\partial w_l} = z_l(w, y)$$

Further the lagrange multiplier of the cost minimization problem is the marginal cost of output:

$$\frac{\partial c(w, y)}{\partial y} = \lambda^*(w, y)$$

Properties of Conditional Factor Demands (2)

6. If $z(w, y)$ is differentiable in w then:

$$\begin{pmatrix} \frac{\partial^2 c}{\partial w_1^2} & \cdots & \frac{\partial^2 c}{\partial w_1 \, \partial w_h} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 c}{\partial w_h \, \partial w_1} & \cdots & \frac{\partial^2 c}{\partial w_h^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial z_1}{\partial w_1} & \cdots & \frac{\partial z_1}{\partial w_h} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_h}{\partial w_1} & \cdots & \frac{\partial z_h}{\partial w_h} \end{pmatrix}$$

is a *symmetric* and *negative semi-definite* matrix.

Properties of the Cost Function and Cond. Factor Dem. (2)

7. If $f(x)$ is homogeneous of degree one (i.e. exhibits constant returns to scale), then $c(w, y)$ and $z(w, y)$ are homogeneous of degree one in y.

Proof: Let $k > 0$ and consider:

$$c(w, k\,y) \;=\; \begin{array}{ll} \min\limits_{x} & w\,x \\ \text{s.t.} & f(x) \geq k\,y \end{array} \tag{1}$$

Recall that by definition of $c(w, y)$ defining x^* to be the solution to

$$\begin{array}{ll} \min\limits_{x} & w\,x \\ \text{s.t.} & f(x) \geq y \end{array} \tag{2}$$

we obtain

$$y = f(x^*)$$

Hence by homogeneity of degree 1 of $f(x)$ we obtain:

$$k\ y = k\ f(x^*) = f(k\ x^*)$$

which implies that $k\ x^*$ is *feasible in Problem (1)*.

Therefore:

$$k\ c(w, y) = k\ [w\ x^*] = w\ (k\ x^*) \geq c(w, k\ y).$$

Let now $\hat{x}$ be the solution to Problem (1). Necessarily:

$$f(\hat{x}) = k\ y$$

or, by homogeneity of degree 1:

$$(1/k)\ f(\hat{x}) = f[(1/k)\ \hat{x}] = y$$

which implies that $[(1/k)\ \hat{x}]$ is *feasible in Problem (2)*.

Therefore we get:

$$c(w, k\ y) = w\ \hat{x} = k\ w\ [(1/k)\ \hat{x}] \geq k\ c(w, y)$$

which concludes the proof. □

Constant Returns to Scale

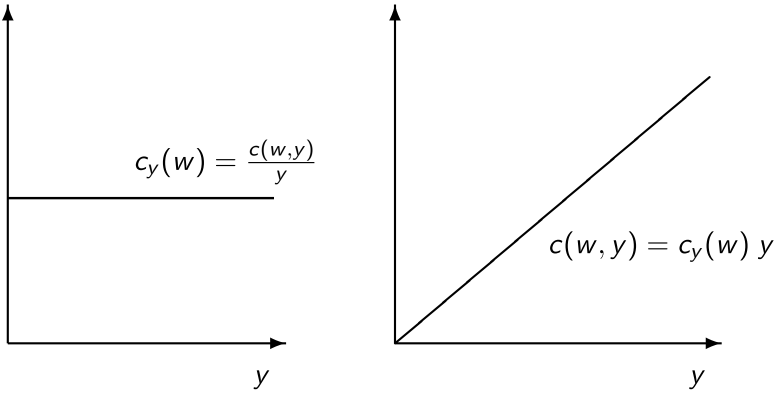
8. A technology that exhibits CRS has a cost function that is linear in y: $c(w, y) = c(w)y$.

9. A technology that exhibits CRS has a constant marginal $(\partial c(w, y)/\partial y)$ and average cost function:

$$(\partial c(w, y)/\partial y) = (c(w, y)/y).$$

Proof: Homogeneity of degree 1 in y implies linearity of $c(w, y)$ in y. By Euler theorem $c_y(w)y = c(w, y)$ or $c_y(w) = c(w, y)/y$. □

Constant Returns to Scale (2)



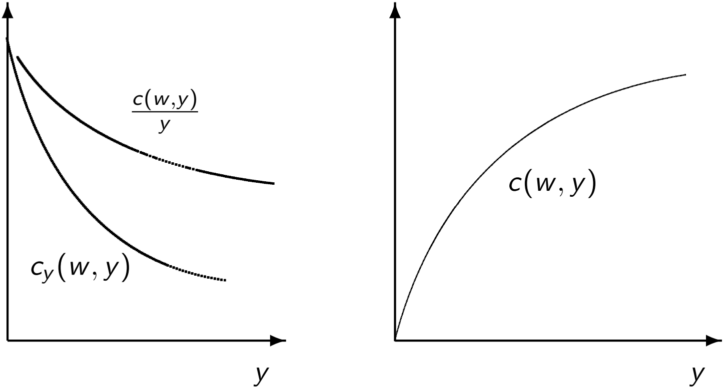
Increasing Returns to Scale

10. If $f(x)$ is convex (IRS technology), then $c(w, y)$ is concave in y.

11. A technology that exhibits IRS has a decreasing marginal cost function $(\partial c(w, y)/\partial y)$ and average cost function:

$$(\partial c(w, y)/\partial y) \leq (c(w, y)/y)$$

Increasing Returns to Scale (2)



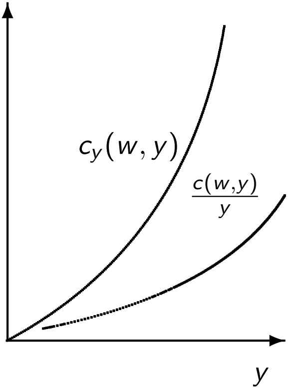
Decreasing Returns to Scale

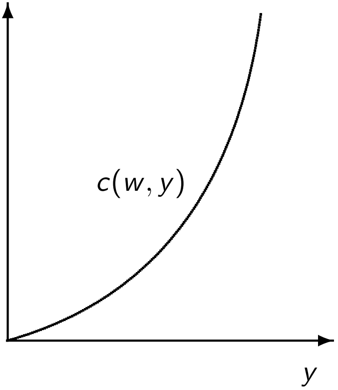
12. If $f(x)$ is concave (DRS technology), then $c(w, y)$ is convex in y.

13. A technology that exhibits DRS has an increasing marginal cost function $(\partial c(w, y)/\partial y)$ and average cost function:

$$(\partial c(w, y)/\partial y) \geq (c(w, y)/y)$$

Decreasing Returns to Scale (2)





Profit Maximization (5)

Assume that the output market is competitive.

The profit maximization problem is then:

$$\max_{y} \quad p\, y - c(w, y)$$

The necessary FOC are:

$$p - \frac{\partial c(w, y^*)}{\partial y} \leq 0$$

with equality if $y^* > 0$.

Profit Maximization (6)

The sufficient SOC conditions for a local maximum:

$$\frac{\partial^2 c(w, y^*)}{\partial y^2} > 0$$

Clearly SOC imply at least local DRS in a neighborhood of y^*.

Notice that if $y^* > 0$ the optimal choice of the firm is:

$$p = \frac{\partial c(w, y^*)}{\partial y} = \mathrm{MC}(y^*)$$

in words, price equal to marginal cost.

This condition defines the solution to the profit maximization problem: *the supply function: $y^*(w, p)$*

Profit Maximization (7)

The two profit maximization problems produce the same outcome for equal (w, p). Indeed:

$$\max_{y} \quad py - c(w, y)$$

where

$$c(w, y) = \min_{x} \quad w\,x \quad \text{s.t.} \quad f(x) \geq y$$

yields

$$\max_x \quad p\,y - w\,x \quad \text{s.t.} \quad f(x) = y$$

the very first problem we considered.

Long Run and Short Run

We now explicitly include *long run* and *short run* considerations in the profit maximization problem (flow variables).

Short run: one or more inputs may be fixed, ass. $x_h = \bar{x}_h$, while the remaining inputs may be varied at will.

The short run variable cost function:

$$
\begin{aligned}
c^S(w, y, \bar{x}_h) \quad = \quad & w_h \, \bar{x}_h + \min_{x_1, \ldots, x_{h-1}} \sum_{l=1}^{h-1} w_l \, x_l \\
& \text{s.t.} \quad f(x_1, \ldots, x_{h-1}, \bar{x}_h) \geq y
\end{aligned}
$$

Long Run and Short Run (2)

Alternatively:

$$c^S(w, y, \bar{x}_h) = \min_x \quad w\,x \\ \text{s.t.} \quad f(x) \geq y \\ x_h = \bar{x}_h$$

Recall $z(w, y)$ denote the long run *conditional factor demands*, that solve:

$$c(w, y) = \min_x w\,x \quad \text{s.t.} \quad f(x) \geq y$$

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_h)$ be the input vector that achieves the minimum long run cost of producing $\bar{y}$:

$$\bar{x} = (\bar{x}_1, \ldots, \bar{x}_h) = z(\bar{w}, \bar{y})$$

Long Run and Short Run (3)

We characterize the relationship between short and long run total costs, or alternatively, short run and long run variable costs (more familiar).

Notice that

$$c(w, y) \equiv c^S(w, y, z_h(w, y))$$

or

$$\frac{c(w, y)}{y} \equiv \frac{c^S(w, y, z_h(w, y))}{y}$$

moreover

$$\frac{\partial c(w, y)}{\partial y} \equiv \frac{\partial c^S(w, y, z_h(w, y))}{\partial y} \tag{3}$$

by Envelope Theorem.

Long Run and Short Run (4)

We shall now focus on a neighborhood of $(\bar{w}, \bar{y})$ and set $\bar{x}_h = z_h(\bar{w}, \bar{y})$.

Recall that Envelope Theorem implies that only the *first order effect* is zero.

Since (3) is an identity in (w, y) we can differentiate both sides with respect to y:

$$\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y^2} + \frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \, \partial \bar{x}_h} \frac{\partial z_h(w, y)}{\partial y} = \frac{\partial^2 c(w, y)}{\partial y^2}$$

Long Run and Short Run (4)

and with respect to w_h:

$$\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \, \partial w_h} + \frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \, \partial \bar{x}_h} \frac{\partial z_h(w, y)}{\partial w_h} = \frac{\partial^2 c(w, y)}{\partial y \, \partial w_h}$$

Now

$$\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \, \partial w_h} = 0$$

since

$$\frac{\partial c^S(w, y, \bar{x}_h)}{\partial w_h} = \bar{x}_h$$

is independent of y.

Long Run and Short Run (5)

Hence by Shephard's Lemma:

$$\frac{\partial^2 c^S(w,y,\bar{x}_h)}{\partial y\ \partial \bar{x}_h} = \frac{\partial z_h(w,y)/\partial y}{\partial z_h(w,y)/\partial w_h}$$

which implies by substitution:

$$\frac{\partial^2 c^S(w,y,\bar{x}_h)}{\partial y^2} + \frac{(\partial z_h(w,y)/\partial y)^2}{\partial z_h(w,y)/\partial w_h} = \frac{\partial^2 c(w,y)}{\partial y^2}$$

which delivers:

$$\frac{\partial^2 c^S(w,y,\bar{x}_h)}{\partial y^2} \geq \frac{\partial^2 c(w,y)}{\partial y^2}$$

since

$$\frac{(\partial z_h(w,y)/\partial y)^2}{\partial z_h(w,y)/\partial w_h} \leq 0$$

Le Chatelier Principle

This allows us to conclude that the function:

$$l(w,y) = c(w,y) - c^S(w,y,\bar{x}_h) \leq 0$$

reaches a local maximum at $\bar{x}$.

By definition of $\bar{x}$, FOC are satisfied:

$$\frac{\partial c^S(w,y,\bar{x}_h)}{\partial y} = \frac{\partial c(w,y)}{\partial y}$$

While we just proved that the SOC hold:

$$\frac{\partial^2 c(w,y)}{\partial y^2} \leq \frac{\partial^2 c^S(w,y,\bar{x}_h)}{\partial y^2}$$

Le Chatelier Principle (2)

A similar approach proves:

$$0 \geq \frac{\partial z_h^S}{\partial w_i} \geq \frac{\partial z_h}{\partial w_i}$$

Moving to profit maximization:

$$0 \geq \frac{\partial x_h^S}{\partial w_i} \geq \frac{\partial x_h}{\partial w_i}$$

and

$$0 \leq \frac{\partial y^S}{\partial p} \leq \frac{\partial y}{\partial p}$$

All these results are summarized under the name of: *Le Chatelier Principle*.

Aggregation

- The question we address is when can we speak of *an aggregate demand* and *aggregate supply function?*
- We start from *aggregate demand*.
- In particular the way this question is usually stated is:

  When can we treat the aggregate demand function as if it were generated by a fictional representative consumer whose preferences satisfies the standard axioms of choice?

- This would also imply that the aggregate Marshallian demand will satisfy the standard properties of Marshallian demands we have analyzed up to now.

Aggregate Demand

- Assume there are I consumers.
- Consider the *aggregate Marshallian demand:*

$$X(p, m^1, \ldots, m^I) = \sum_{i=1}^{I} x^i(p, m^i)$$

- The main question is when can we state the aggregate demand as a function of aggregate income, only:

$$X\left(p, \sum_{i=1}^{I} m^i\right) = X(p, m^1, \ldots, m^I)$$

Aggregate Demand (2)

- This implies that *the aggregate demand has to be invariant to any redistribution of income that sums to the same level.*
- In other words, for every pair of allocations of income: $(m^1, \ldots, m^I)$ and $(\hat{m}^1, \ldots, \hat{m}^I)$ such that

$$\sum_i m^i = \sum_i \hat{m}^i$$

it has to be the case that

$$X(p, m^1, \ldots, m^I) = X\left(p, \hat{m}^1, \ldots, \hat{m}^I\right)$$

or

$$X(p, m^1, \ldots, m^I) - X\left(p, \hat{m}^1, \ldots, \hat{m}^I\right) = 0$$

Aggregate Demand (3)

- Alternatively, for any initial allocation $(m^1, \ldots, m^I)$ and any differential change
$$(dm^1, \ldots, dm^I)$$
such that
$$\sum_{i=1}^{I} dm^i = 0$$
it must be the case that for every commodity $l \in \{1, \ldots, L\}$:
$$dX(p, m^1, \ldots, m^I) = \sum_{i=1}^{I} \frac{\partial x_l^i(p, m^i)}{\partial m^i}\, dm^i = 0$$

Aggregate Demand (4)

- Notice that this condition holds if and only if *the coefficients of the different dm^i are equal:*

$$\frac{\partial x_l^i(p, m^i)}{\partial m^i} = \frac{\partial x_l^j(p, m^j)}{\partial m^j}$$

for every commodity l, every pair of consumers i, j, and every initial income distribution $(m^1, \ldots, m^I)$.

- In other words, *the income effect at p must be the same* whatever consumer we look at and whatever his level of income.

Geometrically we require that all consumers' income expansion paths are parallel, straight lines.

Aggregate Demand (5)

- A special case in which this is true is when all consumers have identical and *homothetic* preferences.
- Preferences are *homothetic* if the indifference curves have the same slope at every point of any ray from the origin.
- Homothetic preferences can be represented by a monotonic transformation of an homogeneous of degree 1 utility function.
- An other special case is when all consumers have preferences that are *quasi-linear* with respect to the same good.

Aggregate Demand (6)

Result

In general a necessary and sufficient condition for the set of consumers to exhibit parallel, straight income expansion path at any price p is that preferences admit indirect utility functions of the Gorman form:

$$v^i(p, m^i) = a^i(p) + b(p)\ m^i$$

where $b(p)$ is common to all consumers.

Property

If every consumer's Marshallian demand satisfies the uncompensated law of demand so does the aggregate demand.

Clearly the problems associated with aggregation arise from *income effects*

Aggregate Supply

- The absence of a budget constraint implies that individual firms' supply are not subject to income effects.
- Hence aggregation of production theory is *simpler and requires less restrictive conditions.*
- Consider J production technologies:

$$(Z^1, \ldots, Z^J)$$

Let $z^j(p, w) = \begin{pmatrix} -x^j(p, w) \\ y^j(p, w) \end{pmatrix}$ be firm j's production plan.

Aggregate Supply (2)

- We define the following *aggregate optimal production plan:*

$$z(p,w) = \sum_{j=1}^{J} z^j(p,w) = \begin{pmatrix} -\sum_j x^j(p,w) \\ \sum_j y^j(p,w) \end{pmatrix}$$

- We have seen that the matrix of cross and own price effects on production plan $z^j(p.w)$:

$$Dz^j(p,w)$$

is *symmetric and positive semi-definite*: the *law of supply*.

- Since both properties are preserved under sum then

$$Dz(p,w)$$

is also *symmetric and positive semi-definite.*

Aggregate Supply (3)

In other words an aggregate law of supply holds.

Result (Existence of the Representative Producer)

In a purely competitive environment the maximum profit obtained by every firm maximizing profits separately is the same as the profit obtained if all J firms were to coordinate their choices in a joint profit maximization:

$$\pi(p, w) = \sum_{j=1}^{J} \pi^j(p, w)$$

Clearly, the intersection of aggregate supply and aggregate demand gives us a *Market equilibrium.*

Competitive Equilibrium

- Consider the entire economy, in which three main activities occur: *production*, *consumption* and *trade*.
- We focus first on a *pure exchange* economy (two activities, consumption and trade).
- Consumers are born with endowments of commodities.
- They can either consume the endowments or trade them.
- Consider $I = 2$ consumers and $L = 2$ commodities.

Edgeworth Box Economy

- In such case the *consumption feasible set* for every consumer is $X^i \in \mathbb{R}^2_+$ and consumer i's *endowment* is:

$$\omega^i = \begin{pmatrix} \omega^i_1 \\ \omega^i_2 \end{pmatrix}$$

- The *total endowment* of commodity l available in the economy is:

$$\bar{\omega}_l = \omega^1_l + \omega^2_l > 0 \quad \forall l \in \{1, 2\}$$

- An *allocation* in this economy is then a pair of vectors x such that

$$x = (x^1, x^2) = \left(\begin{pmatrix} x^1_1 \\ x^1_2 \end{pmatrix}, \begin{pmatrix} x^2_1 \\ x^2_2 \end{pmatrix} \right)$$

Edgeworth Box Economy (2)

- An allocation is *feasible* if and only if

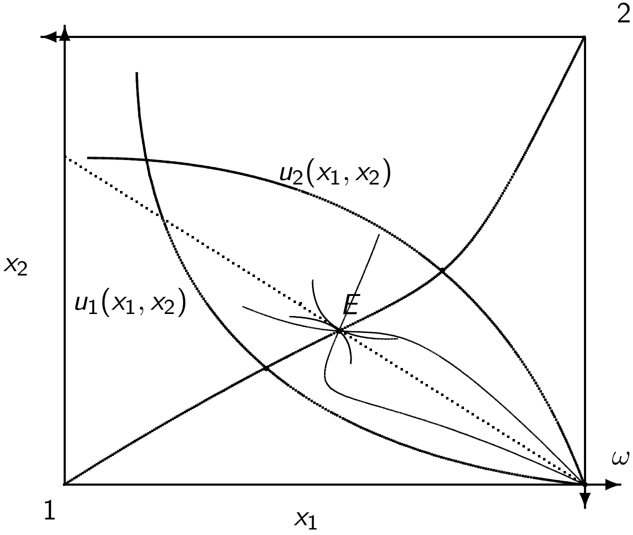
$$x_l^1 + x_l^2 \leq \bar{\omega}_l \quad \forall l \in \{1, 2\}$$

- An allocation is *non-wasteful* if and only if

$$x_l^1 + x_l^2 = \bar{\omega}_l \quad \forall l \in \{1, 2\}$$

- This economy can be represented in *an Edgeworth box.*

Edgeworth Box



Edgeworth Box Economy (3)

- Notice that in such an environment the income of each consumer is the *market value of the consumer endowment:*

$$m^i = p\,\omega^i$$

where however p is determined in equilibrium.

- The budget set of consumer i is then:

$$B^i(p) = \left\{x^i \in \mathbb{R}^2_+ \mid p\,x^i \leq p\,\omega^i\right\}$$

- For a vector of equilibrium prices p the budget sets of both consumers are two complementary sets in the Edgeworth box (slope of the separating line $-\dfrac{p_1}{p_2}$).

Edgeworth Box Economy (4)

- The preferences of the two consumers are represented by two maps of indifference curves.
- For any given level of prices we can represent the *offer curve* of each consumer: the consumption bundle that represent the optimal choice for each consumer.
- The offer curve necessarily passes through the endowment point.
- Indeed the allocation

$$\omega = (\omega^1, \omega^2) = \left(\begin{pmatrix} \omega_1^1 \\ \omega_2^1 \end{pmatrix}, \begin{pmatrix} \omega_1^2 \\ \omega_2^2 \end{pmatrix} \right)$$

is always affordable hence each consumer must choose an optimal consumption bundle that makes him/her at least *as well off* as at ω.

Edgeworth Box Economy (5)

- Given the preferences of the two consumers the only candidate to be an *equilibrium price vector* (if it exists) is a unique price vector that defines a unique budget constraint in the Edgeworth box tangent to indifference curves of both consumers.
- However if the tangency occur at two distinct points on the budget constraint then there will exist *excess supply* in one good $l = 1$ and excess demand in the other good $l = 2$.
- The allocation represented by the two tangency point is then *not* feasible.
- We define a *market equilibrium* as a situation in which *markets clear*, the consumers fulfil their *desired purchases* and the allocation obtained is *feasible.*

Edgeworth Box Economy (6)

Definition

A *Walrasian (competitive) equilibrium* for the Edgeworth box economy is a price vector p^* and an allocation $x^* = (x^{1,*}, x^{2,*})$ such that

$$u_i(x^{i,*}) \geq u_i(x^i) \quad \forall x^i \in B^i(p^*)$$

and

$$x_l^{1,*} + x_l^{2,*} = \bar{\omega}_l \quad \forall l \in \{1, 2\}$$

This corresponds to an *intersection of the two offer curves*.

It also corresponds to a point in which the indifference curves of the two consumers are tangent to the unique budget constraint.

Edgeworth Box Economy (7)

Property

The price vector p^* is identified up to a degree of freedom: only the relative price matters.

Proof: If the preferences of both consumers are locally non-satiated then the budget constraint of both consumers will be binding:

$$p^* x^{i,*} = p^* \omega^i \quad \forall i \in \{1, 2\}$$

If we sum the two budget constraint across consumers we get:

$$p^* \left(x^{1,*} + x^{2,*} \right) = p^* \bar{\omega}$$

which exhibits a linear dependence among the vectors of the equilibrium allocation (from here the degree of freedom). □

Edgeworth Box Economy (8)

- The above property is known as *Walras Law*, (it only depends from binding budget constraints).
- Two main problems with a Walrasian equilibrium: *existence and uniqueness.*
- Uniqueness is in general *not a property* of Walrasian equilibria.
- A Walrasian equilibrium *might not exists* (non-convexity of preferences, unbounded demand).