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EC9D31 Advanced Microeconomics
Final Exam 2020-21 - Section A
Questions and Answers

Question 1. Suppose preferences take the form:

$$u(x_1, x_2) = \min\{2x_1, x_2/2\}.$$

(a) Derive the Marshallian demands $x_i(p, m)$, $i = 1, 2$. Are the goods Marshallian complements or substitutes? **(5 marks)**

(b) Derive the indirect utility function $v(p, m)$. Show that it is homogeneous of degree zero in prices and income. **(4 marks)**

(c) Derive the expenditure function $e(p, U)$. Show that it is homogenous of degree 1 in prices. **(4 marks)**

(d) Derive the Hicksian demands $h_i(p, U)$, $i = 1, 2$. Are the goods Hicksian complements or substitutes? **(4 marks)**

(e) Suppose that a third good x_3 becomes available, such that preferences take now the form

$$u(x_1, x_2, x_3) = \min\{2x_1 + x_2, x_3/2\}.$$

Derive Marshallian demands, $x_i(p, m)$, and Hicksian demands, $h_i(p, U)$, $i = 1, 2, 3$. [Hints. For what prices does the consumer simultaneously consume goods 1 and 3? What happens for all other prices?] **(8 marks)**

Answers to Q1 We proceed in sequence as follows.

(a) The optimal choice occurs at $2x_1 = x_2/2$, and hence $x_2 = 4x_1$, so that the budget constraint budget constraint $p_1x_1 + p_2x_2 = p_1x_1 + p_2(4x_1) = y$.

Solving out, $x_1(p, y) = y/(p_1 + 4p_2)$, and hence $x_2 = 4y/(p_1 + 4p_2)$.

(b) Substituting the Marshallian demands into the utility formula: $v(p,y) = 2x_1(p,y) = 2y/(p_1 + 4p_2)$.

(c) By setting $v = u$ and $y = e$ in v(p,y) and solving for e(p,u) we get $e(p,u) = p_1u/2 + p_2 2u = (p_1/2 + 2p_2)u$.

(d) Using Shepard Lemma $\frac{de(p,u)}{dp_1} = u/2 = h_1(p,U)$ and $\frac{de(p,u)}{dp_2} = 2u = h_2(p,U)$.

(e) At the optimum $x_1 + 2x_2 = x_3/2$ and the consumer consumes $x_1 = 0$ if $p_1 > p_2/2$ and $x_2 = 0$ if $p_1 < p_2/2$.

In the first case, $2x_2 = x_3/2$, and hence $x_3 = 4x_2$, so that the budget constraint $p_1x_1 + p_2x_2 + p_3x_3 = p_2x_2 + p_3(4x_2) = y$.
Solving out, $x_2(p,y) = y/(p_2 + 4p_3)$, and hence $x_3 = 4y/(p_2 + 4p_3)$ and $v(p,y) = 2x_2(p,y) = 2y/(p_2 + 4p_3)$.
Further, because $u = 2x_2$, it follows that $h_2 = u/2$, and hence that $x_3 = 2u$ and $e(p,u) = p_2u/2 + p_3 2u = (p_2/2 + 2p_3)u$.

In the second case, $x_1 = x_3/2$, so that the budget constraint $p_1x_1 + p_2x_2 + p_3x_3 = p_1x_1 + p_3(2x_1) = y$.
Solving out, $x_1(p,y) = y/(p_1 + 2p_3)$, and hence $x_2 = 2y/(p_1 + 2p_3)$ and $v(p,y) = x_2(p,y) = y/(p_2 + 2p_3)$.
Further, because $u = x_1$, it follows that $h_1 = u$, $h_3 = 2u$ and $e(p,u) = p_1u + p_3(2u) = (p_1 + 2p_3)u$.

Question 2. Consider a Cobb-Douglas Production function:

$$f(x) = x_1^\alpha x_2^\beta$$

where $\alpha > 0$, $\beta > 0$ and make no assumptions on $\alpha + \beta$.

(a) Set up the cost minimization problem and write up the Lagrangian. **(5 marks)**

(b) Derive the conditional factor demands $h_1(w,y)$ and $h_2(w,y)$. **(5 marks)**

(c) Find the 2×2 matrix of marginal price effects. Confirm the signs (and, where appropriate, relative magnitudes) of these effects. **(5 marks)**

(d) Find the cost function $c(w,y)$. Confirm its properties. **(5 marks)**

(e) Prove the following result: A technology exhibits CRS if and only if the production function $f(x)$ (if available) is homogeneous of degree 1. **(5 marks)**

Answers to Q2 We proceed in sequence as follows.

(a) The cost minimization problem is:

$$\min_{x_1,x_2} w_1x_1 + w_2x_2 \qquad \text{s.t.} \quad x_1^\alpha x_2^\beta \geq y.$$

The consequent Lagrangian is:

$$\mathcal{L} = w_1x_1 + w_2x_2 - \lambda(x_1^\alpha x_2^\beta - y)$$

(b) The conditional factor demands are:

$$h_1(w_1, w_2, y) = \left(\frac{\alpha}{w_1}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{\beta}{w_2}\right)^{\frac{-\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}}$$

and

$$h_2(w_1, w_2, y) = \left(\frac{\alpha}{w_1}\right)^{\frac{-\alpha}{\alpha+\beta}} \left(\frac{\beta}{w_2}\right)^{\frac{\alpha}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}}.$$

(c) The matrix of marginal price effects is:

$$\begin{bmatrix} \frac{\partial h_1}{\partial w_1} & \frac{\partial h_1}{\partial w_2} \\ \frac{\partial h_2}{\partial w_1} & \frac{\partial h_2}{\partial w_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{w_1}\frac{\beta}{\alpha+\beta}h_1 & \frac{1}{w_2}\frac{\beta}{\alpha+\beta}h_1 \\ \frac{1}{w_1}\frac{\alpha}{\alpha+\beta}h_2 & -\frac{1}{w_2}\frac{\alpha}{\alpha+\beta}h_2 \end{bmatrix}.$$

(d) The cost function is:

$$c(w, y) = w_1h_1(w, y) + w_2h_2(w, y) = y^{\frac{1}{\alpha+\beta}} \, (\alpha + \beta) \, \left(\frac{w_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$$

(e) Assume CRS: this implies that if $z \in Z$ then $t\ z \in Z$, for all $t \geq 0$. By definition, $z \in Z$ means $y \leq f(x)$ and $t\ z \in Z$ means $t\ y \leq f(t\ x)$. By definition of $f(x)$ choose z, and hence x and y, so that $y = f(x)$. We can then re-write the condition above as: $t\ f(x) \leq f(t\ x)$. We need to prove that the equality holds. Suppose it does not. Then there exists y' such that $t\ f(x) < y' < f(t\ x)$ Now $y' < f(t\ x)$ implies by definition of Z that $\begin{pmatrix} -t\ x \\ y' \end{pmatrix} \in Z$ and by CRS we get $\frac{1}{t}\begin{pmatrix} -t\ x \\ y' \end{pmatrix} \in Z$, or $\begin{pmatrix} -x \\ \frac{1}{t}\ y' \end{pmatrix} \in Z$ which means $(1/t)\ y' \leq f(x)$, or $y' \leq t\ f(x)$.

This latter inequality contradicts $t\ f(x) < y'$.

The opposite implication is an immediate consequence of the definition of homogeneity of degree 1.

Question 3. There are two consumers A and B with the following utility functions and endowments, with $\omega_1 \geq \omega_2$, $\alpha \in [0,1]$ and $\beta \geq 1$:

$$u_A = \alpha \ln x_{1A} + (1-\alpha)\ln x_{2A}, \quad \boldsymbol{\omega}_A = (0, \omega_2)$$
$$u_B = \min\{\beta x_{1B}, (1-\beta)x_{2B}\}, \quad \boldsymbol{\omega}_B = (\omega_1, 0).$$

(a) Derive the Marshallian demands $x_i(p,m)$, $i = A, B$. **(5 marks)**

(b) Calculate the market clearing prices and the equilibrium allocations. **(5 marks)**

(c) Explain how the Walrasian equilibrium price of good 1 varies with α and β. **(5 marks)**

(d) Calculate the effect of an increase in ω_1 or ω_2 on the equilibrium price of good 1. **(5 marks)**

(e) In general, an allocation $(x_1, x_2, ..., x_L)$ in an exchange economy is said to be Pareto-efficient if there does not exist another feasible allocation $(x'_1, x'_2, ..., x'_L)$ such that: (a) $u_l(x'_l) \geq u_l(x_l)$, for all l; and (b) $u_l(x_l) > u_l(x_l)$, for some l. Prove that a Walrasian equilibrium allocation $(x^*_1, x^*_2, ..., x^*_L)$ is Pareto-efficient. **(5 marks)**

Answers to Q3 We proceed in sequence as follows.

(a) Let p be the price of good 1 and normalize $p_2 = 1$.

Given price p, consumer A chooses $\mathbf{x}_A$ so that

$$\max \alpha \ln x^1_A + (1-\alpha)\ln x^2_A \qquad s.t. \qquad px^1_A + x^2_A = \omega_2.$$

Hence,

$$\max \alpha \ln x^1_A + (\omega_2 - \alpha)\ln(\omega_2 - px^1_A),$$

first-order conditions are:

$$\frac{\alpha}{x_A^1} = p\frac{(1-\alpha)}{\omega_2 - px_A^1},$$

solving out, $x_A^1 = \alpha\omega_2/p$, substituting back, we obtain: $x_A^2 = \omega_2(1-\alpha)$.

Given price p, consumer B chooses $\mathbf{x}_B$ so that

$$\max\min\{\beta x_B^1, (1-\beta)x_B^2\} \qquad s.t. \qquad px_B^1 + x_B^2 = p\omega_1.$$

The consumer chooses $\beta x_B^1 = (1-\beta)\, x_B^2$, solving this together with $px_B^1 + x_B^2 = p\omega_1$ yields:

$$x_B^1 = \frac{p\,(1-\beta)\,\omega_1}{p\,(1-\beta)+\beta},\ x_B^2 = p\beta\frac{\omega_1}{p\,(1-\beta)+\beta}$$

(b) Market clearing condition, therefore, is:

$$x_A^1 + x_B^1 = \frac{\alpha\omega_2}{p} + \frac{p\,(1-\beta)\,\omega_1}{p\,(1-\beta)+\beta} = \omega_1$$

Hence the equilibrium price is:

$$p = \frac{\alpha\beta\omega_2}{\beta\omega_1 - \alpha\omega_2(1-\beta)}$$

and the equilibrium allocations are

$$x_A^1 = \omega_1 - \alpha\omega_2\frac{(1-\beta)}{\beta}, \quad x_A^2 = \omega_2(1-\alpha),$$
$$x_B^1 = \alpha\omega_2\frac{1-\beta}{\beta},\ x_B^2 = \alpha\omega_2.$$

(c) The price p of good 1 is:

$$p = \frac{\alpha\beta\omega_2}{\beta\omega_1 - \alpha\omega_2(1-\beta)},$$

differentiating with respect to α and β, I obtain:

$$\frac{\partial}{\partial\alpha}\left(\frac{\alpha\beta\omega_2}{\beta\omega_1 - \alpha\omega_2(1-\beta)}\right) = \frac{\beta^2\omega_1\omega_2}{(\beta\omega_1 - \alpha\omega_2(1-\beta))^2} > 0$$

$$\frac{\partial}{\partial\beta}\left(\frac{\alpha\beta\omega_2}{\beta\omega_1 - \alpha\omega_2(1-\beta)}\right) = -\frac{\alpha^2\omega_2^2}{(\beta\omega_1 - \alpha\omega_2(1-\beta))^2} < 0.$$

The equilibrium price of good 1 increases in α and decreases in β.

(d) Differentiating with respect to ω_1 and ω_2, I obtain:

$$\frac{\partial}{\partial \omega_1}\left(\frac{\alpha\beta\omega_2}{\beta\omega_1 - \alpha\omega_2(1-\beta)}\right) = -\frac{\alpha\beta^2\omega_2}{(-\alpha\omega_2 + \beta\omega_1 + \alpha\beta\omega_2)^2} < 0,$$

$$\frac{\partial}{\partial \omega_2}\left(\frac{\alpha\beta\omega_2}{\beta\omega_1 - \alpha\omega_2(1-\beta)}\right) = \frac{\alpha\beta^2\omega_1}{(-\alpha\omega_2 + \beta\omega_1 + \alpha\beta\omega_2)^2} > 0.$$

The equilibrium price of good 1 decreases in ω_1 and increases in ω_2.

(e) Assume that the result is not true. There exists an allocation x such that $\sum_{i=1}^{I} x^i \leq \bar{\omega}$, $u_i(x^i) \geq u_i(x^{i,*})$ for all i and $u_i(x^i) > u_i(x^{i,*})$ for some i.

Then, let's first show that, for all i,

$$p^* x^i \geq p^* x^{i,*}. \tag{1}$$

Assume that this is not true and there exists i such that $p^* x^i < p^* x^{i,*}$. From $p^* x^{i,*} = p^* \omega^i$ we then get $p^* x^i < p^* \omega^i$. This implies that there exists $\varepsilon > 0$ such that if we denote e^T the vector $e^T = (1, \ldots, 1)$, then $p^*\,(x^i + \varepsilon\; e) < p^* \omega^i$. Monotonicity of preferences then implies that $u_i(x^i + \varepsilon\; e) > u_i(x^i)$ which together with the contradiction hypothesis gives: $u(x^i + \varepsilon\; e) > u_i(x^{i,*})$. This contradicts $x^{i,*} = x^i(p^*)$.

Since for some i we have $u_i(x^i) > u_i(x^{i,*})$ then let's show that, for the same i,

$$p^*\; x^i > p^* x^{i,*}. \tag{2}$$

Assume this is not the case. Then there exists a consumption bundle x^i which is affordable for i: $p^* x^i \leq p^* x^{i,*} = p^*\; \omega^i$ and yields a higher level of utility: $u_i(x^i) > u_i(x^{i,*})$. This is a contradiction of the hypothesis $x^{i,*} = x^i(p^*)$.

Adding up Conditions (1) and (2) across consumers we obtain: $\sum_{i=1}^{I} p^* x^i > \sum_{i=1}^{I} p^* x^{i,*}$ or $\sum_{i=1}^{I} p^* x^i > \sum_{i=1}^{I} p^* x^{i,*} = p^* \bar{\omega}$. This is a contradiction of the feasibility of the allocation x.

Question 4. There are three individuals in society, $\{1, 2, 3\}$, three alternatives, $\{x, y, z\}$, and the domain of preferences is unrestricted. Suppose that the social preference relation, R, is given by pairwise majority voting (where voters break any indifferences by voting for x first then y then z) if this results in a transitive social order. If this does not result in a transitive social order the social order is $xPyPz$. Let f denote the social welfare function that this defines.

(a) Consider the following profiles, where P_i is individual i's strict preference relation:

Individual 1: xP_1yP_1z

Individual 2: yP_2zP_2x

Individual 3: zP_3xP_3y

What is the social order? **(3 marks)**

(b) What would be the social order if individual 1's preferences in (a) were instead yP_1zP_1x? or instead zP_1yP_1x? **(5 marks)**

(c) Prove that f satisfies the Pareto property, WP. **(3 marks)**

(d) Prove that f is non-dictatorial. **(3 marks)**

(e) Conclude that f does not satisfy IIA. **(3 marks)**

(f) Prove the following result: A social welfare rule is majoritarian if and only if it is neutral, anonymous, and positively responsive. **(8 marks)**

Answers to Q4 We proceed in sequence as follows.

(a) The preferences xP_1yP_1z, yP_2zP_2x, zP_3xP_3y determine a Condorcet cycle, hence the social order is $xPyPz$.

(b) With preferences yP_1zP_1x, yP_2zP_2x, zP_3xP_3y, the social order is $yPzPx$. With preferences zP_1yP_1x, yP_2zP_2x, zP_3xP_3y, the social order is $zPyPx$

(c) The social choice function f satisfies Weak Pareto: if xP_iy for all i, then x and y cannot be part of a Condorcet cycle, and xPy. Thus, $y \neq f(R)$.

(d) The social choice function f is not dictatorial: consider any agent i and pair of alternatives x, y such that xP_iy. Consider the profile of opponents' preferences R_{-i} such that y is at the top of R_j and x is at the bottom, for all $j \neq i$. Then x and y cannot be part of a Condorcet cycle, and yPx.

(e) The social choice function f cannot satisfy IIA, or else this would be a violation of Arrow impossibility theorem.

(f) Suppose that there are only two alternatives: x is the status quo, and y is the alternative. Each individual preference $R(i)$ is indexed as q in $\{-1, 0, 1\}$, where 1 is a strict preference for x. The social welfare rule is a functional $F(q(1), ..., q(N))$ in $\{-1, 0, 1\}$.

The social rule F is *anonymous* if for every permutation p, $F(q(1), ..., q(N)) = F(q(p(1)), ..., q(p(N)))$.

The social rule F is *neutral* if $F(q) = -F(-q)$.

The rule F is *positively responsive* if $q \geq q'$, $q \neq q'$ and $F(q') \geq 0$ imply that $F(q) = 1$.

A social welfare rule F is *majoritarian* if:

. $F(q) = 1$ if and only if: $n^+(q) = \#\{i : q(i) = 1\} > n^-(q) = \#\{i : q(i) = -1\}$,

. $F(q) = -1$ if and only if $n^+(q) < n^-(q)$,

. $F(q) = 0$ if and only if $n^+(q) = n^-(q))$.

May's Theorem A social welfare rule is majoritarian if and only if it is neutral, anonymous, and positively responsive.

Proof: Clearly, majority rule satisfies the 3 axioms.

By anonimity $F(q) = G(n^+(q), n^-(q))$. If $n^+(q) = n^-(q)$, then $n^+(-q) = n^-(-q)$, and so, by neutrality, $F(q) = G(n^+(q), n^-(q)) = G(n^+(-q), n^-(-q)) = F(-q) = -F(q)$. This implies that $F(q) = 0$. If $n^+(q) > n^-(q)$, pick q' with $q' < q$ and $n^+(q'^-(q')$. Because $F(q') = 0$, by positive responsiveness, it follows that $F(q) = 1$. When $n^+(q) < n^-(q)$, it follows that $n^+(-q) > n^-(-q)$, hence $F(-q) = 1$ and by neutrality, $F(q) = -1$.