# Advanced Economic Theory
# Models of Elections
# Lecture 2

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## Downsian elections with ideological candidates

. Suppose there are two candidates $i = L, R$ with ideologies $b_i$ such that $b_L < m < b_R$, and $m - b_L < b_R - m$.

. The utility of candidate $i$ if policy $x$ is implemented is $u_i(x, b_i) = L(|x - b_i|)$, with $L' < 0$.

**Theorem** The unique Nash Equilibrium is such that candidates $i$ choose $x_i = m$, and tie (although candidates are ideological).

*Proof.* For any $x_L \neq x_R$, if $x_i < x_j$, candidate $i$'s vote share is $F(\frac{x_L + x_R}{2})$, and candidate $j$'s is $1 - F(\frac{x_L + x_R}{2})$.

. Suppose that $x_L < m$, then candidate $R$ wins and implements $x_R$ by choosing $x_R$ in $(x_L, 2m - x_L)$.

. Hence, if $x_L < 2m - b_R$, $R$'s best response $BR_R(x_L) = \{b_R\}$, and if $2m - b_R < x_L < m$, then $BR_R(x_L)$ is empty.

. But if $x_R = b_R$, then $BR_L(x_R)$ is empty.

. If $m < x_L < b_R$, then $BR_R(x_L) = [x_L, +\infty)$.

. If $x_L > b_R$, then $BR_R(x_L) = \{b_R\}$.

. But if $x_R > x_L > m$ or $x_R = b_R$, then $x_L \notin BR_L(x_R)$.

. Hence, there is no Nash Equilibrium with $(x_L, x_R) \neq (m, m)$.

. Suppose that candidate i chooses $x_i = m$.

. Then, implemented policy is $m$ regardless of $x_j$, and $BR_j(x_i) = (-\infty, +\infty)$.

. We conclude that the unique Nash Equilibrium is $x_L = x_R = m$, and the election is tied.

## Citizen candidate models

. Key assumption of Downsian models is that politicians can commit to any policy platform, regardless of their ideology.

. Convergence to median obtains with office-motivated candidates, but also with policy motivations (if voters' preferences are known).

. What happens if politicians cannot commit and can only implement their preferred policy?

. Say voters vote for the candidate with platform they prefer.

. Then, there exist equilibria in which two or more candidates differentiate platforms.

. If voters coordinate not to vote for losing candidates, then exactly two candidates run in the election.

## Osborne and Slivinski 1996

. Policy space is $X = \mathbb{R}$ and there is a continuum of citizens $i$.

. The citizens' ideal platforms $b_i$ empirical distribution $F$ is continuous with unique median $m$.

. Each citizen $i$ chooses to run or not in the election, $e_i \in \{E, N\}$.

. If a citizen $i$ enters, she becomes a "candidate" with platform $x_i = b_i$ (citizens cannot commit to a different platform).

. After all citizens have simultaneously chosen on entry, they vote.

. Voting is "sincere:" each voter $i$ with bliss point $b_i$ votes for the candidate(s) $j$ whose platform $x_j$ is closest to $b_i$.

. Votes are split equally if multiple candidates platforms coincide.

. A citizen who chooses $E$ incurs the cost $c > 0$, and derives benefit $w > 0$ if she wins.

. Let the platform of the election winner be $x_W$.

. If citizen $i$ with ideal platform $b_i$ chooses $N$ then $i$'s payoff is

$$u_i(N, e) = -|x_W - b_i|.$$

. If citizen $i$ with ideal platform $b_i$ chooses $E$,

then her payoff is $u_i(E, e) = w - c$ if she wins,

and $u_i(E, e) = -|x_W - b_i| - c$ if she loses.

. If no citizen enters, then they all obtain the payoff of $-\infty$.

## Results

**Proposition** There is a one-candidate equilibrium iff $w \leq 2c$.
If $c \leq w \leq 2c$, then the candidate's platform is $x_W = m$.
If $w < c$, then $x_W \in [m - \frac{c-w}{2}, m + \frac{c-w}{2}]$.

. If $w > 2c$, then a second candidate would enter even just to tie.

. If $x = m$, then no entrant can defeat the candidate.

. If $w < c$, and $|m - x_W| \leq \frac{c-w}{2}$, then no-one who can defeat the candidate would strictly benefit by entering.

**Proposition** In any 2-candidate equilibrium the platforms are $x_A = m - e$ and $x_B = m + e$ for some $e \in (0, \bar{e}(F)]$.
Any such equilibrium exists if and only if $2e \geq c - w/2$, $c \geq |m - s(e, F)|$ and either $e < \bar{e}(F)$ or $e = \bar{e}(F) \leq 3c - w$.

. $s(e, F)$ is the platform such that $A$ and $B$ still tie their votes if a third candidate $C$ enters with $x_C = s(e, F)$.

. $\bar{e}(F)$ is the value of $e$ such that $A$ and $B$ lose to $C$ iff $e > \bar{e}(F)$.

. If $e > \bar{e}(F)$, then a third candidate enters and wins.

. If $e = \bar{e}(F) > 3c - w$, then a third candidate enters and ties.

. If $e < c - w/2$, then one of the two candidates drops out.

. If $c < |m - s(e, F)|$, then an entrant may want to enter and lose.

**Proposition** Every 3-candidate equilibrium is such that:
. either the election is a 3-way tie, and the platforms are $x_A = t_1 - e_1$, $x_B = t_1 + e_1 = t_2 - e_2$, $x_C = t_2 + e_2$ for some $e_1, e_2 \geq 0$, where $t_1 = F^{-1}(1/3)$, $t_2 = F^{-1}(2/3)$.
. or candidates $A$ and $C$ tie the election and $B$ loses for sure, and the platforms are $x_A < x_B < x_C$.

. A necessary condition for 3-way tie is $w \geq 3c + 2|e_1 - e_2|$.

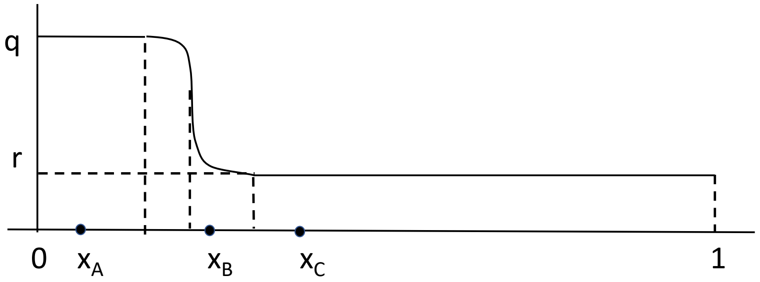
. In the 2-way tie equilibrium, candidate 2 enters to lose the election and induce a tie.

. If $B$ did not enter, her worst candidate would win for sure.

. A necessary conditions for 2-way tie is $w \geq 4c$ and $c < t_2 - t_1$:

  . if $c > t_2 - t_1$, then $B$ would not enter,

  . if $w < 4c$, then one of the two winning candidates drops out.




. Candidate B enters to lose the election.

. B's entry makes A and C tie: $q(x_A + x_B)/2 = r[1-(x_B + x_C)/2]$.

. By entering B steals more votes to A than to C.

. B is closer to C than to 1: $x_C - x_B < x_B – x_A$.

**Proposition** A necessary condition for the existence of an equilibrium in which $k \geq 3$ candidates tie for first place is $w \geq kc$. A necessary condition for the existence of an equilibrium in which there are three or more candidates is $w \geq 3c$.

. There may be multiple candidates elections.

. These equilibria generalize the logic of the 3-way tie equilibrium in the previous proposition.

. Each pair of contiguous candidates is symmetrically located around an ideologically $k$-tile, $t_1$, $t_2$, ..., $t_{k-1}$.

## Besley and Coate 1997

. Besley and Coate 1997 assume that voters vote strategically.

. Voters do not waste vote on candidates who are ideologically close to their bliss point, but have no chance to win.

. As there is a continuum of voters, no voter is pivotal.
This assumption requires coordination among voters.

. There are no equilibria in which 3 or more candidates tie election.

. There are no equilibria in which a candidate enters the election and loses for sure.

. These equilibria are upset by strategic voters who vote second best candidate, to break a tie with a candidate they dislike more.

# Probabilistic voting

. In Downsian elections, winning probabilities jump discontinuously because voters preferences are known.

. Probabilistic voting models smooth out discontinuities by adding "noise" to voters' preferences.

. If candidates maximize probability to win, then platforms converge to the expected median platform.

. If candidates maximize vote share, then platforms converge to an weighted average platform.

. Platforms may converge also in multi-dimensional policy spaces.

## Aggregate uncertainty

. Candidates maximize the probability of winning majority.

. Voters' preferences do not vary independently.
Median platform depends on a random common state.

. Each voter $j$ with bliss point $b_j \in \mathbb{R}$ has utility $L(|x - b_j|)$,
with $L' < 0$, $L'' < 0$, and $\lim_{z \downarrow 0} L'(z) = 0$, $\lim_{z \uparrow \infty} L'(z) = -\infty$.

. Each ideal point $b_j$ is decomposed as: $b_j = m + \delta_j + e_j$:

- . $\delta_j$ is the fixed $j$'s bias relative to the median platform $m$,
  the empirical distribution of $\delta_j$ across $j$ has median zero;
- . $m$ is the random median platform, with c.d.f. $F$ and median $\mu$;
- . $e_j$ is noise, i.i.d. over $j$, with symmetric density and $E[e_j] = 0$.

. As in the Downsian model there are two candidates, $i = A, B$ who care only about winning the election.

. Candidates $i$ simultaneously commit to policies $x_i \in \mathbb{R}$ if elected.

. After candidates choose platforms, each voter votes, and the candidate with the most votes wins.

. If $x_A = x_B$, then the election is tied.

**Proposition** In the unique Nash equilibrium of the probabilistic model with aggregate uncertainty, the candidates $i = 1, 2$ choose $x_i$ equal to the median $\mu$ of the distribution of the median policy $m$ and tie the election.

*Proof:* Suppose that $x_i < x_j$, then candidate $i$ wins the election if $m < (x_A + x_B)/2$ and $j$ wins if $m > (x_A + x_B)/2$.

. The probability $q_i(x_i, x_j)$ that $i$ wins the election is

$$q_i(x_i, x_j) = \begin{cases} \frac{F(x_A+x_B)}{2} & \text{if } x_i < x_j, \\ 1/2 & \text{if } \ x_i = x_j, \\ 1 - \frac{F(x_A+x_B)}{2} & \text{if } x_i = x_j. \end{cases}$$

. Given $x_j$, candidate $i$ chooses $x_i$ to maximize $q_i(x_i, x_j)$.

. Suppose that $x_j < \mu$. Then, $q_i(x_i, x_j) > 1/2$ and strictly decreasing in $x_i$ for $x_i > x_j$. $i$'s best response is empty.

. Likewise, if $x_j > \mu$, then $i$'s best response is empty.

. If $x_j = \mu$, then $q_i(x_i, x_j) < 1/2$ and strictly increasing in $x_i$ for $x_i < x_j$, $q(\mu, x_j) = 1/2$, and $q_i(x_i, x_j) < 1/2$ and strictly decreasing in $x_i$ for $x_i > x_j$. $i$'s best response is $x_i = \mu$.

. Hence, there is a unique equilibrium: $x_A = x_B = \mu$.

## Vote share maximization

- There are $G$ groups of voters $g$ with $s_g$ share of voters in each $g$.
- Candidates $i = A, B$ simultaneously announce platforms $x_i$ in $\mathbb{R}^d$.
- The payoff of voter $k$ in group $g$ is: $u_k(x, i) = L_g(x) + \eta_{ki}$
- $L_g$ is a continuously differentiable loss function, strictly decreasing in the distance $||x - b_g||$ from a bliss point $b_g$ in $\mathbb{R}^d$.
- $\eta_{ki}$ are non-policy benefits for $k$ if $i$ is in power.
- Let $\sigma_k = \eta_{kB} - \eta_{kA}$, drawn independently across individuals, with cumulative distribution $H_g$ on $\mathbb{R}$ and density $h_g$.
- Let $q_{gi}$ be fraction of voters in $g$ that vote candidate $i = A, B$.
- Candidate $i$ picks $x_i$ to maximize vote share $q_i = \sum_{g=1}^{G} s_g q_{gi}$.

## Results

- Each voter $k$ in group $g$ votes for $A$ if $L_g(x_A) - L_g(x_B) > \sigma_k$.
- Vote share for $A$ in group $g$ is $q_{gA} = H_g(L_g(x_A) - L_g(x_B))$.
- Suppose that
  - $q_A = \sum_{g=1}^{G} s_g H_g(L_q(x_A) - L_q(x_B))$ is strictly concave in $x_A$
  - $q_B = \sum_{g=1}^{G} s_g [1 - H_g(L_q(x_A) - L_q(x_B))]$ str. concave in $x_B$.
- Then the equilibrium $(x_A, x_B)$ solves the FOC:

$$\sum_{g=1}^{G} s_g h_g(L_q(x_A) - L_q(x_B)) DL_g(x_A) = 0$$

$$\sum_{g=1}^{G} s_g h_g(L_q(x_A) - L_q(x_B)) DL_g(x_B) = 0,$$

where $DL_g(x_i) = (\frac{\partial L_g}{\partial x_{i1}}, ...., \frac{\partial L_g}{\partial x_{in}})^T$.

**Proposition** If a pure strategy equilibrium $(x_A, x_B)$ of probabilistic voting model exists, then $x_A = x_B = x$ such that

$$\sum_{g=1}^{G} s_g h_g(0) DL_g(x) = 0.$$

. Nash-equilibrium corresponds to solution to maximization of weighted utilitarian social welfare function:

$$\sum_{g=1}^{G} s_g w_g DL_g(x) = 0,$$

with group weights $w_g = h_g(0)$.

. Group weight corresponds to group size and responsiveness to policy changes $h_g(0)$, i.e. share of unbiased voters/swing voters.

. When do pure strategy equilibria exist?

  . Strict concavity of $q_i$ in $x_i$ for $i = A, B$ is hard to check.

  . A sufficient condition is that for each group $g$, $H_g(L_g(x_A) - L_g(x_B))$ is strictly concave in $x_A$ and $x_B$.

## Summary

- I have presented the main alternative spatial models of elections.
- Suppose candidates have policy preferences and cannot credibly commit to platforms.
- Then there exist equilibria in which platforms "diverge" from the median policy.
- If office motivated candidates are uncertain about the voters' preferences, then platforms converge to the expected median.
- Equilibrium exist in multi-dimensional policy spaces, if candidates maximize vote shares and voters' preferences are uncertain.
- This equilibrium is Pareto efficient for the electorate.

## Next lecture

. I will introduce candidates with policy preferences in the aggregate uncertainty model.

. Because of uncertainty, equilibrium platforms diverge.

. If voters' preferences may change during campaigns, then platform divergence improves electorate welfare.

. I present a model without voter preference uncertainty, in which policy-motivated candidates diverge from median.

. By diverging, candidates signal they care about policy and will exert effort if elected.