Final Exam

Part A. Consider the following game of dissolving a partnership that owns an asset. The value to partner i of owing the whole asset is denoted v_i and is private information to i. It is common knowledge that these values are independently and uniformly distributed on $[0, 1]$. Partner 1 initially has an ownership share of s, and partner 2 has the share of $1 - s$. As is common in such cases, partners have developed a high level of animosity towards each other so one of them has to go. They have decided to play the following two-stage game. Partner 1 makes an offer of b to partner 2. If partner 2 accepts, partner 1 pays 2 the amount $(1 - s)b$ and then 1 owns 100 percent of the asset. If 2 rejects the offer, then 2 pays 1 the amoung sb and 2 owns 100 percent of the asset.

a. [5 marks] Write down the payoffs of each player.

b. [45 marks] Find the perfect Bayesian equilibrium of the game.

Part B.

Question 1. Three committee members must pick one (and only one) candidate from a list of four candidates, $\{A, B, C, D\}$, to fill a position. The strict ordinal preferences of the committee members over these four candidates are as follows:

$$C \succ_1 B \succ_1 D \succ_1 A$$

$$B \succ_2 C \succ_2 A \succ_2 D$$

$$A \succ_3 B \succ_3 C \succ_3 D$$

where $X \succ_i Y$ means that committee member i strictly prefers candidate X to Y.

The committe has agreed to use the "successive veto" procedure to select a candidate: First, member 1 is allowed to veto (ie, eliminate) any one candidate from the list; member 2 then vetoes one candidate from the remaining list; and finally, member 3 vetoes one candidate from the remaining list. The candidate who has not been eliminated at the end of this process is selected to fill the position.

a. [20 marks] Draw the extensive form of this game and find all subgame perfect equilibria assuming the game has perfect information (you don't need to draw the normal form).

b. [20 marks] How would the set of subgame perfect equilibria change if the order of play were 1 then 3 then 2?

c. [10 marks] Find a pure strategy Nash equilibrium of the game in (a) which supports B as the outcome.

Question 2. Consider a duopoly in which the profit functions of the two firms are:

$$\pi_1 = 200p_1 - 15{p_1}^2 + 10p_1p_2$$
$$\pi_2 = 200p_2 - 15{p_2}^2 + 10p_1p_2$$

a. [25 marks] Find the Bertrand pricing (firms choosing prices simultaneously) equilibrium for this game.

b. [25 marks] Is the above equilibrium Pareto efficient? If not, show how it can be improved (through collusive strategies).

Answers

Part A.

a. If 2 accepts an offer, their payoffs are $(v_1 - (1-s)b, (1-s)b)$. If 2 rejects the payoffs are $(sb, v_2 - sb)$.

b. Given the payoffs in (a), Partner 2's best response to an offer of b is to accept if $(1-s)b > v_2 - sb$, reject if $(1-s)b < v_2 - sb$, and mix 50:50 if $(1-s)b = v_2 - sb$. Now, expected utility of 1 is

$$EU_1(b) = v_1 - (1-s)b\,\Pr(2\text{ accepts}) + sb\,\Pr(2\text{ rejects})$$

$\Pr(2\text{ accepts}) = \Pr((1-s)b \geq v_2 - sb)$ and $\Pr(2\text{ rejects}) = 1 - \Pr(2\text{ accepts})$. Hence

$$EU_1(b) = \begin{cases} (v_1 - (1-s)b)b + sb(1-b) & \text{if } b \leq 1 \\ v_1 - (1-s)b & \text{if } b > 1 \end{cases}$$

The latter case is impossible since for $b > 1$, $v_1 - (1-s)b > sb \Rightarrow v_1 > b$. Hence $b \leq 1$ and $EU_1(b) = (v_1 + s)b - b^2$ Maximizing over b we obtain $b^* = \frac{v_1+s}{2}$ as partner 1's optimal offer.

Hence in a perfect Bayesian equilibrium we have $\sigma_1^*(b = \frac{v_1+s}{2}|v_1) = 1$,

$$\sigma_2^*(\text{ accept } |v_2, b) = \begin{cases} 1 & \text{if } v_2 < b \\ \frac{1}{2} & \text{if } v_2 = b \\ 0 & \text{if } v_2 > b \end{cases}$$

and the supporting belief is $\mu_2(v_1|b) = 1$.

Part B.

1a. Subgame perfect equilibria are the set of all strategies (in terms of notation, the capital letter means "remove the respective candidate") that are consistent with the following path: $(1aB, 2bA, 3dD)$ with C as an outcome. In particular, the full list would include $(1aB, 2aC, 3aC)$, $(1aB, 2aC, 3bD)$, $(1aB, 2aC, 3cC)$, $(1aB, 2aC, 3dD)$, $(1aB, 2aC, 3eD)$, $(1aB, 2aC, 3fC)$, $(1aB, 2aC, 3gD)$, $(1aB, 2aC, 3hD)$, $(1aB, 2aC, 3iB)$, $(1aB, 2aC, 3jC)$, $(1aB, 2aC, 3kC)$, $(1aB, 2aC, 3lB)$, ... (the rest is obtained by replacing $2aC$ with $2bA, 2cA$ and $2dA$, leaving everything else unchanged).The equilibria follow as a result of each player maximizing her payoff in every subgame.

1b. Subgame perfect equilibria are the set of all strategies that are consistent with the following paths: $(1aA, 3aC, 2bD)$, $(1aA, 3aD, 2cC)$, $(1aD, 3dA, 2jC)$, $(1aD, 3dC, 2lA)$ with B as an outcome in all cases.

1c. For instance, the following could be the required equilibrium: $(1aA, 2aD, 3cC)$ with B as an outcome. This is a Nash equilibrium, because if any two players use their suggested strategies, the third cannot benefit by playing some other strategy.

2a. Firm 1 maximizes profit with respect to its price taking firm 2's price as given:

$$\max_{p_1} 200p_1 - 15{p_1}^2 + 10p_1p_2$$

The FOC is $200 - 30p_1 + 10p_2 = 0$. This gives the best response function $p_1 = \frac{20}{3} + \frac{1}{3}p_2$. As the firms are identical, the second firm's best response function is $p_2 = \frac{20}{3} + \frac{1}{3}p_1$. Solving the two gives the NE $p_1 = p_2 = 10$. When the two firms use these strategies the profits are $\pi_1 = \pi_2 = 1500$.

2b. The total profit of the two firms is:

$$\max_{p_1,p_2} \pi_1 + \pi_2 = (200p_1 - 15{p_1}^2 + 10p_1p_2) + (200p_2 - 15{p_2}^2 + 10p_1p_2) \tag{1}$$

The FOC with respect to p_1 gives $200 - 30p_1 + 20p_2 = 0$ and FOC with respect to p_2 gives $200 - 3p_2 + 20p_1 = 0$. Solving simultaneously gives $p_1^c = p_2^c = 20$ giving each firm a profit of $\pi^c = 2000$. Now we show that this profit can be achieved in the long run with an appropriate discount factor. Now consider the following strategy: firm i plays the collusive strategy ($p_i = 20$) in the first period. If firm i observes firm j playing collusively in $t - 1$ then firm i will continue to play the collusive strategy. If firm i observes j playing any other strategy in period t, then firm i will play the Nash strategy by setting $p_i = 10$ from period t on. To show that the collusive solution can be supported, one must show that if one firm plays this strategy then the other's best response is also to follow the same strategy. Without loss of generality, suppose that firm 2 follows the same strategy. If firm 1 follows the trigger strategy then it will earn 200 each period forever. This gives a present value of $\frac{2000}{1-\delta}$. If firm 1 deviates from this strategy it should do so immediately and should deviate in the optimal way by setting $p_1 = \frac{40}{3}$ and earning $\pi_1 = \frac{8000}{3}$. It will foresee that after the first period, the outcome will be the Nash outcome forever. The present value of profit if firm 1 deviates is then $\frac{8000}{3} + \delta(\frac{1500}{1-\delta})$. Firm 1's best response is to follow the trigger strategy if:

$$\frac{2000}{1-\delta} > \frac{8000}{3} + \delta(\frac{1500}{1-\delta}) \tag{2}$$

which reduces to $\delta > \frac{4}{7}$. This is the range of δ for which the trigger strategy supports the collusive (and more efficient) solution.