Advanced Economic Theory
Models of Elections
Lecture 8

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Paradox of voting (Riker and Ordeshook 1968)

. Let $b > 0$ be a voter's payoff difference between her favored candidate's and the opponent's victory.

. Let $c > 0$ be the cost of voting.

. Let p_n be the probability that her vote changes the outcome of the election, when there are $2n + 1$ voters.

. $p_n = \Pr(\text{tie}) \leq \binom{2n}{n} \frac{1}{2^n} \frac{1}{2^n} = \frac{2n!}{n!n!} \frac{1}{2^{2n}} \to 0$, as $n \to \infty$.

. Voters should not bother voting in large elections: $p_n b < c$.

. This is true also if voters are altruistic and b increases in n. Let $b_n = \bar{b}(2n + 1)$, $\lim_{n \to \infty} p_n b_n = \lim_{n \to \infty} \frac{2n!}{n!n!} \frac{2n+1}{2^n} \bar{b} = 0$.

. Riker and Ordeshook say voters get benefit $d > 0$ from voting, that yields from fulfilling civic duty or for expressing own opinion.

- Palfrey and Rosenthal (1983, 1985) calculated the equilibrium turnout in a game of costly voting with fixed candidate positions:
  - high turnout equilibria are in mixed strategy, and require identical expected voting share across the two candidates;
  - they are not robust to uncertainty over voters' preferences.
- Group voting models explain turnout with group leadership:
  - Voting behavior is determined by small number of leaders;
  - each exerts high mobilization effort, leading to high turnout.
- Ethical voting explains turnout with consistent ethical rules:
  - each candidate's supporters follow a common rule,
  - the rule dictates it is ethical to vote if cost is not too high;
  - if abided by all supporters, the rule maximizes their welfare.

Group-based voting (Shachar and Nalebuff 1999)

. There two electoral leaders A, B, who favor policies a, b.

. Fraction r of voters who prefer a is unknown, with c.d.f. F.

. Each voter i's voting cost c_i is drawn from uniform dist. on $[0, \bar{c}]$.

. If A exerts effort e_a then voters i who favor a vote iff $c < p(e_a)$.

. A fraction $p(e_a)/\bar{c}$ of voters who support a vote.

. If B exerts effort e_b then fraction $p(e_b)/\bar{c}$ of b supporters vote.

. The function p and the elasticity p'/p are increasing in e.
(Note effectiveness of effort is independent of number of voters.)

. Hence, $\Pr(a \text{ wins}|e_a, e_b) = \Pr(rp(e_a)/\bar{c} > (1-r)p(e_b)/\bar{c})$

$$= 1 - F\left(\frac{p(e_b)}{p(e_a)+p(e_b)}\right).$$

- The leaders' payoffs are:

$$u_A(e) = w_a \Pr(a \text{ wins}|e_a, e_b) - e_a = w_a(1 - F(\tfrac{p(e_b)}{p(e_a)+p(e_b)})) - e_a,$$

$$u_B(e) = w_b \Pr(b \text{ wins}|e_a, e_b) - e_b = w_b F(\tfrac{p(e_b)}{p(e_a)+p(e_b)}) - e_b.$$

- Suppose payoff functions are quasi-concave and smooth.
- The equilibrium conditions are:

$$w_a F'(\tfrac{p(e_b)}{p(e_a)+p(e_b)}) \tfrac{p(e_b)}{(p(e_a)+p(e_b))^2} p'(e_a) = 1,$$

$$w_b F'(\tfrac{p(e_b)}{p(e_a)+p(e_b)}) \tfrac{p(e_b)}{(p(e_a)+p(e_b))^2} p'(e_b) = 1.$$

- Simplifying, we get: $\frac{w_a p(e_b) p'(e_a)}{w_b p(e_a) p'(e_b)} = 1$, or $\frac{w_a p'(e_a)}{p(e_a)} = \frac{w_b p'(e_b)}{p(e_b)}$.
- Say $w_a = w_b$. Then $e_a = e_b$, as p'/p is monotonic.
- Hence, in equilibrium: $w_a \frac{p(e_a)}{p'(e_a)} = w_b \frac{p(e_b)}{p'(e_b)} = \frac{1}{4} F'(\frac{1}{2})$.

. When $e_a = e_b$, $F'(\frac{1}{2})$ is the p.d.f. of a electoral tie.

. Turnout $p(e_a)/\bar{c} + p(e_b)/\bar{c}$ decreases in voting cost (lower $\bar{c}$), and increases in how close the election is expected $(F'(\frac{1}{2})$ large).

. These empirical predictions are verified in the data.

. Individuals are certainly influenced by efforts of parties and other organizations with a stake in outcome of election.

. But model does not specify mechanism by which voters are influenced: how does effort of leaders translate into votes?

. And in the model, voters are not strategic.

. But evidence suggests that voter voting behavior is at least partly strategic.

Rule utilitarianism (Feddersen and Sandroni 2006)

. Another approach to explaining turnout, based on idea that voters are motivated to vote by "ethical" concerns.

. In Nash equilibrium, each voter chooses an action that maximizes her own payoff, given other voters' actions.

. In symmetric game, a rule utilitarian chooses an action that, if chosen by every voter, maximizes the sum of the voters' payoffs.

. For example, action chosen by two rule utilitarians playing prisoner's dilemma is cooperation.

. Voting game is not symmetric—voters disagree about best alternative—and application of idea of rule utilitarianism is not straightforward.

The model

. There are two alternatives, a and b.

. There is a continuum of voters, some favor a and others favor b.

. Fraction r of voters who favor a is unknown.

. Each voter i can vote for a, vote for b, or abstain.

. Each voter's cost of voting c_i is drawn from c.d.f. G, with support $[0, \bar{c}]$ and density g.

. Each voter i knows her own voting cost c_i, but not the voting cost of any other voter.

. Because of continuum of voters, a single vote is irrelevant.

. Voters who care only about their own payoff abstain.

. Some voters are "ethical," they vote if their cost is not too high.

. The fractions q_a, q_b of ethical voters who favor a, b are unknown. They are drawn independently from c.d.f F.

. An ethical voter i who favors $x = a, b$, votes iff $c_i \leq \hat{c}_x$.

. Given choices of abstainers and of ethical voters on the opposite side, each ethical voter follows rule that maximize social welfare (according to her views) if all ethical voters in her group follow it.

. Given rule $\hat{c}_y$ for $y \neq x$, ethical voters who favor x follow the rule $\hat{c}_x$ that would maximize social welfare (as they perceive it):

$$W_x(\hat{c}_x, \hat{c}_y) = w \Pr(x \text{ wins} | \hat{c}_a, \hat{c}_b) - \theta(\phi(\hat{c}_a, \hat{c}_b),$$

where ϕ is the expected total voting cost of all voters, and θ is an increasing convex function.

. A pair of rules $(\hat{c}_a, \hat{c}_b)$ is "consistent" if for each $x = a, b$, $W_x(\hat{c}_x, \hat{c}_y) \geq W_x(\hat{c}'_x, \hat{c}_y)$ for all rules $\hat{c}'_x$ and for $y = a, b$, $y \neq x$.

Results

. We view consistent rules as equilibrium of a two-player game.

. Each player $x = a, b$ chooses $\hat{c}_x$ and receives payoff:

$$U_x(\hat{c}_x, \hat{c}_x) = W_x(\hat{c}_x, \hat{c}_y) = w\,\Pr(x \text{ wins}|\hat{c}_a, \hat{c}_b) - \theta(\phi(\hat{c}_a, \hat{c}_b)).$$

Lemma A pair of rules is consistent $(\hat{c}_a, \hat{c}_b)$ if and only if $(G(\hat{c}_a), G(\hat{c}_b))$ is a Nash equilibrium of this two-player game.

. Expected social cost of voting is:

$$\phi(\hat{c}_a, \hat{c}_b) = rE(q_a)\int_0^{\hat{c}_a} cg(c)dc + (1-r)E(q_b)\int_0^{\hat{c}_b} cg(c)dc.$$

. Winning probabilities are:

$$\Pr(a \text{ wins}|\hat{c}_a, \hat{c}_b) = \Pr(rq_aG(\hat{c}_a) \geq (1-r)q_bG(\hat{c}_b))$$
$$= Pr((1-r)q_b/rq_a \leq G(\hat{c}_a)/G(\hat{c}_b)),$$
$$\Pr(b \text{ wins}|\hat{c}_a, \hat{c}_b) = Pr(rq_a/(1-r)q_b \leq G(\hat{c}_b)/G(\hat{c}_a)).$$

. If r, q_a, q_b were fixed, then consistent rules would not exist.

. If election is tied, voters of either group $x = A, B$ would be better off slightly increasing $\hat{c}_x$ to win election outright.

. If the election is not tied, voter of winning group can slightly decrease $\hat{c}_x$ and still win, but at lower social cost.

. If r, q_a, and q_b are random, do consistent pair of rules exist?

. It is convenient to change strategic variable from cutoff value $\hat{c}_x$ of cost to fraction $d_x = G(\hat{c}_x)$ of voters who vote.

. Then $\Pr(a \text{ wins}|\hat{c}_a, \hat{c}_b) = Pr((1-a)q_b/rq_a \leq d_a/d_b)$

$$\Pr(b \text{ wins}|\hat{c}_a, \hat{c}_b) = Pr(rq_a/(1-r)q_b \leq d_b/d_a).$$

. Each player $x = a, b$ chooses $d_x \in [0, 1]$ and receives payoff:

$$U_x(d) = w \Pr(x \text{ wins}|d_a, d_b) - \theta(\psi(d_a, d_b))$$

where $\psi(d_a, d_b) = \phi(G^{-1}(d_a), G^{-1}(d_b))$.

Proposition The strategic game $(I, (S_i)_{i\in I}, (U_i)_{i\in I})$ has a pure strategy Nash equilibrium s if for all $i \in I$,

- the strategy set S_i of each player i is a nonempty compact convex subset of a Euclidean space,
- the payoff function U_i is continuous and quasiconcave on S_i.

. U_i is quasiconcave on S_i if $\{s_i' \in S_i : U_i(s_i', s_{-i}) \geq u_i(s)\}$ is convex for every $s \in \times_{j\in I} S_j$.

. The payoff functions are:

$$U_a(d_a, d_b) = w \Pr((1-r)q_b/rq_a \leq d_a/d_b) - \theta(\psi(d_a, d_b)),$$
$$U_b(d_a, d_b) = w \Pr(rq_a/(1-r)q_b d_b/d_a) - \theta(\psi(d_a, d_b))$$

where $\psi(d_a, d_b) = rE(q_a)\int_0^{H(d_a)} cg(c)dc$
$+(1-r)E(q_b)\int_0^{H(d_b)} cg(c)dc,$

defining $H = G^{-1}$.

. ψ is increasing and convex in d_a given d_b:

$$\begin{aligned}\frac{\partial\psi(d_a,d_b)}{\partial d_x} &= xE(q_x)H(d_x)g(H(d_x))H'(d_x) \\ &= xE(q_x)H(d_x)g(H(d_x))/g(H(d_x)) \\ &= xE(q_x)H(d_x) > 0.\end{aligned}$$

$$\frac{\partial^2\psi(d_a,d_b)}{\partial d_x^2} = \frac{\partial xE(q_x)H(d_x)}{\partial d_x} = xE(q_x)/g(H(d_x)) > 0.$$

. But the winning probability $\Pr((1-r)q_y/rq_x \leq d_x/d_y)$ of $x = a, b$ is not in general concave in d_x.

. Need to make assumptions on distribution functions.

Lemma If the cumulative distribution functions of the random variables $\frac{(1-r)q_b}{rq_a}$ and $\frac{rq_a}{(1-r)q_b}$ are concave, then for $x = a, b$, $y \neq x$, the payoff function U_x is concave in d_x for any d_y.

Proposition If the cumulative distribution functions of the random variables $\frac{(1-r)q_b}{rq_a}$ and $\frac{rq_a}{(1-r)q_b}$ are concave, then the two-player game has a unique Nash equilibrium d_a, d_b, and hence a unique consistent pair of rules $\hat{c}_a, \hat{c}_b$ exists.

. Apart from a technical issue with boundary values of d, concavity ensures existence.

. Assumption on c.d.f. of $\frac{(1-r)q_b}{rq_a}$ and $\frac{rq_a}{(1-r)q_b}$ is satisfied if one of following conditions is satisfied:

- . distribution functions of q_a and q_b are concave,
- . distribution functions of r and q_b are concave,
- . distribution functions of $1 - r$ and q_a are concave.

Example

. The fraction r of voters favoring a is deterministic.

. The fraction of ethical voters q_a, q_b are independently distributed uniformly on $[0, 1]$.

. Distribution G of voting costs is uniform on $[0, \bar{c}]$.

. Function θ that values social cost of voting is linear.

. These conditions are sufficient for existence of consistent rules.

. Turnout rate $\hat{c}_x$ smaller for larger group x ("underdog effect").

. Turnouts are $r\hat{c}_a$ and $(1 - r)\hat{c}_b$. Turnout higher for larger group.

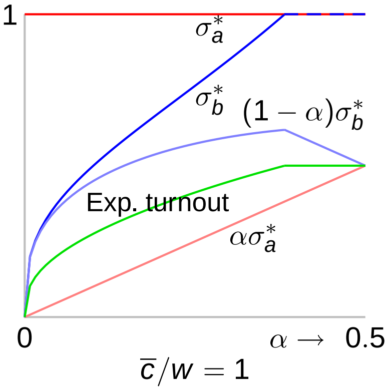
. Overall turnout $r\hat{c}_a + (1 - r)\hat{c}_b$ larger when groups closer in size.

. Margin of victory smaller when groups more similar in size.

Group rule utilitarian voting model

Example

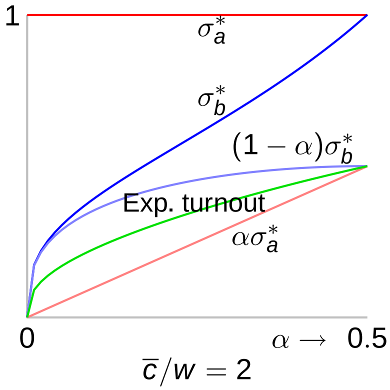
α: fraction of type *a* individuals in population
σ_t^*: fraction of type *t* ethicals who vote
$\alpha\sigma_a^*$, $(1-\alpha)\sigma_b^*$: fraction of type *a*, *b* individuals who vote



Group rule utilitarian voting model

Example

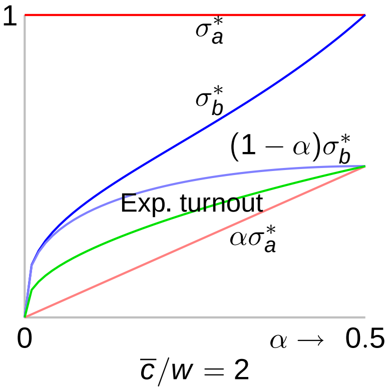
- Turnout rate smaller for larger group ("underdog effect")
- Total expected turnout higher for larger group $\Rightarrow$ larger group more likely to win election



Group rule utilitarian voting model

Example

- Turnout larger when groups more similar in size; zero when only one group
- Expected margin of victory smaller when groups more similar in size



Summary

. Voters should not bother voting in large elections.

. The probability that one vote changes the outcome is negligible.

. However, it may be that voters get a direct benefit from voting, from fulfilling civic duty or for expressing own opinion.

. Further, I have presented a group mobilization model in which voters follow a small set of leaders.

. Leaders exert high mobilization effort, leading to high turnout.

. I have presented a model of ethical voting rules: each candidate's supporter votes if her own cost is not too high.

. If obeyed by all supporters, such rules maximize their welfare.

Next Lecture

. We consider legislative bargaining.

. Repeated bargaining over fixed resources with random proposer nomination yields a unique stationary equilibrium.

. Agreement is reached after the first proposal.

. The proposer obtains the largest share, but her advantage is smaller with an open amendment rule.

. Under closed amendment rule, the proposer's advantage increases in number of legislators.

. Bargaining over policies leads to change of policies with inertia.

. An endogenous status quo induces more moderate proposals, and provides insurance to the legislators.