EC9D3 Advanced Microeconomics, Part I: Lecture 8

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What is Social Choice?

- Normative economics. What is right or wrong, fair or unfair for an economic environment.
- Axiomatic approach. Appropriate axioms describing efficiency, and fairness are introduced.
- Allocations and rules are derived from axioms.
- Beyond economics, social choice applies to politics, and sociology.

Main Questions

- How does a society/committee/group solve conflicts of interest among its members?
- Is there an "optimal" policy for given interests in society? (I.e. can we find a desirable social choice function?)
- If there is an "optimal" policy, will it always be implemented?
  - Observations seem to suggest that welfare improving policies are oftentimes not the outcome of political decision processes.
  - Examples: industrial bargaining, international conflicts.
- How do institutions shape political outcomes?
- How do political institutions affect economic outcomes?

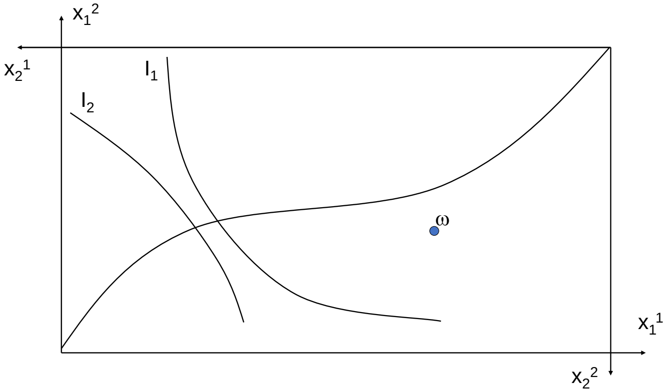
Social Preferences

- Consider a set of social alternatives X, in a society of N individuals.
- Each individual i has preferences over X, described by the binary relation $R(i)$, a subset of X^2.
- The notation $xR(i)y$ means that individual i weakly prefers x to y.
- Strict preferences $P(i)$ are derived from $R(i)$:
  $xP(i)y$ corresponds to $xR(i)y$ but not $yR(i)x$.
- Indifference relations $I(i)$ are derived from $R(i)$:
  $xI(i)y$ corresponds to $xR(i)y$ and $yR(i)x$.

Social Preferences (2)

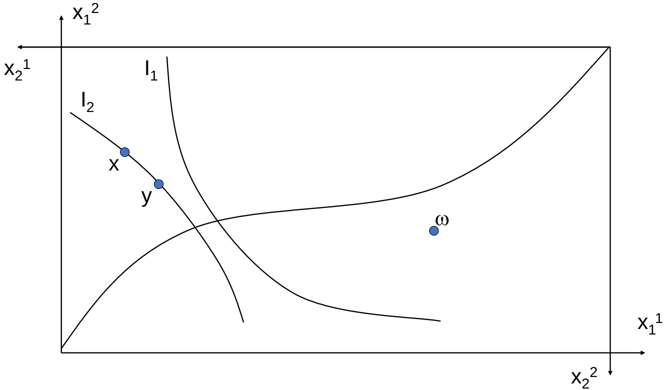
- The relation $R(i)$ is complete: for any x, y in X, either $xR(i)y$ or $yR(i)x$, or both.
- The relation $R(i)$ is transitive: for any x, y, z in X, if $xR(i)y$ and $yR(i)z$, then $xR(i)z$.
- Social preference relation: a complete and transitive relation $R = F(R(1), ..., R(N))$ over the set of alternative X.
- Social preferences R aggregate the preferences $R(i)$ for all $i = 1, ..., N$, and satisfies appropriate efficiency and fairness axioms.

Example: Exchange Economy



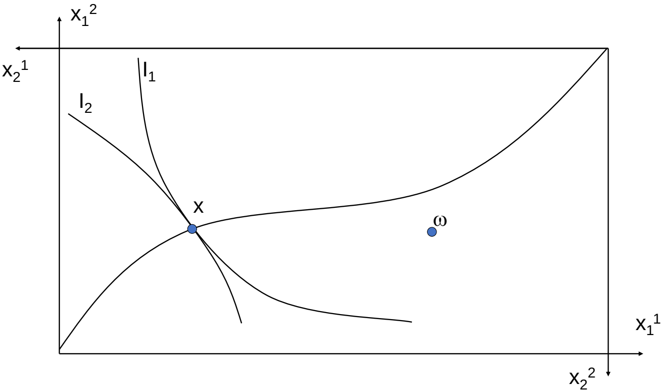
- In the exchange economy with 2 consumers and 2 goods, x are such that $x_j^1 + x_j^2 = \omega_j^1 + \omega_j^2$, where ω is the endowment and j is the good.

Example: Exchange Economy (2)



- For individual i, $xI(i)y$ if (x_j^1, x_j^2) and (y_j^1, y_j^2) are on the same indifference curve I_i.
- The relations $R(i)$ and $P(i)$ are described by the contour sets of the utilities u_i.

Example: Exchange Economy (3)



- The line of contracts describes all Pareto optimal alternatives.
- One possible social preference is R such that xPy if and only if x is on the line of contracts, and y is not.

Example: Pliny the Younger

- Consul Africanius Dexter found slain and his freedmen were accused of killing him.
- Roman Senate decides whether to
  a. acquit them
  b. banish them
  d. put them to death
- Senators hold one of three sets of preferences:
  1. $a \succ_i b \succ_i d$ (Pliny was among these senators)
  2. $b \succ_i a \succ_i d$
  3. $d \succ_i b \succ_i a$
- Consider situation where
  3 senators have preferences under 1.,
  2 senators possess preferences 2.,
  2 senators possess preferences 3.

Example: Pliny the Younger (2)

- Voting procedure:
  (1) vote whether guilty (b or d) or not guilty (a)
  (2) if not guilty, vote whether b or d

  Outcome: b

- Plurality Voting: all alternatives are put to a vote simultaneously.

  Outcome: a

- Institutional context of decisions (here voting rule) + individual preferences affects profoundly what outcome people choose.

Condorcet Paradox

- 2 individuals possess preferences under 1. $(a \succ_i d \succ_i b)$,
- 2 individuals possess preferences under 2. $(b \succ_i a \succ_i d)$,
- 2 individuals possess preferences under 3. $(d \succ_i b \succ_i a)$
- Pairwise voting:

  I. first vote a against b, then winner against d. Outcome: d.

  II. first vote a against d, then winner against b. Outcome: b.

  III. first vote b against d, then winner against a. Outcome: a.
- Agenda-setter (fixing voting rule) determines the outcome.

Condorcet Paradox (2)

- Transitivity of preferences: for all i, if $a \succ_i b$ and $b \succ_i d$, then $a \succ_i d$.
- On social/group level, pairwise majority voting yields $a \succ_G d \succ_G b$.
- Transitivity would suggest: $a \succ_G b$.
- However, pairwise majority voting gives $b \succ_G a$.
- We obtain Condorcet Cycle: $a \succ_G d \succ_G b \succ_G a$
- Condorcet Paradox: Although individual preferences are transitive, there need not exist a transitive social preference.
- Condorcet Winner: An alternative x is called a Condorcet Winner, if it beats any other alternative in a pairwise vote.

Relevance of Condorcet Cycles

- Thought experiment: n persons, m alternatives.
- We assign preferences over the m alternatives randomly to all persons.
- What is the likelihood that we encounter a Condorcet Cycle in pairwise voting among all possible pairs of alternatives?

	$n = 3$	$n = 5$	$n = 25$	$\cdots$	$n = \infty$
$m = 3$	0.056	0.069	0.084		0.088
$m = 5$	0.16	0.2	0.242		0.251
$m = 8$	0.271	0.334	0.359		0.415
$m = 49$					0.841

The Case of Two Alternatives

- Suppose that there are only two alternatives:
  x is the status quo, and y is the alternative.

- Each individual preference $R(i)$ is indexed as q in $\{-1, 0, 1\}$, where 1 is a strict preference for x.

- The social welfare rule is a functional

$$F(q(1), ..., q(N)) \text{ in } \{-1, 0, 1\}.$$

May's Axioms

- AN: The social rule F is anonymous if for every permutation p,
$$F(q(1), ..., q(N)) = F(q(p(1)), ...q(p(N))).$$

- NE: The social rule F is neutral if $F(q) = -F(-q)$.

- PR: The rule F is positively responsive if $q \geq q'$, $q \neq q'$ and $F(q') \geq 0$ imply that $F(q) = 1$.

May's Theorem

- A social welfare rule F is majoritarian if:
  - $F(q) = 1$ if and only if:
    $n^{+}(q) = \#\{i : q(i) = 1\} > n^{-}(q) = \#\{i : q(i) = -1\}$,
  - $F(q) = -1$ if and only if $n^{+}(q) < n^{-}(q)$,
  - $F(q) = 0$ if and only if $n^{+}(q) = n^{-}(q)$).

Theorem

A social welfare rule is majoritarian if and only if it is neutral, anonymous, and positively responsive.

May's Theorem

Proof: Clearly, majority rule satisfies the 3 axioms.

By AN, $F(q) = G(n^+(q), n^-(q))$.

If $n^+(q) = n^-(q)$, then $n^+(-q) = n^-(-q)$, and so, by NE:
$F(q) = G(n^+(q), n^-(q)) = G(n^+(-q), n^-(-q)) = F(-q) = -F(q)$.

This implies that $F(q) = 0$.

If $n^+(q) > n^-(q)$, pick q' with $q' < q$ and $n^+(q') = n^-(q')$.

Because $F(q') = 0$, by PR, it follows that $F(q) = 1$.

When $n^+(q) < n^-(q)$, it follows that $n^+(-q) > n^-(-q)$,
hence $F(-q) = 1$ and by NE, $F(q) = -1$. □

Multiple Alternatives: Arrow's Axioms

- U. Unrestricted Domain. The domain of f must include all possible $(R(1), ..., R(n))$ over X.
- WP. Weak Pareto Principle. For any x, y in X, if $xP(i)y$ for all i, then xPy.
- ND. Non-Dictatorship. There is no individual i such that for all x, y, if $xP(i)y$, then xPy, regardless of the relations $R(j)$, for j other than i.

Arrow's Axioms

- IIA. Independence of Irrelevant Alternatives. Let $R = F(R(1), ..., R(N))$, and $R' = F(R'(1), ..., R'(N))$.

  For any x, y, if every individual i ranks x and y in the same way under $R(i)$ and $R'(i)$, then the ranking of x and y must be the same under R and R'.

- This axiom requires some comments.

- In some sense, it requires that each comparison can be taken without considering the other alternatives at play.

- The axiom fails in some very reasonable voting rules.

Borda Count

- Suppose that X is a finite set.
- Let $B_i(x) = \#\{y : xP(i)y\}$.
- The Borda Rule is: xRy if and only if $B_1(x) + ... + B_N(x) \geq B_1(y) + ... + B_N(y)$.
- This rule does not satisfy IIA.
- Consider 2 agents and $\{x, y, z\}$ alternatives.

  $xP(1)zP(1)y$, $yP(2)xP(2)z$ yields xPy.

  $xP^*(1)yP^*(1)z$, $yP^*(2)zP^*(2)x$ yields yP^*x.

Arrow Impossibility Theorem

Theorem

If there are at least 3 alternatives in X, then the axioms of Unrestricted Domain, Weak Pareto and Independence of Irrelevant Alternatives imply the existence of a dictator.

Proof: (Geanakoplos 1996).

Step 1. Consider any arbitrary alternative c. Suppose that $xP(i)c$ for any x other than c, and for any i.

By Weak Pareto, it must be that xPc for all such x.

Arrow Impossibility Theorem (2)

Step 2. In order, move c to the top of the ranking of 1, than of 2, all the way to n.

Index these orders as $(P_1(1), ..., P_1(N)), ..., (P_N(1), ..., P_N(N))$.

By WP, as $cP_N(i)x$ for all i and x other than c, it must be that cP_Nx for all x other than c.

The alternative c is at the top of the ranking.

Because c is at the top of the ranking P after raising it to the top in all individual i' s ranking $P(i)$, there must be an individual n such that c raises in P, after raising c to the top in all rankings $P(i)$ for i smaller or equal to n.

We let this ranking be $(P_n(1), ..., P_n(N))$.

Arrow Impossibility Theorem (3)

We now show that c is raised to the top of P_n for the ranking $(P_n(1), ..., P_n(N))$, i.e. when raising c to the top of $P(i)$, for all $i < n$.

By contradiction, say that aP_nc and cP_nb.

Because c is at the top of $P_n(i)$ for $i < n$, and at the bottom of $P_n(i)$ for $i > n$, we can change all i's preferences to $P^*(i)$ so that $bP^*(i)a$.

By WP, bP^*a.

By IIA, aP^*c and cP^*b.

By transitivity aP^*b, which is a contradiction.

This concludes that c is at the top of P_n, i.e. when raising c to the top of $P(i)$ for all individuals $i < n$.

Arrow Impossibility Theorem (4)

Step 3. Consider any a, b different from c.

Change the preferences $(P_n(1), ..., P_n(N))$ to $(P^*(1), ..., P^*(N))$ such that $aP^*(n)c\ P^*(n)b$, and for any other i, a and b are ranked in any way, as long as the ranking of c (either bottom or top), did not change.

Compare P_{n-1} to P^*: by IIA, aP^*c.

Compare P_{n+1} to P^*: by IIA, cP^*b.

By transitivity, aP^*b, for all a, b other than c.

Because a, b are arbitrary, we have: if $aP^*(n)b$, then aP^*b.

n is a dictator for all a, b other than c.

Arrow Impossibility Theorem (5)

Step 4. Repeat all previous steps with an arbitrary allocation d, playing the role of c.

Because, again, it is the ranking P_n of n which determines whether d is at the top or at the bottom of the social ranking, we can reapply step 3.

Again, we have: if $aP^*(n)b$, then aP^*b.

n is a dictator for all a, b other than d.

Because n is a dictator for all a, b other than c, and n is a dictator for all a, b other than d, we obtain that n is a dictator for all comparisons. $\square$

A Diagrammatic Proof

- We assume that preferences are continuous.
- Continuity. For any i, x, the sets $\{y : yR(i)x\}$ and $\{y : xR(i)y\}$ are closed.
- Complete, transitive and continuous preferences $R(i)$ can be represented as continuous utility functions u_i.
- We aggregate the utility functions u into a social welfare function $V(x) = F(u_1(x), ..., u_N(x))$.
- PI. Pareto indifference. If $u_i(x) = u_i(y)$, for all i, then $V(x) = V(y)$.
- If V satisfies U, IIA, and P, then there is a continuous function W such that: $V(x) > V(y)$ if and only if $W(u(x)) > W(u(y))$.

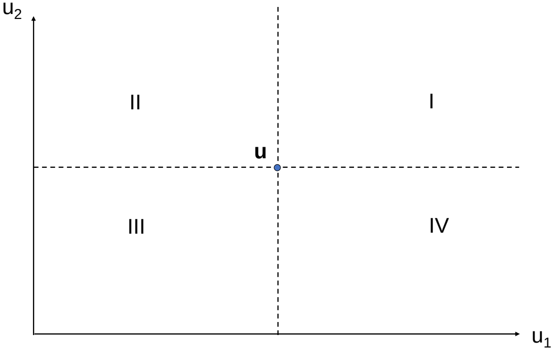
A Diagrammatic Proof (2)

- The welfare function depends only on the utility ranking, not on how the ranking comes about.
- The axioms of Arrow's theorem require that the utility is an ordinal concept (OS) and that utility is interpersonally noncomparable (NC).
- If u_i represents $R(i)$, then so does any increasing transformation $v_i(u_i)$. All transformations v_i must be allowed, independently across i.
- The function W aggregates the preferences $(u_i)_{i=1,\dots,n}$ if and only if the function $g(W)$ aggregates the preferences $(v_i(u_i))_{i=1,\dots,n}$ where g is an increasing transformation.

A Diagrammatic Proof (3)

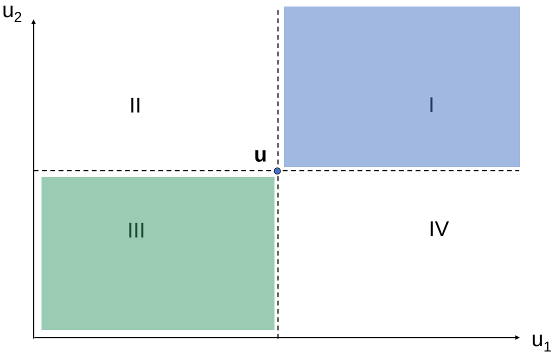
Proof: Suppose that $N = 2$ (when $N > 2$ the analysis is a simple extension)

Pick an arbitrary utility vector u.



A Diagrammatic Proof (4)

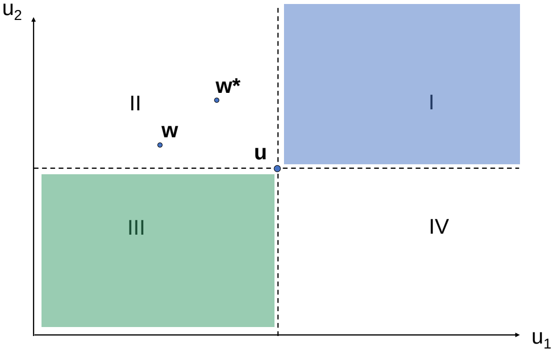
By weak Pareto, $W(u) > W(w)$ for all utility indexes w in III, and $W(w) > W(u)$ for all utility indexes w in I.



A Diagrammatic Proof (5)

Suppose that $W(u) > W(w)$ for some w in II.

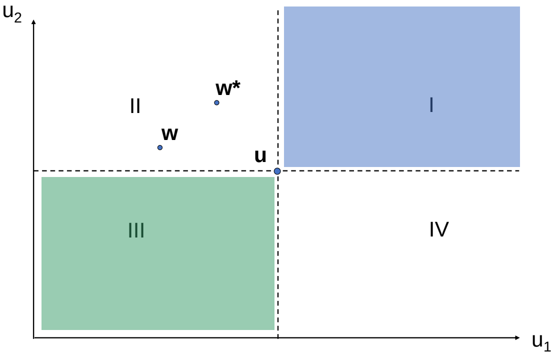
Applying the transformation $v_1(w_1) = w_1^*$ and $v_2(w_2) = w_2^*$, the OS/NC principle implies that $W(u) > W(w^*)$.



A Diagrammatic Proof (6)

This concludes that for all w in II, either $W(u) < W(w)$, $W(u) = W(w)$, or $W(u) > W(w)$.

It cannot be that $W(u) = W(w)$ because transitivity would imply $W(w^*) = W(w)$.

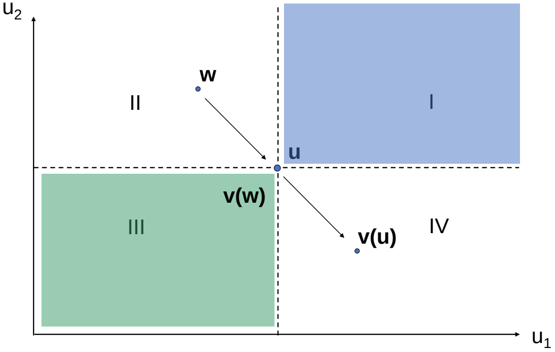


A Diagrammatic Proof (7)

Suppose that $W(u) > W(w)$. In particular, $W(u) > W(u_1 - 1, u_2 + 1)$.

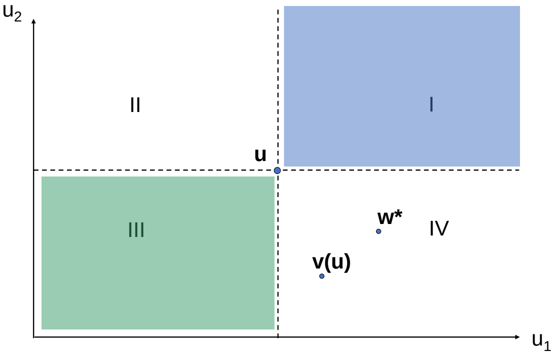
Consider the transform $v_1(u_1) = u_1 + 1, v_2(u_2) = u_2 - 1$.

By the OS/NC principle, $W(v(u)) > W(v_1(u_1 - 1), v2(u_2 + 1)) = W(u)$.



A Diagrammatic Proof (8)

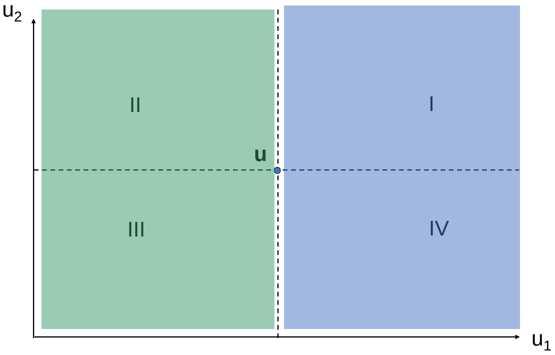
By repeating the step of quadrant II, the transform $v_1(w_1) = w_1^*$ and $v_2(w_2) = w_2^*$, OS/NC principle implies $W(u) < W(w^*)$ for all w^* in IV.



A Diagrammatic Proof (9)

The indifference curves are either horizontal or vertical. Hence there must be a dictator.

In the figure, agent 1 is the dictator.



Social Choice Functions

- The Arrow theorem concerns the impossibility to find a social preference order that satisfies minimal requirements.
- But the object of social choice may be restricted to find simply a socially optimal alternative rather than to rank all possible alternatives.
- We will show that a similar impossibility result holds when considering social choice functions.

Social Choice Functions (2)

- Let $\bar{R}$ be a subset of the set of all profile of preferences $R = (R(1), ...R(N))$.
- A social choice function f defined on $\bar{R}$ assigns a single element to every profile R in $\bar{R}$.
- The function f is monotonic if, for any R and R' such that $x = f(R)$ maintains its position from R to R' we have that $x = f(R')$.
- The alternative x maintains its position from R to R', if $xR(i)y$ implies $xR'(i)y$ for all i and y.

Social Choice Functions

- The social choice function f is weakly Paretian if $y \neq f(R)$ whenever there is x s.t. $xR(i)y$ for all i.

- An individual n is dictatorial for f if, for every R, $f(R) \in \{x : xR(n)y$ for all $y \in X\}$.

Theorem

Suppose that there are at least three distinct alternatives. Then any weakly Paretian and monotonic social function defined on the whole domain of preferences is dictatorial.

Arrow Theorem for Social Choice Functions

Proof. The result is shown as a corollary of Arrow impossibility theorem.

We derive a social rule $F(R)$ from f, using the properties of f on all $\bar{R}$.

We show that F satisfies all Arrow axioms but ND.

Step 1. Given a subset X' of X and R, we say that R' takes X' to the top from R, if for every i,

. $xP'(i)y$ for all x in X' and y not in X';

. $xR(i)y$ if and only if $xR'(i)y$ for all x, y in X'.

Arrow Theorem for Social Choice Functions (2)

Step 2. If both R' and R'' take X' to the top from R, then $f(R') = f(R'')$.

In fact, by WP, $f(R')$ is in X'.

Because $f(R')$ maintains its position from R' to R'', $f(R') = f(R'')$ by monotonicity.

Step 3. For every R, we let $xF(R)y$ if $x = f(R')$ when R' is any profile that takes $\{x, y\}$ to the top of profile R.

By Step 1, F is well defined.

Arrow Theorem for Social Choice Functions (3)

Step 4. Because f is weakly Paretian, if R' takes $\{x, y\}$ to the top from R, then $f(R)$ in $\{x, y\}$.

By step 1 and 2, either $xF(R)y$ or $yF(R)x$ (but not both).

Hence F is complete.

Step 5. Suppose that $xF(R)y$ and $yF(R)z$.

If R'' takes $\{x, y, z\}$ to the top of R, then $F(R'')$ in $\{x, y, z\}$, because f is weakly Paretian.

Say by contradiction that $y = f(R'')$.

Consider R' that takes $\{x, y\}$ to the top from R''.

By monotonicity, $f(R') = y$.

Arrow Theorem for Social Choice Functions (4)

But R' also takes $\{x, y\}$ to the top from R.

This contradicts $xF(R)y$. In sum, $y \neq f(R'')$.

Similarly $z \neq f(R'')$, by contradiction with $yF(R)z$.

Thus $x = f(R'')$.

Let R' take $\{x, z\}$ to the top from R''.

By monotonicity, $x = f(R')$.

But R' also takes $\{x, z\}$ to the top from R, and hence $xF(R)z$.

Hence $F(R)$ is transitive.

Arrow Theorem for Social Choice Functions (5)

Step 6. The social rule $F(.)$ satisfies Weak Pareto.

In fact, if $xR(i)y$ for all i, then $x = f(R')$ whenever R' takes $\{x, y\}$ to the top from R.

Thus $xF(R)y$, by step 1.

Step 7. The social rule $F(.)$ satisfies IIA.

In fact, if R and R' have the same ordering for x, y, and R'' takes $\{x, y\}$ to the top of R, then it also takes $\{x, y\}$ to the top of R'.

Hence, $x = f(R'')$ implies: $xF(R)y$ and $xF(R')y$.

Arrow Theorem for Social Choice Functions (6)

Step 8. The social choice function f is dictatorial.

By Arrow theorem, there is an agent i such that for every profile R, we have $xF(R)y$ when $xR(i)y$.

Thus if $xR(i)y$ for all y, then $x = f(R)$.

This final step concludes the proof. □

We have shown how to construct the social rule $F(R)$ starting from the social choice function f.

Applying Arrow's theorem, we have shown that there is a dictator for $F(R)$, and so f is dictatorial.