Patent Rights and Innovation Disclosure[*]

Hugo Hopenhayn
University of California
at Los Angeles[†]

Francesco Squintani
University of Warwick[‡]

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Abstract

This paper studies optimal patents with respect to the timing of innovation disclosure. In a simple model, we identify forces that lead firms to either suboptimally patent too early or too late in equilibrium, and we determine conditions so that stronger patents induce earlier or later equilibrium disclosure. Then, by solving an infinite multi-stage patent game with a more explicit structure, we describe innovation growth, and derive detailed predictions that can be used for policy experiments. As an application, we calibrate our multi-stage game using summary statistics from the seeds-breeding industry. We find that weaker patent rights may result in welfare gains of 46% relative to the status quo. The gains are achieved because weaker patents reduce competition, thus leading firms to postpone patenting.

1 Introduction

The traditional debate on patents contrasts their positive incentives on investment in R&D with their negative effects on competition. According to 'Schumpeterian' wisdom, R&D firms find it worthwhile to innovate only if they can secure monopolistic property rights on their intellectual findings. Others underline the negative effects of patents on social welfare through monopoly pricing and on the incentives for future innovations.[1] This paper studies optimal patent rights from a different angle: The incentives of patents with respect to the timing of innovation disclosure.[2]

We first study a simple and widely applicable continuous-time patent race model. Individual firms pay a flow cost to compete in the R&D of the same innovation. Each firm makes no progress

[*]We are grateful for their comments to David Levine, Glenn Mac Donald, and several seminar audiences. The usual caveat applies.

[†]Economics Department, UCLA, Bunche Hall 9353, UCLA Box 951477, Los Angeles, CA 90095-1477, USA.

[‡]Economics Department, University of Warwick, Coventry CV4 7L, United Kingdom

[1]Some of the strongest opponents of patent rights and intellectual monopoly, e.g. Boldrin and Levine (2007), provocatively challenge the views that patents are needed to remunerate R&D activity.

[2]We later discuss in detail earlier studies on the timing of innovation disclosure (e.g. Horstmann, MacDonald and Slivinsky, 1985; Scotchmer and Green, 1990; and Matutes, Regibeau and Rockett, 2000).

until it achieves a breakthrough, whose arrival time is exponentially distributed and is not publicly observed. After the breakthrough, a firm's innovation value grows at a deterministic rate which decreases over time, due to diminishing returns. At any moment in time, any firm may patent its innovation, and end the game. It earns the stream of profits associated with its innovation, but it discloses its technology to potential future competitors. We make no restrictive assumptions on the firms' payoffs at the end of the game, but we summarize patent strength in a single parameter.[3]

In order to assess the welfare effects of patent policy, we first compare equilibrium strategies with the socially-optimal strategies adopted by firms who internalize the consequences of their decisions on consumer welfare and their competitors' payoffs. This comparison leads us to single out three forces that can lead firms to suboptimally patent innovations either too early or too late in equilibrium. The first force leads to the most innovative insight of our paper: While the literature on the timing of innovation disclosure is mainly concerned with the possibility that firms keep innovations secret and suboptimally delay innovation disclosure, our results warn that the opposite possibility may also be detrimental to welfare. Competing firms may patent their innovations too early for fear of being preempted in a patent race. This makes research durations too short, hindering the optimal development of innovations and distorting incentives for future research.[4]

Because this 'fear of preemption force' is novel to the academic R&D literature, there are no systematic empirical studies assessing its importance.[5] But as we discuss later, there are many anecdotes in R&D history in which this force played a major role, and some research suggests that stronger patents induce early disclosure of inventions, and patents of lower value (see, e.g., Gallini, 2002). Further, it is likely that the switch from the first-to-invent to the first-to-file system, which

[3] Our basic patent race is designed within the theoretical framework developed by Hopenhayn and Squintani (2011), Botcheff and Mariotti (2012), and Botcheff, Bolte and Mariotti (2013). But unlike them we move beyond one-stage preemption games.

[4] Baker and Mezzetti (2005) uncover an independent reason for why firms' innovation disclosure policy may be socially excessive. By disclosing technologies not covered by patents, market leading firms may make it more difficult for competitors to patent their innovations, as they may cease to be sufficiently novel. This disclosure practice may be socially harmful if it reduces R&D incentives.

[5] However, in an empirical piece studying data from the U.S. Patent and Trademark Office, Cohen and Ichii (2005) establish a connection between patent interferences and patent races. They find evidence that competition in patent races expedites innovation disclosure. Similarly, Cohen et al. (2000) surveyed 1478 R&D labs in the U.S. manufacturing sector, and found evidence of disclosure lags when firms do not compete in patent races.

took effect in the Spring of 2013 in the US, increased the relevance of fear of preemption. (Under the old first-to-invent system, a firm would have been less concerned with preemption, if in the position of verifying the timing of its innovation in Court.)

Further, we identify a 'duplication cost' force that leads firms to suboptimally delay disclosure, because they do not internalize the costs paid by competitors in their decision of when to end the game. Finally, we uncover a 'real-option tradeoff' that may make firms stop too early or too late depending on whether society values the option of delaying disclosure more or less than individual firms do, relative to the value of patenting the innovation.[6]

We then turn to policy analysis. Contrary to usual presumptions, we find that stronger patents may make patents less frequent. We identify a simple condition that discriminates whether stronger patent rights make firms patent earlier or later, in the equilibrium of our model. This condition depends on whether stronger patents increase the private option value for delaying disclosure more or less than they increase the patent value.

We argue that our simple model is widely applicable. One application is a class of two-stage patent races that have been the subject of many studies on sequential innovations. In the first stage, firms compete on a 'basic' innovation that has no commercial value per se, but that leads to valuable applications developed in the second stage.[7] Our basic model can be interpreted as the first stage of one such race, where the payoffs are thought of coming from the solution of such a second stage. Our main finding that firms may patent their innovations too early may bear important implications. Because early disclosure depresses the value of investing in basic research, we predict that it may be under-provided even in the presence of strong patent rights.

In the second part of the paper, we expand and detail our findings by fully solving an infinite multi-stage patent game with a more explicit structure than the simple game we described earlier. Unlike in the first part of the paper, the study of the infinite multi-stage patent game allows us to explicitly describe innovation growth, and formally distinguish it from social welfare. Each stage

[6] As we discuss in the literature review, our real option tradeoff is related to insights by Matutes, Regibeau and Rockett (1996).

[7] Among those who study this class of games, see Green and Scotchmer (1995), Scotchmer (1996), Kremer (1998), and Denicolo' (2000). Unlike us, their papers do not study the timing of innovation disclosure.

is a single patent race on a novel innovation that builds on previous innovations. We allow for intertemporal turnover: Each competing firm may drop out of the game at the beginning of a new race. This may happen, for example, if a firm finds out that it does not have the capabilities to compete on the new innovation. Those who exit are replaced by new competitors.[8,9] Here, the payoff of the firm ending a stage includes the expected present discounted value of the patented innovation net of development costs, but also the firm's 'continuation payoff' for ending the stage and moving on to the next one. Likewise, the other firms' payoffs include the continuation payoffs for moving on to the following stages in the game.

This game is primarily of interest for R&D applications in which new innovations do not displace earlier ones from the market.[10] There, the value for ending a stage game does not necessarily equal the market value of the patented innovation. It may also include the expected license fees paid by other firms which market improvements in the future, or the profit for innovations covered by continuation patents. Further, we argue that under some assumptions this game may also be applied to environments in which new innovations displace earlier ones from the market, as in the quality ladder literature where all innovations apply to the same product (see, e.g., Aghion and Howitt, 1992). These specific assumptions are that patents have infinite length, and that the innovator makes a take it or leave it offer to the holders of earlier patents when bargaining over the licensing fee needed to market its improved product, which includes earlier patented technology. Under these assumptions, there is no hold up problem among innovators of different stages. While the earlier quality ladder literature focused on this hold up problem as the only source of market inefficiency, we identify a novel source of inefficiency by studying the timing of

[8] Allowing for the possibility of exit relates our model to an important insight in the R&D literature, called 'standing on the shoulder of giants' effect (as in, for example, Scotchmer, 1991) or 'intertemporal spillover' effect. In that literature, firms may choose not to patent their innovations, due to the limited capability to appropriate the value of innovation spillovers on future research. For the same reasons, here, firms may choose to delay patenting their innovations.

[9] Some R&D models presume that innovators cannot participate to future races (e.g., Green and Scotchmer, 1995; and Scotchmer, 1996; O'Donoghue, Scotchmer and Thisse, 1998), others make the opposite assumption (e.g., Scotchmer and Green, 1990, and quality ladder models such as Aghion and Howitt, 1992).

[10] For example, Scotchmer (1991) discusses the case of a technology developed while researching for a drug may later be applied to develop a drug that cures a different disease, whereas Green and Scotchmer (1995) discuss a surgical device developed for humans that may lead to a different surgical device for pets. The possibility that sequential innovations do not market displace previous ones is often acknowledged in the R&D literature (see, e.g., Green and Scotchmer, 1995; DeNicolo', 2000; Scotchmer, 1996; O'Donoghue et al., 1998).

innovation disclosure.[11] We briefly mention in the conclusion how to expand the model to account for different patent length and allocations of bargaining power in the licensing negotiations.

The analysis of this infinite multi-stage game allows us to derive precise and detailed predictions. The 'real-option tradeoff' we identified in our basic model takes always the form of 'real-option delay,' here. Because firms may exit the game, they do not fully internalize the intertemporal spillovers of their innovation on the future lines of research. Specifically, they do not internalize the value of disclosing their innovations to firms that will build on their research to achieve innovations that they may not have the capability to research. So, the private option value of improving the current innovation dominates the social option value, and this pushes firms to patent too late in equilibrium. Further, this 'intertemporal spillover force' is stronger as the probability of exiting the game increases. In fact, the socially optimal strategies are independent from intertemporal turnover, because the choices of firms that maximize social value are unaffected by changes in the firms' identities. Instead, the self-interested firms who actually compete in our multi-stage game delay stopping further when their exit probability increases.

Because intertemporal spillovers are counteracted by fear of preemption, firms may disclose innovations too early or too late in equilibrium relative to the social optimum. We show that the sign of the distortion depends on whether turnover is above or below a precise threshold, function of the number of firms competing in the game. In fact, fear of preemption is stronger as the number of competitors increases in the game. Further, we identify conditions under which stronger patents slow down innovation growth by delaying equilibrium patenting. When turnover is stronger than competition, patent strength is more likely to make firms delay equilibrium disclosure, as the innovator is more likely to exit the game when patenting its innovation.[12]

On the basis of these results, we identify three possibilities for patent policy. Weak patents are optimal when turnover is sufficiently stronger than competition, so that firms patent too late and stronger patents would make them delay disclosure even further. At the same time, weak patents

[11] Among studies on optimal patent policy in innovation ladder models, see, e.g. O'Donoghue (1998), O'Donoghue et al. (1998), and Hopenhayn, Llobet and Mitchell (2006).

[12] Note also that, because stronger patents encourage R&D, they may make innovators postpone disclosure to avoid competition by future innovators who build their research on the innovator's patent, and whose R&D activity is encouraged by stronger patents.

are optimal also when turnover is sufficiently weaker than competition, so that firms patent too early and stronger patents would make them patent even earlier. It is only for intermediate values of turnover and competition that strong patents are optimal in our model. And, perhaps surprisingly, we find that this is not because strong patents induce firms to disclose innovations early, but because they induce firms to extend their R&D durations, and this improves social welfare. In the discussion of our results, we explore how to determine whether strong or weak patents are optimal in different R&D industries, as well as the regulatory implications of our analysis.

We conclude the paper by arguing that our rich and detailed model can be used for policy experiments. As an illustration, we calibrate our multi-stage game using summary statistics from the US seeds-breeding industry in the late 1980's. We set the cost parameters, interest rate, arrival rate of the breakthrough, the number of competitors and the likelihood to exit the game to match observed summary statistics. We cannot observe innovation value growth parameters, but we calibrate them so as to approximate the durations and rates of return found in the data. The best calibrated time spent in research of a new seed is significantly smaller than the socially optimal time: The optimal research duration is 43% longer. This result reveals a strong fear of preemption effect, which translates into a consistent welfare loss: The social value of the innovations actually patented is only 62% of the value of the socially optimal innovations.

Our policy experiments show that this welfare loss can be greatly reduced by weakening patent rights. The benefits of this policy are two-fold. First, holding the number of competing firms fixed, we show that making patents weaker leads firms to lengthen research durations. Second, weaker patents reduce competition and thus fear of preemption, as they reduce the number of firms that can enter the game and achieve a non-negative profit. We find that the first effect is negligible in our exercise: Weakening patents while keeping constant the number of firms making positive profits yields a welfare gain of only 5% over the status quo. On the other hand, the second effect turns out to be powerful. By reducing patent strength so that the industry can support only two of the four firms competing in the status quo, a planner may achieve 87% of the social optimum and a 46% gain relative to the status quo. We conclude the numerical study with a robustness analysis that verifies that our results are not sensitive to changes in parameters of the model.

The paper is presented as follows. After the literature review, section 3 presents and studies our basic model. Section 4 details the analysis of our multi-stage model, and leads to section 5 that reports our numerical results. Section 6 concludes. Omitted proofs are in the appendix.

2 Literature Review

The main contribution of our paper is the formulation of a rich framework to study innovation disclosure timing in details. This question is largely unexplored in the R&D literature and in the literature on optimal patent rights (see, e.g. Rockett, 2010, for a review of this literature). We discuss some exceptions below. The main novelties distinguishing our work from previous papers are our results that firms may patent too early relative to the social optimum, and that stronger patents may delay innovation disclosure. These results follow from our rich dynamic framework, in which the innovation value increases over the time spent by firms on R&D. Importantly, the possibility that firms may patent too early in equilibrium is verified in our numerical analysis of Section 5, and leads to our finding that reducing patent strength may lead to significant welfare gains.

An early paper studying the timing of innovation disclosure is Horstmann, Macdonald and Slivinsky (1985). In a simple model, they recognize that firms may choose not to patent their products, for fear of disclosing technology that may lead to imitation and development of competing products.

Closer to our work, Scotchmer and Green (1990) set the agenda on sequential R&D and patent races in a model with two stages and two firms as well as different institutional rules. When considering institutions analogous to the ones we consider in our paper, they find instances in which the first innovation is not patented in equilibrium, so that firms suboptimally disclose innovations too late. In contrast, firms may never patent too late in our infinite multi-stage model if they exit the game with zero probability, as it assumed in their paper. Interestingly, this difference is due to their assumption that the second innovation displaces the first one from the market. It can be shown that our results would extend to the framework by Scotchmer and Green (1990), if they were to adopt our assumption that innovations do not displace earlier ones from the market.

Also related to our work, Matutes, Regibeau and Rockett (1996) study optimal patent design in a model where a firm that has completed basic research must decide when to disclose its results. By keeping them secret, it enjoys a lead in the race to develop applications of the completed research, but it pays the cost of waiting to market already developed applications. So, their paper identifies a real option delay effect: firms may suboptimally disclose basic research too late because of the option value of waiting to develop applications. Our paper completes this insight by uncovering the possibility of also real option anticipation, that takes place when the social option value of delaying disclosure is larger than the private value. These two opposite effects are known in the literature on real options. For example, Weeds (2002) identifies a real option tradeoff in a stopping game in which two players choose when to make an irreversible investment decision. Instead, we study the timing of innovation disclosure; and we also consider an infinite multi-stage game in which the option value for waiting is mitigated by the possibility of participating to future stages of the game.

The other forces that we single out when comparing equilibrium and socially optimal patent times are novel within the literature on the timing of innovation disclosure. However, they relate to insights from the broader R&D literature, that we now briefly discuss.

Preemption games have been adopted by Fudenberg et al. (1983) and Harris and Vickers (1985) to assess the optimality of competitors' exit decisions from races towards a finish line. They find that preemptive forces may lead firms to withdraw from races when they believe they lag behind. Relatedly, Hoppe and Lehmann-Grube (2005) study timing games of product and process innovation adoption and show that the competition may take the form of a preemption game, leading to the introduction of a less well developed technology. Instead, we study innovation disclosure and find that preemptive forces may lead firms to suboptimally patent their innovations too early. Further, Gilbert and Newbery (1982) show that an incumbent may spend too much R&D, from a social point of view, for fear of preemption by a potential entrant who may win the race to the patent office. But while this leads to socially excessive R&D investment, our preemption effect leads to socially inadequate investment because the leading firm stops R&D too soon.

In our model, we single out other forces that may counteract fear of preemption. One of these forces are duplication costs, well known in R&D models, although not considered in the context of innovation disclosure timing. For example, Loury (1979) shows that firms may overinvest in R&D, in a simple strategic game in which firms choose experimentation intensity. Among the earliest dynamic games illustrating duplication costs are the differential games put forth in Reinganum (1981, 1982), where the players may increase the arrival rate of a Poisson distributed innovation at a cost. Choi (1985) extend the analysis to the case of uncertain arrival rate. While their strategic variable is the time of quitting, the strategic choice in our stopping game is the time of patenting.

3 The Basic R&D Race

The Model We study a continuous-time patent race with $N+1$ firms. Each firm i incurs a flow cost c to compete in the R&D of the same innovation. The firm makes no progress until it achieves a breakthrough in the R&D activity. The time arrival of each breakthrough τ_i is exponentially distributed, independent across firms, with arrival rate λ identical across firms. After experiencing a breakthrough, firm i's innovation improves deterministically. Firms cannot observe whether a competitor achieved a breakthrough or not. The race ends when any of the firms patents its innovation.[13] If ending the race at time t past its breakthrough arrival, firm i obtains the payoff $\overline{U}(t;\alpha)$ at time t, whereas each one of its competitors receives the payoff $\underline{U}(t;\alpha)$. As is standard, payoffs are discounted with interest rate r.

The first-mover payoff $\overline{U}(t;\alpha)$ includes the expected present discounted value of the patented innovation net of the development costs, but it might also include a continuation payoff for ending the race, as modelled in Section 4. The expected present-discounted value of the patent

[13] The possibility that firms exit the game without patenting is not part of the strategies considered in our simple stopping game. To ensure that this possibility does not disturb our analysis, it is sufficient to assume that the flow cost c is not too large relative to the breakthrough arrival rate λ. We report the precise form of our parametric restriction after calculating the equilibrium.

does not necessarily equal the market profit for the patented product. It may also include the expected license fees paid by other firms which market improvements in the future, or the profit for innovations covered by continuation patents. At the same time, the payoff $\overline{U}(t;\alpha)$ is to be taken as net of licensing fees paid to earlier patent holders, in the case that the patented innovation is an improvement on earlier patents. The race-losers' payoffs $\underline{U}(t;\alpha)$ include the continuation payoff accrued by losing competitors when the race is ended, as well as positive spillovers of the patented innovation on the products of the other competing firms, or negative effects on their market shares induced by the patented innovations. Because race losers have the option to leave the marker after the race is ended, we assume that $\underline{U}(t;\alpha) \geq 0$.

We assume that $\overline{U}(t;\alpha) > \underline{U}(t;\alpha)$ at least for t sufficiently large: The firm winning the race gains the positive profits associated to the innovation. Further, we assume that $\overline{U}_1(t;\alpha) > 0$ and that $\overline{U}_{11}(t;\alpha) \leq 0$, with $\overline{U}_1(t;\alpha) \to 0$ as $t \to \infty$. The first-mover payoff does not grow until the firm makes a breakthrough. The firm experiences rapid progress right after the breakthrough, to later hit diminishing returns so that its payoff growth decreases over time. We do not impose restrictive requirements on $\underline{U}_1(t;\alpha)$. In some contexts, i's patent reduces each competitor j's market share, so that $\underline{U}_1(t;\alpha) < 0$; in other applications, innovation spillovers induce such positive externalities that $\underline{U}_1(t;\alpha) > 0$. But even in this case we assume that $\overline{U}_1(t;\alpha) - \underline{U}_1(t;\alpha) > 0$. The parameter $\alpha \in [0,1]$ describes patent strength, and we discuss it later in detail.

Each firm i's strategy prescribes when i should patent its innovation, given the 'calendar' time $\hat{t}_i$ since the beginning of the race, and given the amount of time t past its breakthrough. We consider Perfect Bayesian Equilibria with strategies that are symmetric across firms, and that do not depend on calendar time. Any such a strategy can be identified by a stopping time T: each firm i stops the game and patents its innovation when the time past its breakthrough t equals T. In other terms, each firm i patents the innovation at calendar time $\hat{t}_i = T + \tau_i$.

Our main research question is determining whether firms disclose innovations too early or too late in equilibrium, relative to the social optimum. Hence, we also calculate the stopping time T^* associated with the symmetric, calendar-time independent strategies that maximize the social value of the race. Following Marschak and Radner (1972), T^* is found by calculating the Pareto-dominant equilibrium of the game (the so-called 'team problem') in which each firm i internalizes the positive spillovers that her strategy induces on competitors and consumers, instead of just maximizing its own payoff. We denote by $U^*(t)$ the social value achieved when firm i stops the team problem at time t after i's breakthrough arrival. Again, we assume that $U_1^*(t) > 0$ for all t, and that $U_1^*(t)$ is a strictly decreasing function of t, with $U_1^*(t) \to 0$ as $t \to \infty$.

The parameter α represents the strength of patent rights. We say that patents are stronger when they allocate a larger share of the social value of the patent to the patent holder, and do not distinguish among the institutions that determine a higher α. The social value of the race $U^*(t)$ is independent of α, because the team problem represents an environment in which each firm maximizes the social value of R&D, independently of the identity of those who appropriate of social value. We do not need to assume that $\overline{U}_2(t;\alpha) > 0$ in our analysis. Likewise, there is no need to assume that $\underline{U}_2(t;\alpha) \leq 0$ for our purposes. Nevertheless, it is always the case that $\overline{U}(t;\alpha) + N\underline{U}(t;\alpha) \leq U^*(t)$ for all $\alpha \in [0,1]$, because consumers' welfare is not accounted for in the sum of the firms' values $\overline{U}(t;\alpha) + N\underline{U}(t;\alpha)$.[14]

Our model immediately applies to first-to-file patent systems. It may also be relevant for first-to-invent systems, whenever the timing of innovation cannot be precisely verified in Court.[15]

Analysis We begin solving our basic model by calculating equilibrium stopping time T. We let $V(t)$ be the expected equilibrium value of any arbitrary firm i at time t after i's breakthrough, given that the other N firms adopt the stopping time T. We show in the appendix that, for any t such that firm i weakly prefers to remain in the game, the flow value $rV(t)$ obeys the following standard dynamic programming equation:

$$rV(t) = -c + \dot{V}(t) + h\left(\hat{t}_i\right)\left[\underline{U}(T;\alpha) - V(t)\right]. \tag{1}$$

[14] In some applications, it is reasonable to assume that $\bar{U}(t;\alpha) + N\underline{U}(t;\alpha) = U^*(t)$. But this will not be the case in the multi-stage patent race we develop in the next section. In fact, the continuation values associated with the equilibrium stopping time T and included in $\bar{U}(t;\alpha) + N\underline{U}(t;\alpha)$ will be strictly smaller than the continuation value associated with the optimal stopping time T^* and included in the social payoff $U^*(t)$.

[15] However, it can be shown that there does not exist a symmetric pure-strategy equilibrium for our model, under a first-to-invent system, when the timing of innovation disclosure can be fully and precisely verified.

The first two terms in the right-hand side are the flow cost $c(t)$, and the time increment in the equilibrium value $\dot{V}(t)$. The third term is also expressed in flow terms and represents the expected loss for losing the race when another firm patents its innovation. Specifically, this term consists of the hazard rate of losing the race $h(\hat{t}_i)$, times the change in payoff induced by this event, $\underline{U}(T;\alpha) - V(t)$. Because the race is lost whenever any of the N firms other than i achieves a breakthrough earlier than i, and because breakthrough arrivals are independent and exponentially distributed, we prove in the appendix that the hazard rate $h(\hat{t}_i)$ equals $N\lambda$, whenever $\hat{t}_i \geq T$.[16]

At the equilibrium stopping time T, the firm is indifferent between patenting its innovation or not. Because the value for ending the game is the first-mover payoff $\overline{U}(t;\alpha)$, standard value-matching and smooth-pasting conditions (see, e.g., Dixit and Pindyck, 1994) yield:

$$rV(T) = r\overline{U}(t;\alpha), \; \dot{V}(T) = \overline{U}_1(T;\alpha).$$

Substituting these equalities in equation (1), we obtain the equation pinning down the equilibrium stopping time T, displayed in the following result.

Proposition 1 *There exists a unique symmetric, calendar-time independent equilibrium. When* $\overline{U}_1(0;\alpha) > c + r\overline{U}(0;\alpha) + N\lambda\left[\overline{U}(0;\alpha) - \underline{U}(0;\alpha)\right]$, *its associated equilibrium stopping time T is the unique solution of*

$$r\overline{U}(T;\alpha) = -c + \overline{U}_1(T;\alpha) - N\lambda\left[\overline{U}(T;\alpha) - \underline{U}(T;\alpha)\right], \tag{2}$$

otherwise, the equilibrium stopping time T is zero.[17]

The expression on the left-hand side of equation (2) represents the opportunity cost of waiting to patent, the first-mover payoff $\overline{U}(T;\alpha)$ expressed in flow terms. The expression on the right-

[16] For $\hat{t}_i < T$, we show in the appendix that the hazard rate $h(\hat{t}_i)$ equals zero. Of course, this case is irrelevant for our analysis, as i's competitors do not ever patent before $\hat{t}_i$ equals the equilibrium stopping time T, in any symmetric calendar-time independent equilibrium.

[17] As anticipated in footnote 13, to ensure that our conclusions are robust when allowing firms to exit the race without patenting their innovations, it is sufficient to assume:

$$c/\lambda \leq \overline{U}(T;\alpha)\,\frac{r+N\lambda}{r+(N+1)\lambda}e^{-(r+N\lambda)T} + N\underline{U}(T;\alpha). \tag{3}$$

This condition can be easily checked ex-post, after calculating the equilibrium stopping time T. It is evident that it is not vacuous, as it always holds when $c = 0$. The derivations leading to condition (3) are not insightful. We omit them and make available upon request.

hand side represents the net gains for waiting: The firm pays the flow cost c, gains the value increment $\overline{U}_1(T;\alpha)$, but exposes itself to the risk of being beaten in the patent race. The hazard rate of this last event is $N\lambda$ and the net loss is $\overline{U}(T;\alpha) - \underline{U}(T;\alpha)$.

We now turn to determine whether firms disclose innovations too early or too late in equilibrium. We calculate the stopping times T^* that maximize social value in the team problem that we defined earlier. As we detailed, these strategies are found by stipulating that each competing firm internalizes the positive spillovers that its strategy induces on competitors and consumers. So, letting $V^*(t)$ be each firm i's equilibrium value in the team problem at time t past its breakthrough, firm i's dynamic programming equation now reads:

$$rV^*(t) = -(N+1)c + \dot{V}^*(t) + N\lambda[V^*(T^*) - V^*(t)]. \tag{4}$$

We highlight the modifications with respect to equation (1). First, the flow cost term is multiplied by the number of competing firms, because i internalizes the flow cost borne by its competitors. Second, if any of i's competitors patents its innovation, the social gain is the value $V^*(T^*)$, to be traded against the social value $V^*(t)$ lost by firm i.

At the optimal stopping time T^*, $V(T^*) = U^*(T^*)$. So, T^* is pinned down by the following equation:

$$rU^*(T^*) = -(N+1)c + U_1^*(T^*), \tag{5}$$

provided that $U_1^*(0) > (N+1)c + rU^*(0)$; and otherwise $T^* = 0$.

We now compare the equilibrium stopping time T, pinned down by equation (2), with the optimal stopping time T^*. We focus on the case in which both T and T^* are strictly positive, for which this comparison is easier to explain. By subtracting equation (5) calculated at the equilibrium stopping time T from equation (2), and rearranging the resulting expression, we obtain the following decomposition.

Proposition 2 *Whether firms patent too early $(T < T^*)$ or too late $(T > T^*)$, relative to the socially optimal strategies, depends on whether the quantity*

$$-Nc - \left\{ \left[\overline{U}_1(T;\alpha) - r\overline{U}(T;\alpha)\right] - \left[U_1^*(T) - rU^*(T)\right] \right\} + N\lambda \left[\overline{U}(T;\alpha) - \underline{U}(T;\alpha)\right] \tag{6}$$

is strictly positive or negative.

This decomposition allows us to identify the following three different forces which jointly determine whether the firms disclose their innovations too early or too late in equilibrium, relative to the socially optimal strategies.

First, we note that the flow cost is c in equation (2), whereas it equals $(N+1)\,c$ in equation (5); hence, the net difference is Nc as reported in the first term of expression (6). As long as $N > 0$, this difference pushes each firm to stop too late in equilibrium, as the firm does not internalize the flow costs paid by its competitors while the race continues. While novel in the context of the disclosure timing problem, this force can be related to the 'duplication cost effect' that has been studied in the broader R&D literature, as we earlier pointed out.[18]

The second term in expression (6) compares the quantity $\overline{U}_1(T;\alpha) - r\overline{U}(T;\alpha)$ in equation (2), to the quantity $U_1^*(T) - rU^*(T)$ in equation (5). The former is the difference between the private payoff growth $\overline{U}_1(T;\alpha)$ and the current private payoff flow $r\overline{U}(T;\alpha)$. Thus, it represents the private tradeoff between continuing and stopping at T. Likewise, the term $U_1^*(T) - rU^*(T)$ represents the social tradeoff between continuing and stopping. If the term $\left[\overline{U}_1(T;\alpha) - r\overline{U}(T;\alpha)\right] - \left[U_1^*(T) - rU^*(T)\right]$ is positive, then the society values the option of not ending the race less than the individual firms, so that the competing firms stop too late in equilibrium; and viceversa. As this comparison is a tradeoff known in the literature on real options, we dub 'real-option tradeoff' the force identified here.

Whether the real option tradeoff pushes firms to patent too early or too late in equilibrium depends on the R&D industry considered. As we explain in details later, the real option tradeoff often takes the form of delay in industries where the innovator does not have the capability to develop and market applications of the innovation it patents. For example, pharmaceutical innovators often sell their innovation to larger companies, with better development technologies and facilities. On the other hand, the real option tradeoff likely pushes firms to patent too early in

[18]The empirical relevance of duplication costs is well known. For example, the white paper prepared for the WIPO-IFIA International Symposium on Inventors and Information Technology (1998) reported that 30 percent of all R&D expediture in Europe and US duplicated patented research.

equilibrium when their innovations make competitors' established technologies obsolete. So, the private option value of not patenting the innovation is smaller than the social value, because the innovator fully appropriates of the competitors' market shares as long as its technology is superior, regardless of its additional social value. Instead, the patent holder of an established technology is usually reluctant to patent the new, superior technology for fear of cannibalization.[19]

Finally, the term $N\lambda\left[\underline{U}(T;\alpha)-\overline{U}(T;\alpha)\right]$ identifies a 'fear of preemption' force that pushes each firm to patent too early in equilibrium, concerned that a competitor beats it in the race to the patent office. The identification of this force is the most innovative result of our analysis. In fact, the literature on the timing of innovation disclosure usually presumes that early disclosure is beneficial to the society, as it speeds up economic growth. Innovation secrecy is seen as the main evil. But, as we shall argue with our calibration exercise in Section 5, the fear of preemption force we identify here may have an important role in the optimal design of patent policy; and it may give a justification for policies that reduce patent strength.

While this fear of preemption force is novel to the academic literature, there are many anecdotes in R&D history where concerns about preemption by competitors led inventors to anticipate patenting. As reported by Lemley (2012), the most notable example is the telephone. Alexander Graham Bell was aware not only of competitors working to develop a telephone but of the filing of patent applications by those competitors. He rushed his application to the patent office before he finished his invention in order to avoid being preempted by them.[20] Further, it is likely that the switch from the first-to-invent to the first-to-file system, which took effect in the Spring of 2013 in the US, increased the relevance of fear of preemption. As a result, one can expect an increase of patent rate, and a decrease in the proxy variables used to assess patent value, such as citation numbers. Indeed, some legal scholars rendered these concerns manifest in the debate

[19] Fear of cannibalization by the market leader and early patenting by competitors are quite common, see for example the study by Igami (2013) on new generation and old generation hard disk drives, as well as the reluctance to patent and commercialize digital camera technology by Kodak, the holder of the technological superior chemical print technology.

[20] Even then, Bell did not beat his rivals to the patent office, Elisha Gray filed a caveat on the same day. Similarly, Eli Whitney was expressly warned that competitors were working on similar inventions and this seems to have spurred him to file his patent application. Other anecdotes in which inventors feared the possibility of preemption involve Edison, who was aware of the work of others on the lightbulb, the Wright Brothers, who recognized that they were in competition with other inventive teams, as well as Watson and Crick, who knew they were racing Linus Pauling to discover the helical structure of DNA.

that took place before the harmonization of the US system with the first-to-file system adopted worldwide.[21]

The decomposition presented in Proposition 2 provides effective guidelines to establish whether firms patent their innovations too early or too late in specific R&D industries. For example, because duplication costs are larger when the research cost c increases, whereas fear of preemption is more relevant when the likelihood of innovation λ is larger, one can expect suboptimally early patenting in industries characterized by low R&D costs and fast innovation achievements, and suboptimally late patenting in R&D competitions with high costs and low chances of success. Further, the role played by real-option tradeoff can be gauged by assessing the intertemporal effect of innovations on competitors' profits and on social value, analogously to the cases we discussed explained above. In Proposition 2, the role of competition is ambiguous, as a larger competition parameter N makes both duplication costs and fear of preemption more relevant. However, we later show that competition leads unambiguously to suboptimally early patenting in the multi-stage game we develop in the next section.

Once determined whether competing firms disclose their innovations too early or too late in equilibrium, a natural question is whether stronger patent rights will induce earlier or later equilibrium disclosure. The usual presumption in the literature is that increasing α and making patents stronger induces earlier equilibrium disclosure. Proposition 3 below shows that the reverse can also be true. Stronger patent rights make firms patent earlier in equilibrium when a greater α increases $\overline{U}_1 (T, \alpha)$, the option value for letting the innovation value grow further, more than it increases $r\overline{U} (T; \alpha) + N\lambda \left[\overline{U} (T; \alpha) - \underline{U} (T; \alpha) \right]$, the value for ending the game plus the expected loss in the event of preemption, both expressed in flow terms.

Proposition 3 *An increase of the patent strength parameter α increases the equilibrium stopping*

[21] For example, McFadyen (2007) writes: "Because the objective of a first-to-file patent system is, in effect, to reward early filing and punish late filing, many believe the first-to-file system endorses a race to the patent office. [...] The inventor no longer has the opportunity to develop the invention through its entirety, [...] Such a process of awarding priority of inventorship will precipitate hasty application drafting with limited experimental exemplification or support. In short, the first-to-file system encourages speculative filing of applications on unproven inventions by 'idea men' rather than actual development of useful commercial inventions, and would retard rather than promote progress." See also Pedersen and Braginsky (2006), and Conley (1991).

T if

$$\overline{U}_{12}(T;\alpha) > (N\lambda + r)\,\overline{U}_2(T;\alpha) - N\lambda\underline{U}_2(T;\alpha), \tag{7}$$

and T decreases in α if the inequality is reversed.

Our insights on the relationship between patent strength α and the equilibrium stopping time T may be quite relevant for policy. The strengthening of U.S. patent protection over the 80s and 90 led to the doubling the number of new patents granted per year to domestic inventors between 1985 and 1999 (see, for example, Gallini, 2002). However, Lerner (2002) examines 177 policy shifts in 60 countries from 1850 to 2000 and finds some support for an "inverted-U" relationship between patent strength and the number of patent applications. His findings challenge the predominant views of the R&D literature, but are broadly consistent with our theoretical results.[22]

Our insights on the relationship between patent strength α, the equilibrium stopping time T and the socially optimal stopping time T^* may be broadly applicable. One class of games to which they may be applied is the infinite multi-stage game presented in the next section. We conclude this section by briefly discussing how to apply our insights to another class of games that has been studied in the R&D literature. These games feature two-stages. In the first one, individual firms are engaged in basic research: They compete to develop an innovation that does not have market value per se. But once this basic innovation is patented, it leads to different applications that can be marketed and generate significant profits. Crucially, these models presume that the innovator does not have the capability to develop and commercialize applications of the innovation it patents. To apply our basic model to these games, suppose that our $N+1$ firms compete on basic research, and interpret the first-mover payoff $\overline{U}(t;\alpha)$ as the share of profits generated by future applications that is captured by the basic research patent holder through licensing agreements and other arrangements.

One fundamental insight of the literature is that there is a hold up problem across first and second stage innovators. Awarding too strong patent rights to the basic research patent holder

[22] Other studies found no clear relationship between patent strength and patent numbers; see, e.g., Kortum and Lerner, (1998) on US R&D from 1980 to 2000, Hall and Ziedonis (2001) on U.S. semiconductor R&D from 1979 to 1995, Branstetter and Sakakibara (2001) on Japan R&D from 1988 to 1998, Arora et al. (2008) on U.S. R&D from 1990 to 2002.

may stifle R&D on applications, and decrease the market value of the research line. It is not difficult to recover this hold up problem in our model by assuming that $\overline{U}(t;\alpha)$ is decreasing in α for values of α close to one. Further, this hold up problem may lead to real-option delay, as we anticipated earlier. When selling the patent to the second stage firm, the first innovator is paid the stock value of the patented innovation, but it is unlikely remunerated also for the full option value of improving the innovation further before patenting it. As a result, the private option value dominates the social option value, and the real-option tradeoff takes the form of delay.

Most importantly, when the fear of preemption force we earlier identified is sufficiently strong in these two-stage race games, the expression (6) may take positive values, despite the counteracting real-option delay force discussed above. As a result, basic research may sometimes be disclosed too early, leading to important welfare distortions. Interestingly, these distortions may also lower the value of basic research $\overline{U}(T;\alpha)$, and may induce lower incentives to invest resources in studying basic research in the first place.[23]

Hence, our model identifies a novel reason behind the common wisdom, dating back at least to Nelson (1959), that basic research is underproduced. Our novel insight is that firms that compete on basic research may patent their innovation too early, for fear of being preempted in the patent race. This practice may stifle incentives for further development, and reduce the payoffs and incentives for engaging in basic research in the first place. Crucially, this distortion may be worsened by stronger patents, when they make competing firms disclose their innovation earlier, as it is usually presumed in the literature on innovation disclosure. Thus, we conclude, strengthening patent rights need not necessarily provide efficient incentives to conduct basic research, and may even worsen the underproduction.

[23] In simple derivations available upon request, we expand our basic model and endogeneize both the entry decision (extensive margin) and the investment choice which determines a firm's arrival rate λ (intensive margin). We determine precise conditions for fear of preemption to lead to lower incentives in both margins.

4 The Multistage Game

4.1 Model

We consider an infinite multi-stage game where each stage is a patent race for a different innovation, which is technologically feasible only when the previous innovations have been developed. Each stage of the game can be studied using the methodology developed in the previous section. There are $N+1$ competing firms, paying cost c until the stage ends. Each firm i starts making progress only after making a breakthrough, which has an exponentially distributed time arrival τ_i, independent across firms, with arrival rate λ identical across firms. The time τ_i is not publicly observed.

If a firm ends the stage at time t since its breakthrough, its first-mover payoff $\overline{U}(t;\alpha)$ takes here the explicit form $\alpha x(t) - c_0 + U^L$. The term $x(t)$ denotes the expected present discounted social value of the patented innovation, a fraction α of which is earned by the innovator. The term c_0 denotes the development costs for consolidating the research outcomes so that a successful patent application can be presented. For simplicity, c_0 is taken to be independent of t, and we assume that $\alpha x(0) \leq c_0$: it is not enough just to achieve a breakthrough for the innovation to be profitable. In our main applications, c_0 may be reinterpreted as also including the costs for marketing the patented innovation and any other fixed costs incurred after patenting.[24] The term U^L is the continuation value of the firm patenting the innovation and ending the stage game; we will derive it precisely later.

In line with Section 3, we assume that the growth rate $\dot{x}(\cdot)$ is a strictly decreasing, positive function, with $\alpha\dot{x}(0) > c$, $\dot{x}(t) \to 0$ as $t \to \infty$; but with $\alpha x(t) > c/r + c_0$ for t sufficiently large. The payoff of all losing firms $\underline{U}(t;\alpha)$ is here assumed to consist only of the continuation value U^F, which we derive later. Finally, each firm's social payoff in the team problem $U^*(t)$ consists of $x(t) - c_0 + U^S$: the social value of the patented innovation $x(t)$ less the development cost c_0,

[24] A large fixed development cost is incurred in medical research, due to the expensive medical trials to test and finess a chemical compound or treatment. These trials are integral part of R&D; for example, they may induce a complete reassessment of earlier engineered compounds (as was the case for Sildenafil, originally researched to cure hypertension). Indeed, according to the PhRMA 2002 Industry Profile, for every one drug that reaches the market, approximately 250 drugs are tested in pre-clinical animal trials.

plus the social continuation value U^S that we derive later.

We focus the solution on stage-stationary equilibria in which (as in the previous section) strategies are symmetric and calendar-time independent, so that they can be summarized by a stopping time T. Unlike in the previous section, we can here explicitly describe innovation growth. Given the equilibrium stopping time T, we let the expected time-averages equilibrium innovation growth g equal $E[x(T)/(T+\hat{\tau})]$, where $\hat{\tau}$ be the random time of the first breakthrough arrival among the $N+1$ competitors of any given stage. We show in the appendix that whenever λ is not too small and x is sufficiently concave, a larger equilibrium stopping time T corresponds to a smaller equilibrium innovation growth g. In order to assess equilibrium welfare (as in the previous section) we also study the stopping time T^*, associated with the stage-stationary, calendar time independent, symmetric strategies that maximize social welfare.

As discussed in the introduction, our multi-stage game covers primarily the case in which the product invented in any stage does not displace earlier innovation from the market. While each innovation is possible only because of previously disclosed technology advancements, there is no market interaction between the different patented products, and hence the expected value $\alpha x(t) - x_0$ can be defined independently of what happens in future stages of the game. Also, for these applications it is immaterial whether the cost c_0 is paid before or after the patent application is filed, so that the term c_0 may include the costs for marketing the innovation as well as its development costs.

However, our game may also be applied to contexts in which each innovation concerns the same product and fully displaces all previous innovations from the market. Under the assumptions of infinite patent length and full bargaining power to license buyers, we may interpret $x(t)$ as the increment in the expected present discounted social value of the product generated by the current innovation, and $\alpha x(t)$ as the fraction of this value earned by the innovator when marketing the improved product. Because its innovation is based on prior technology, the innovator must buy licenses for all the patents covering the currently marketed product before marketing the improved product. Suppose that the innovator can make a take-it-or-leave-it offer to the current license and patent holders. Then, the payment exactly equals the expected present discounted market value of the unimproved product. So, net of this payment, the innovator's expected present discounted market profit for its innovation equals $\alpha x(t)$ exactly, as in the case of no-market displacement.[25]

As discussed in the introduction, some R&D multi-stage models presume that innovators cannot participate in future races, whereas others make the opposite assumption. Here, we do not impose any such restriction. After an innovation is patented, we say that a firm may find itself incapable to compete on the next innovation, and may exit the game with some exogenous probability. For example, the next innovation may concern a product that the firm would not be able to market; or for the case of full market displacement, the next innovation may be an improvement that requires a technological capability outside of the firm's expertise.

To avoid notation clutter, we assume that each firm's payoff when leaving the game is zero.[26] We let the exit probability be $1-\rho$ for the stage winner, and $1-\chi$ for all the other players. As we explain in details later, the turnover parameter $1-\rho$ represents the strength of the 'intertemporal spillover' effect in our game. We assume that $\chi \leq \rho$ as, intuitively, the stage winner is less likely to find its technology inappropriate for the next innovation. When a firm exits the game, it is promptly replaced by an entrant so that exactly $N+1$ firms compete in each stage of the game. This assumption can be rationalized by supposing that potential entrants pay a fixed cost to set up the research capability to participate in the game, and that this cost is such that a firm does not find it profitable to enter the game if (and only if) there are already $N+1$ competitors at the beginning of the stage. (We will later show that the equilibrium payoff is decreasing in N.)

4.2 Equilibrium

Unlike in Section 3, the equilibrium stopping time T now enters the analysis in determining the firms' continuation values U^L and U^F when each stage of the game ends. Because the equilibrium is stage-stationary and symmetric; each firm expects that the stopping time in each future stage

[25] Note that this interpretation of our model does not require that the expected present discounted market value of the unimproved product is the same in each stage of the game, or that it is the same on and off the equilibrium path.

[26] In practice, this is only a normalization assumption. The qualitative properties of our game would not change if this payoff were strictly positive. This would be the case, for example, if a firm leaving our game achieved a positive payoff by improving already patented innovations.

will be equal to the current stopping time T. Let $Q(T)$ be each firm's expected value at the beginning of a new stage of the game (when all players adopt the stopping time T) so that $U^L(T) = \rho Q(T)$ and $U^F(T) = \chi Q(T)$ because a stage winner (respectively, loser) participates in the next stage with probabilities ρ and χ respectively. We prove in the appendix that $Q(T)$ takes the form:

$$Q(T) = \left\{\beta(T)\frac{\alpha x(T) - c_0}{N+1} - \frac{c}{r}[1-\beta(T)]\right\}\frac{1}{1-\xi\beta(T)}, \tag{8}$$

where

$$\xi = \frac{N}{N+1}\chi + \frac{1}{N+1}\rho$$

is the expected probability of a firm not leaving the game at the next stage, calculated at the beginning of any stage, where each player expects to win the stage with equal probability $1/(N+1)$. Further, letting $\hat{\tau}$ be the random time of the first breakthrough arrival among the $N+1$ competitors of any given stage, $\beta(T) \equiv E\left[e^{-r(\hat{\tau}+T)}\right]$ is the expected discount factor for the random time $\hat{\tau}+T$, in which the stage ends (calculated at the beginning of any stage). Because the hazard rate of $\hat{\tau}$ is $(N+1)\lambda$, we prove in the appendix that

$$\beta(T) = e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda + r}.$$

Given these definitions, the expression (8) is intuitive. The term between in the large brackets is the expected value of any stage in which the firm participates, calculated at the beginning of that stage. The firm pays the flow cost c until the random stage-ending time $\hat{\tau}+T$, so that the firm's expected one-off cost is $c[1-\beta(T)]/r$; whereas with probability $1/(N+1)$ the firm wins the stage, achieving payoff $\alpha x(T) - c_0$ discounted by $\beta(T)$. After the next stage, the firm expects to participate in future stages with probability ξ, as it does not know whether it will win the stage or not. Because discount factors can also be interpreted as the probability not to exit a infinite multi-stage game, the term $\xi\beta(T)$ can be understood as the stationary probability of remaining in the game at any future stage, so that the term $1/[1-\xi\beta(T)]$ compounds payoffs as is standard in any infinite multi-stage stage game.

Having calculated the continuation payoffs $U^L(T)$ and $U^F(T)$, we can now can plug them into the expressions $\overline{U}(T;\alpha) = \alpha x(T) - c_0 + U^L(T)$ and $\underline{U}(T;\alpha) = U^F(T)$ that appear in the equilibrium equation (2) of Proposition 1. Doing so, we obtain the following result which fully determines the unique equilibrium stopping time of our multi-stage patent game. The statement distinguishes the value function $Q(T)$, which depends on the stopping time T, from the equilibrium value Q which depends only on the model's exogenous parameters.

Proposition 4 *There exists a unique symmetric, stage-stationary and calendar-time independent equilibrium. Its associated equilibrium stopping time T is the unique solution of*

$$(r + N\lambda)[\alpha x(T) - c_0] + [r\rho + N\lambda(\rho - \chi)] Q(T) = -c + \alpha \dot{x}(T). \tag{9}$$

The stopping time T decreases in N, λ, c, ξ (fixing $\rho - \chi$), and in $\rho - \chi$ (fixing ξ), whereas it increases in c_0. The function $Q(\cdot)$ increases in T. The equilibrium value Q decreases in N, c, and in $\rho - \chi$ (fixing ξ).

Equation (9) implicitly calculates the equilibrium stopping time T. It equalizes the flow value for ending the stage $r[\alpha x(T) - c_0 + \rho Q(T)]$ with the net gains for waiting $-c + \alpha\dot{x}(T) - N\lambda[\alpha x(T) - c_0 + (\rho - \chi) Q(T)]$ which include the flow cost c, the innovation value increment $\alpha\dot{x}(T)$, and payoff loss in case of preemption $\alpha x(T) - c_0 + (\rho - \chi) Q(T)$, which occurs with hazard rate $N\lambda$. The comparative statics are intuitive. Increasing N and λ increases the fear of preemption and makes firms stop earlier, as does a higher research cost c, or a higher $\rho - \chi$. Also intuitive is that a higher development cost c_0 makes firms stop later.

Turning to considering comparative statics with respect to the equilibrium value Q, we find that Q decreases in c because higher costs depress profits and because they shorten the equilibrium stopping time T and thus reduce $Q(T)$. However, holding ξ fixed, changes in $\rho - \chi$ have no direct effect on the equilibrium value Q, so that the negative relationship between Q and $\rho - \chi$ is entirely due to the negative relationship between $\rho - \chi$ and T. Most importantly, we highlight that the equilibrium value Q decreases in N, as anticipated earlier. This is because higher competition both reduces the probability of winning any game stage, and shortens the equilibrium stopping time T, so that $Q(T)$ is lower.

We now turn to consider the relationship between the equilibrium stopping time T and our policy parameter α. As in Proposition 3, we derive a precise condition such that T increases or decreases in α, so that stronger patent rights induce later or earlier innovation disclosure.

Proposition 5 *Increasing α increases T whenever*

$$\dot{x}(T) > (r + N\lambda)x(T) + \frac{r\rho + N\lambda(\rho - \chi)}{N+1}\frac{\beta(T)}{1 - \xi\beta(T)}x(T), \tag{10}$$

and the opposite conclusions hold if the inequality is reversed.

When $\rho = \chi = \xi$, there is a threshold function $\hat{\xi} : N \mapsto \xi$, such that stronger patents induce earlier equilibrium disclosure if $\xi > \hat{\xi}(N)$ and later disclosure if $\xi < \hat{\xi}(N)$.

With the aid of the discussion presented after Proposition 3, the interpretation of this result is simple. Increasing α increases one's value for patenting an innovation $rx(T)$ and continuation utility $r\rho\frac{1}{N+1}\frac{\beta(T)}{1-\xi\beta(T)}x(T)$, as well as the loss if preempted $N\lambda x(T)$ and continuation value loss if preempted $\frac{N\lambda(\rho-\chi)}{N+1}\frac{\beta(T)}{1-\xi\beta(T)}x(T)$. All these terms (calculated as flow values) correspond to the value for disclosing the innovation and proceeding to the next stage. If they are dominated by the option value for waiting $\dot{x}(T)$ which also increases in α, then a higher α increases the equilibrium stopping time T. Further, we note for future reference that when $\rho = \chi = \xi$, there is a threshold function $\hat{\xi} : N \mapsto \xi$, such that stronger patents induce earlier equilibrium disclosure if $\xi > \hat{\xi}(N)$ and later disclosure if $\xi < \hat{\xi}(N)$. As the exit probability $1 - \xi$ increases, stronger patents are more likely inducing later equilibrium disclosure.

Proposition 5 carries relevant implications also for the innovation growth g, as it shows that stronger patents may decrease growth. As well as determining $dT/d\alpha$, condition (10) also pins down the sign of the derivative $dg/d\alpha$, which is the opposite of $dT/d\alpha$ when x is sufficiently concave and λ not too small, as we pointed out earlier. Hence, Proposition 5 provides a novel insight for patent policy. Stronger patents may make patents less frequent and slow down innovation growth. The previous literature presumed that stronger patent rights increase incentives for R&D efforts and speed up innovation growth. Here, we have uncovered a novel counteracting effect. Stronger patents do not only raise the value of the patented innovation, but also the value of waiting to further increase its value. When the second effect dominates the first one, stronger patents may slow down innovation growth. Importantly, our analysis also shows that slowing down innovation growth may be socially beneficial, whenever the equilibrium stopping time T is shorter than the socially optimal stopping time T^*, which we calculate next.

4.3 Socially Optimal Strategies

In order to determine whether firms disclose innovations too early or too late in equilibrium, we now calculate the stopping time T^* associated with the stage-stationary, calendar time independent, symmetric strategies that maximize social welfare. Following the same procedure adopted to calculate T, we first prove in the appendix that the social continuation value U^S takes the following form, as a function of T^*:

$$U^S(T^*) = \frac{\beta(T^*)}{1-\beta(T^*)}\left[x(T^*) - c_0\right] - (N+1)\frac{c}{r}. \tag{11}$$

We compare this formula with the expression (8) which defines the equilibrium payoff $Q(T)$ in the multi-stage patent race. Each firm engaged in the team problem internalizes the research costs c paid by current and future firms throughout the game. This loss is represented by the term $-(N+1)c/r$. The first term in expression (8) represents the social benefit of the patented innovation $x(T^*) - c_0$ discounted by $\beta(T^*)$ at the beginning of the stage and compounded by the multiplier $1/\left[1-\beta(T^*)\right]$.

Now, plugging in the definition $U^*(T^*) = x(T^*) - c_0 + U^S(T^*)$ into equation (5), we fully determine the socially optimal stopping time T^*.

Proposition 6 *There is a unique stage-stationary, calendar-time independent, symmetric strategy profile that maximizes social welfare in our multi-stage patent game. Its associated stopping time T^* is the unique solution of:*

$$\frac{r}{1-\beta(T^*)}\left[x(T^*) - c_0\right] = \dot{x}(T^*). \tag{12}$$

The left-hand side of the above equation represents the social value of the current innovation, calculated in flow terms and compounded with multiplier $1/\left[1-\beta(T^*)\right]$, whereas the right-hand side represents the net social flow value for increasing the innovation value. Note that, because the flow cost c is paid by all firms (current and future ones) it does not appear in the equation characterizing the socially optimal stopping time T^*.

We now compare the equilibrium stopping time T with the optimal stopping time T^*. We are interested in the effect of the competition parameter N and of the intertemporal spillover parameters $1-\rho$, $1-\xi$ and $1-\chi$ on the comparison between T and T^*. These parameters bear a close relationship to the forces we identified in Proposition 2. N determines the strength of the fear of preemption effect; whereas ρ, ξ and χ influence the real option tradeoff as they determine the relationship between the private and social continuation payoffs $U^L(\tau)$, $U^F(\tau)$ and $U^S(\tau)$.[27] Interestingly, the duplication cost effect identified in Proposition 2 does not play a role in the infinite multi-stage model of this section. As we anticipated earlier, this is because the flow cost c does not have any effect on the optimal stopping time as it is paid by all firms (current and future ones), and the number of firms participating in each stage of the race is constant.

In order to simplify the exposition, we focus on the case in which there is no exit probability advantage for the race winner, i.e. $\rho = \chi = \xi$, and study the effect of the competition parameter N and of the turnover parameter $1-\xi$, on the difference between the equilibrium stopping time T and the optimal stopping time T^*.[28] The reported polar cases of extreme ξ and N are derived when $\alpha = 1$, so that the innovator fully appropriates of the stream of profits of the patented innovator. In this case, the equilibrium equation (9) can be directly compared with equation (12).

Proposition 7 *When there is no exit probability advantage for the race winner (i.e., when $\rho = \chi = \xi$), there is a threshold function $\bar{\xi} : N \mapsto \xi$, such that, with strong patents, firms disclose innovations too early in equilibrium (i.e., $T < T^*$) if patents competition dominates turnover — i.e., if $\xi > \bar{\xi}(N)$, whereas innovations are disclosed too late if $\xi < \bar{\xi}(N)$. With full patent strength*

[27] In fact, letting

$$\Pi(\tau) = \beta(\tau)\frac{x(\tau) - c_0}{N+1} - \frac{c}{r}[1-\beta(\tau)]$$

be the per-capita per-period payoff function, the continuation payoffs can be expressed as:

$$U^L(\tau) = \frac{\rho\Pi(\tau)}{1-\xi\beta(\tau)},\ U^F(\tau) = \frac{\chi\Pi(\tau)}{1-\xi\beta(\tau)} \text{ and } U^S(\tau) = \frac{(N+1)\Pi(\tau)}{1-\beta(\tau)}.$$

[28] To assess the effect the parameters ρ and χ, we also establish that, holding the number of competing firms $N+1$ and the exit probability $1-\xi$ fixed, the equilibrium stopping time T decreases relative to the socially optimal time T^* as the race winner exit probability advantage $\rho - \chi$ increases.

$(\alpha = 1)$, *firms patents too early in the presence of competition* $(N > 0)$ *and with no turnover* $(\xi = 1)$, *or with strong competition* (N *is large) and no full turnover* $(\xi > 0)$. *Instead,* $T > T^*$ *when there is no competition* $(N = 0)$ *and some turnover* $(\xi < 1)$, *or with full turnover* $(\xi = 0)$, *regardless of* N.

The analysis of Proposition 7 shows that weak competition (i.e., $N = 0$) or large turnover $(\xi = 0)$ induce firms to patent their innovation suboptimally late, as it is presumed in the existing literature. Instead, strong competition (i.e., high N) or small turnover $(\xi = 1)$ lead firms to patent their innovations too early. In general, more competition (i.e. larger N) induces an earlier equilibrium disclosure time T, relative to the socially optimal stopping time T^*, whereas more turnover (larger $1 - \xi$) makes firms patent their innovations later in equilibrium, relative to T^*.

These results allow us to recover the fear of preemption and intertemporal spillover forces we discussed in the introduction. The competition parameter N determines the strength of fear of preemption, whereas the exit probability $1-\xi$ determines the strength of intertemporal spillovers. In fact, when the turnover parameter $1 - \xi$ is large, current competitors anticipate that they will likely not participate in the next stage of the game, and that the knowledge they patent will spill over to the next stage competitors. As a result, the private option value of improving the innovation further before patenting them is larger than the social option value. So, we determine that fear of preemption leads firms to patent their innovations too early, whereas intertemporal spillovers lead them too patent too late.

The results in Proposition 7 relating turnover to the timing of patents are by and large consistent with the empirical findings by Hall, Jaffe and Trajtenberg (2001). They find that the electronic industry, characterized by high intertemporal turnover, has significantly lower patenting rate than the mechanical industry, which has less turnover, over the 1965-1990 period. Conversely, they find some evidence of higher patent value in the electronics industry over the 1975-1990 period, by comparing patent citation numbers across the two industries. There is also empirical evidence supporting our results in Proposition 7 relating competition to the timing of patents. For example, Magazzini, Pammolli and Riccaboni (2008) find that forward citation indexes of patents are negatively related with the number of competing firms in the R&D industry, in a large database of pharmaceutical and biotechnology patents granted by the USPTO from 1965 to 2005.

Further, some of the polar cases reported in Proposition 7 are of special interest, as they are closely related to the literature. The case of $N = 0$ and $\xi = 0$ is reminiscent of the models by Green and Scotchmer (1995), and O'Donoghue, Scotchmer and Thisse (1998). In these R&D investment models, formulated to study patent design in the presence of intertemporal spillovers, a single, new firm participates in each stage of the game. Here, we find that firms disclose innovations too late in equilibrium, relative to the social optimum, when patent rights are strong. Likewise, the case of $\xi = 1$ and $N > 0$ can be related to the quality ladder models quoted in the introduction. These models provided a canonical framework to study growth in the presence of technological innovation. Interestingly, we here find that firms may disclose innovations too early in equilibrium due to fear of preemption.

Propositions 5 and 7 may have important implications for policy. The latter determines when firms disclose their innovations in equilibrium too early or too late relative to the social optimum, whereas the former determines when strengthening patents induces earlier or later equilibrium disclosure. Hence, these two propositions jointly determine whether weak or strong patent rights would be more effective in correcting the distortions caused by the firms' self-interested equilibrium disclosure strategies.

Proposition 7 shows that, when $\rho = \chi = \xi$, there is a threshold function $\bar{\xi} : N \mapsto \xi$, such that the firms disclose too early in equilibrium if $\xi > \bar{\xi}(N)$, and too late if $\xi < \bar{\xi}(N)$. Proposition 5 shows that, when $\rho = \chi = \xi$, there is another threshold function $\hat{\xi} : N \mapsto \xi$, such that stronger patents induce earlier equilibrium disclosure if $\xi > \hat{\xi}(N)$ and later disclosure if $\xi < \hat{\xi}(N)$. The final result of this section, Proposition 8 below, brings Propositions 5 and 7 together and shows that it is always the case that $\bar{\xi}(N) < \hat{\xi}(N)$. So, we determine when it is the case that weak or strong patents are optimal, as a function of the turnover parameter $1 - \xi$, which pins down the strength of intertemporal spillovers, and of the competition parameter N, which determines the strength of fear of preemption. Further, we determine whether optimal patent policy improves upon equilibrium by making firms delay earlier or later, as a function of the parameters $1 - \xi$ and

N.

Proposition 8 *Suppose that there is no exit probability advantage for the race winner (i.e., that $\rho = \chi = \xi$). When turnover is sufficiently weak relative to competition—i.e., $\xi > \hat{\xi}(N)$, weak patents are optimal as they lengthen suboptimal equilibrium patent times (i.e., because $T < T^*$ and $dT/d\alpha < 0$). When turnover is sufficiently strong relative to competition—i.e., $\xi < \bar{\xi}(N)$, weak patents are optimal as they shorten suboptimal equilibrium patent times (i.e., as $T > T^*$ and $dT/d\alpha > 0$). In the intermediate case in which $\bar{\xi}(N) < \xi < \hat{\xi}(N)$, strong patents are optimal because they lengthen suboptimal equilibrium patent times ($T < T^*$ and $dT/d\alpha > 0$).*

The results in Proposition 8 are powerful. Within our framework, we find that if the turnover parameter $1-\xi$ is sufficiently small relative to the competition parameter N, then the social planner should opt for weak patent rights. This is because fear of preemption dominates intertemporal spillovers, and firms would patent innovations too early, in equilibrium, if strong patent rights were in place. Weaker patents make firms patent their innovations later and this improves social welfare.

Further, weak patents are also optimal when the turnover parameter $1 - \xi$ is sufficiently large relative to the competition parameter N, but for the opposite reasons. Here, intertemporal spillovers dominate fear of preemption, and firms patent their innovation too late in equilibrium, under strong patents. Weaker patents make firms anticipate disclosing their innovations so that social welfare improves.

It is only in the intermediate case in which $1 - \xi$ is neither too small nor too large relative to N, that strong patent rights are optimal. But, perhaps surprisingly, this is not because they make firms disclose their innovation earlier. To the contrary, in this intermediate case, fear of preemption dominates intertemporal spillovers, and firms patent their innovation suboptimally early in equilibrium. Strong patents make firms patent their innovation later, and this improves social welfare.

The above analysis has determined a rich set of predictions that can be taken to the data. The key parameters we identified to assess the optimality of weak or strong patents are turnover (measured by $1-\xi$) and competition (pinned down by N). These parameters are often observable. Also, competition may be related to observable technological or institutional barriers to entry. Hence, one can take to the data also the implication of our analysis that strong patents are optimal only in the case of intermediate entry barriers.

As highlighted in the introduction, our main results concern the case in which competition is sufficiently stronger than turnover —$\xi > \hat{\xi}(N)$, so that fear of preemption dominates intertemporal spillovers, and weak patents are optimal as they lengthen suboptimal equilibrium patent times (i.e., because $T < T^*$ and $dT/d\alpha < 0$). Some evidence of excessive patenting and of low value patents can be found in numerous anecdotes about trivial patents, and in the staggeringly low rate of patents that are commercialized.[29] Furthermore, we later illustrate in Section 5 how our results can be used to assess quantitatively the optimality of weak patents by means of a numerical exercise.

The implication of our results with respect to regulatory policy are manifold. As well known in the literature, there are a number of institutional possibilities to weaken patent rights. A simple one is to reduce patent breadth. Indeed, as patent protection is narrower, competitors are more capable of inventing and patenting competing technologies. But the effect of this policy is not clear in the context of the timing of innovation disclosure. Keeping R&D expenditure constant, in fact, narrower patent protection may lead to a increase in patenting frequency; and this would exacerbate the welfare loss induced by fear of preemption. Other possibilities to weaken patent strength may include weakening the innovator's exclusive right to market the patented innovation.

Within the context of our model, one can explore alternative policy interventions to reduce welfare loss. For example, making patent applications more expensive, or tightening the non-obviousness requirement for granting patents would increase the cost c_0 and lengthen the equi-

[29] For a humourous account of the phenomenon of trivial patents, see http://images.businessweek.com/ss/09/04/0408_ridiculous_patents/ According to the then director of public affairs for the U.S. Patent & Trademark Office, "There [were] around 1.5 million patents in effect [in the US in 2005], and of those, maybe 3,000 were commercially viable," see http://www.businessweek.com/stories/2005-11-09/avoiding-the-inventors-lament Other, complementary, explanations for these phenomena are that private inventors are often overconfident about their inventions' prospects, and that R&D firms often patent innovations that they do not intend to commercialize, to preempt competitors from patenting the same technology or a similar one.

librium patent time T. Whenever a significant welfare loss is due to the excessive patenting, and low-value innovations, the benefits of these policy interventions are easy to appreciate. Another, less obvious, possibility is to reduce fear of preemption by intervening on the competition parameter N. As we shall see in the numerical exercise of Section 5, in fact, weakening patent strength also reduces competition, as it makes R&D industries capable of sustaining a smaller number of firms. And, evidently, other possible policy interventions that reduce N include raising institutional entry barriers in R&D industries, fostering joint ventures across R&D firms, and by providing incentives for patent pools, in which different firms cross-licensed patents relating to a particular technology.[30]

The next section illustrates how our model can be used to calculate optimal patents and assess welfare gains in specific industries by means of a numerical exercise. Before presenting our findings, however, we take a small detour and conduct a few robustness checks on the multi-stage analysis game we studied in this section.

4.4 Robustness checks

This section assesses whether the predictions of our multi-stage patent game are robust when we allow the firms broader plans of action than the ones considered so far. This exercise requires expanding our model beyond the boundaries of the class of stopping games to which our multi-stage patent game belongs. We consider three specific possibilities that may be natural in the contexts of some applications of our model. The first possibility is that, at any time, competing firms may choose to exit the game as well as choose to end the stage by patenting its innovation. The second possibility is that, after ending a stage of our multi-stage patent race, a firm may improve the innovation it has just patented, instead of moving on to participate in the next stage of our game. The third possibility is that, while engaged in one stage of our game, a firm may switch to research a new innovation without disclosing its current innovation, instead of ending the current stage of the game by patenting its innovation.

[30] Among the largest patent pools recently established, about 20 companies active in the radio frequency identification domain formed a Consortium in August 2005, to administer their cross-licensed patents.

Participation. The first robustness requirement we impose on the parameters of our multi-stage patent game is that competing firms do not ever have any incentive to leave the game. We have already introduced this check in the context of the basic model presented in Section 3 where we required that c/λ be sufficiently small to satisfy condition (3). In the context of the multi-stage game this sufficient condition can be refined to obtain the following sufficient condition, which is satisfied with much slack,[31]

$$c/\lambda \leq \left(\frac{r + N\lambda}{r + (N+1)\lambda} e^{-(r+N\lambda)T} + \frac{N}{N+1} \frac{\rho}{1 - \rho\beta\left(T\right)} \right) \left[\alpha x\left(T\right) - c_0\right]. \tag{13}$$

As is the case for condition (3), the above condition can easily be checked after calculating the equilibrium stopping time T, and is evidently not vacuous, as it is satisfied whenever c is sufficiently small relative to λ.

Improvements after patenting. We now consider the possibility that an innovator improves its innovation after patenting it, instead of participating in the next stage of our game. Whether or not this possibility requires any parametric restriction on our multi-stage model depends on the context of its application. As we earlier pointed out, our model may be applied to contexts in which innovations displace previous ones in the market and to contexts in which there is not any market displacement. In the latter case, the possibility that innovations are improved after patenting does not entail any modification of our model. It is sufficient to interpret the private innovation value $\alpha x\left(T\right)$ as including the value of all the innovation's improvements, that is achieved by the first innovator through follow up patents or licensing.

Matters are more complicated in contexts with market displacement. To be sure, in equilibrium, it cannot be the case that any firm i (including the firm who patented the innovation) obtains a payoff larger than the equilibrium value, by deviating to improve a patented innovation instead of participating in the next stage of our game.[32] But this does not imply that the equilibrium play would always conform to our earlier results, when allowing for the possibility that

[31] As the derivation of this condition is tedious and uninformative, we make it available only upon request.

[32] In fact, if this were the case, a different firm j could appropriate of almost all i's payoff, by improving the same innovation and by patenting the improvements slightly before than i. So, even if a single firm would potentially gain over the equilibrium value by improving a patented innovation in absence of competition, this potential gain will be dissipated by competition in equilibrium. As a result, our results are always robust to the possibility that any firm patents its innovation at a time $t < T$ but then keeps improving it, instead of switching to the next stage

innovations patented at time T are improved after they are patented. In fact, to dissipate any potential gain over the equilibrium value, firms would improve patents with positive probability on the equilibrium path. So, to ensure that our results are robust with respect to the possibility of improving innovations after patenting them, we introduce appropriate parametric restrictions that rule out any such potential gains. These restrictions are satisfied when the value function $x(\cdot)$ is sufficiently concave and when the costs c and c_0 are sufficiently high, so that firms would not be able to recoup them if improving an innovation after patenting it.

The specific restrictions are calculated as follows. Because we search for a sufficient condition for the robustness of our analysis, we consider the worse-case scenario in which rejoining the multi-stage game only after improving the patented innovation does not lower the continuation payoff. Further, we abstract from the possibility that competitors diminish the value of improving the patented innovation, by either patenting competing innovations in the next stage of our game, or by directly competing with the innovator in the improvement of the patented innovation.

Under these simplifying worse-case assumptions, a player who improves a patented innovation until any time t and then rejoins our multi-stage game obtains a value function $\hat{V}(t)$ that follows a simple dynamic programming equation: $r\hat{V}(t) = -c + \hat{V}'(t)$. At the optimal time $\hat{T}$ to stop improving the patented innovation, the value is $\hat{V}(\hat{T}) = \alpha[x(\hat{T}) - x(T)] - c_0 + U^L(T)$: the firm obtains the innovation private value increment over the previous patent $\alpha[x(\hat{T}) - x(T)]$, net of the development cost c_0, together with the continuation value for rejoining the game $U^L(T)$. So, provided that $\hat{T}$ is positive, it is the unique solution of:

$$r\left[\alpha x(\hat{T}) - \alpha x(T) - c_0 + \rho Q(T)\right] = -c + \alpha\dot{x}(\hat{T}).$$

Therefore, a sufficient condition (containing much slack) which rules out that firms improve patented innovations instead of moving on to the next stage in the game is that the discounted deviation value $e^{-r(\hat{T}-T)}\hat{V}(\hat{T}) - \left[1 - e^{-r(\hat{T}-T)}\right] c/r$ be not be larger than the equilibrium value. I.e., after simple manipulations it is sufficient that:

$$\alpha[x(\hat{T}) - x(T)] \leq \frac{1 - e^{-r(\hat{T}-T)}}{e^{-r(\hat{T}-T)}} \left[\rho Q(T) + c/r\right] + c_0. \tag{14}$$

of our game. Due to the above value dissipation argument, in fact, the value of this deviation is no larger than $\alpha x^*(t) - c_0 + \rho Q(T)$, the equilibrium value for patenting at time t, which is strictly smaller than $V(t)$.

In words, this condition requires that the maximal private value increment $\alpha\left[x(\hat{T}) - x(T)\right]$ not to be sufficiently large to recoup the development cost c_0, together with the appropriately compounded flow research costs c, and the continuation payoff postponement. As is the case for condition (13), the above condition can easily be checked after calculating the equilibrium stopping time T. Again, condition (14) is not vacuous (e.g., it is satisfied when c_0 is not too small, and x is sufficiently concave), and it can be proved that it does not conflict with earlier parametric restrictions.

Switching without Patenting Finally, we consider the possibility that a firm may switch to research a new innovation keeping its current innovation secret, instead of patenting it as it is required by our multi-stage game model.

First, we note that this possibility is not always feasible in specific applications. For example, in medical research one can often interpret c_0 as the cost of medical trials that conclude the research of a new drug or therapy. Such trials are public by law and they are a fundamental part of the research and development of the innovation. Hence it may not often be possible that a company switches to a new project without disclosing its current innovation. In other applications, the possibility that a firm secretly switches to research a new innovation may disturb our analysis. We now introduce parametric restrictions so that this is not the case; these restrictions are satisfied when the function $x(\cdot)$ is sufficiently concave or when the arrival rate λ is sufficiently small relative to N or r.

Omitting details, we derive our sufficient condition by imposing that no firm i is tempted to switch secretly to the next innovation in the multi-stage sequence at any time t past its breakthrough, regardless of the patenting strategy chosen after the secret switch. Again, we consider a worse-case scenario. We presume that, if firm i achieves a breakthrough on the secretly researched innovation before any competitor patents the previous innovation, then it cannot be preempted on either of the two innovations. Derivations analogous to the ones leading to the determination of $\hat{T}$ in the previous subsection show that firm i improves its secretly researched innovation until the time $\bar{T}$ that solves $r\left[\alpha x\left(\bar{T}\right) - c_0 + \rho Q\left(T\right)\right] = -c + \alpha \dot{x}\left(\bar{T}\right)$. After lengthy calculations presented in an online appendix, we then prove that the following condition is sufficient to ensure that firms have no incentive to secretly switch to research a new innovation before completing the current stage of our multi-stage game:

$$\lambda e^{-r\bar{T}}\left[\alpha x\left(\bar{T}\right)-c_0\right]\leq\left[(N-1)\lambda+r\right]\left[\alpha x\left(T\right)-c_0+(1-\xi)Q\left(T\right)\right]. \tag{15}$$

This condition requires that λ is not too large relative to N and r or that the maximal innovation private value $\alpha x(\bar{T})$ is not much larger than the equilibrium private value $\alpha x\left(T\right)$.[33] As for conditions (13) and (14), the above condition can be checked after calculating the equilibrium stopping time T. Again, condition (15) is not vacuous (e.g., it is satisfied when x is sufficiently concave or λ sufficiently small) and it can be proved that it does not conflict with earlier parametric restrictions.

5 Numerical Analysis

The simulation strategy takes its parameters from the agricultural seed industry as found in Fernandez-Cornejo (2004). The data reveal that aggregate costs for research and development of a new variety are 2.25 million dollars in the late 1980's. Further, the average time from cross-pollination to determination of a new variety is 7.9 years. We take this period as the 'research period'. This period is followed by an average of 3.2 years spent in developing the new variety. Hence the per-year research cost c is set at $2.25/(7.9+3.2) = 0.202$, whereas the cost c_0 is $2.25-(0.202)\,(7.9) = 0.654$. We set the interest rate at $r = 0.095$, the average bank prime rate in the second half of the 1980's. Although our symmetry assumption is unlikely to hold precisely in this industry, it would be too cumbersome to solve an asymmetric version of the model. Hence, we assume equally size firms. The reported Herfindahl index allows us to take the number of competitors as $N+1 = 4$. Because the same competitors compete over time, we take $\rho = \chi = 1$. Following Alston et al. (2000), table 15, we set the social rate of return of innovation to 74.3% and the private rate of return to $r = 0.32$. This gives an estimated value of $\alpha = 1.32/1.743 = 0.76$, significantly higher that the classical estimates by Mansfield et al. (1977), where the social value

[33] Note that the possibility of switching to researching the next innovation without patenting the current one is meaningless when there are no competitors, so that $N = 0$. Hence the right-hand side of condition (15) is always positive, whenever this exercise is relevant.

Parameter	r	c	c_0	N	$\rho = \chi$	$1/\lambda$	α	γ	a_0
Value	0.09	0.20	0.66	3	1	6	0.76	0	0.32

Table 1: Parameter values

of innovation is close to twice as large as the private value. During this time period, competitors researched a very broad variety of plant and vegetable seeds and there is no indication that novel innovations displaced previous seeds in the market.[34]

Further, the data by Fernandez-Cornejo (2004) reveal that in the first stages of the research process (recognition, parent-line preparation, and initial crosses) the value of the innovation does not grow. The value of the innovation begins to grow with the following phase (progeny selection) which starts on average 6 years after the beginning of the research process. With these data, we shall henceforth focus on λ so that the average time for innovation growth to start is $1/\lambda = 6$. We choose an innovation growth process such that $\dot{x}(t)$ decays exponentially: $x(t) = a_0 \frac{1-e^{-\gamma t}}{\gamma}$ so that $\dot{x}(t) = a_0 e^{-\gamma t}$. We note that the half-time of the growth process is given by $\ln 2/\gamma$, this is the amount of time $\mathcal{T}$ needed for the growth rate $\dot{x}(\mathcal{T})$ to reach $a_0/2$.[35]

We recover the parameters γ and a_0 so that the expected research time, $1/\lambda + T$ is as close as possible to 7.9 (we obtain 7.76) and the private rate of return R is as close as possible to 0.32 (we get 0.33). The parameter values are given in Table 1.

Taking the number of firms as given, we calculate the socially optimal stopping rule. Results are given in Table 2. The difference between the best calibrated time spent in research of a new seed, $1/\lambda + T_{0.76} = 7.76$ years, and the socially optimal research time, $1/\lambda + T^* = 11.12$ years, is striking. Our calculations require that firms should wait an average of 43% more before patenting their innovations. The social value of the innovations actually patented is only 62% of the value of the socially optimal innovations. Fear of preemption appears to dominate the other forces we have uncovered in the theoretical analysis, and to yield a large welfare loss.

[34] Among the different plants and vegetables which were the object of R&D, Alston et al. (2000) include barley, beans, cassava, sugar cane, groundnuts, maize, millet, other crops, pigeon pea or chickpea, potato, rice, sesame, sorghum, and wheat, as well as various sorts of tree crops and animal feed. Further, different seed varieties of the same plants were developed for different markets, e.g., some varieties were bred to be resistant to drought and others to excessive rainfall.

[35] One can check that our parametric restrictions are satisfied, as long as $\gamma > 0$, and $a_0 > c$, and that the results presented in Section 4 hold also when $\gamma = 0$, here.

	Expected duration	Social welfare
Competitive	7.70	8.10
Optimal	11.12	13.06
Comp/Opt	0.69	62%

Table 2: Social welfare

N+1	U^*	α^*	$\bar{\alpha}$	$U_{\bar{\alpha}}$
1	10.6	1	1	10.6
2	13.1	0.06	0.34	11.8
3	13.5	0.05	0.47	10.4
4	13.1	0.05	0.60	8.4
5	12.1	0.05	0.72	6.14

Table 3: Optimal number of firms

We find that firms can be induced to choose the optimal stopping time by making patent rights less strong.

First, holding the number of competing firms fixed, our calculations show that optimal research durations can be achieved if and only if the fraction of innovation earned by the innovator is negligible, and equal to 0.05. However, for such a small value of α, the 4 firms participating in the patent race in the status quo would not find it profitable to remain in the race. Instead, the highest level of appropriation that is consistent with 4 firms staying in the race equals 0.60, which achieves 65% of the maximum social welfare, a modest gain relative to the status quo.

Second, social welfare can be improved by making patents weaker so as to reduce the number of firms participating in the game. Table 3 gives social welfare and optimal patent strength as we vary N. Social welfare is maximized by having one firm less than the status quo, though the gains are small. As in the status quo, optimal patent strength at $\alpha = 0.06$ is extremely low. Taking into account the voluntary participation of firms, welfare is maximized with 2 firms and a value of $\bar{\alpha} = 0.34$, achieving 87% of the social optimum, and 46% gain relative to the status quo. Restricting the number of participants in the race by using weak patents leads to considerable gains.

The final part of our numerical investigation is reported in an online appendix. We repeat the analysis by varying the parameters of the model around the values recovered from the data. We demonstrate the robustness of the conclusions of our policy experiments on the calibrated model, and derive basic comparative statics on our variable of interest.

6 Conclusion

We have studied how patent rights shape incentives with respect to the timing of innovation disclosure. In a simple model, we have identified forces that lead firms to either patent innovations too early or too late in equilibrium, relative to the socially optimal strategies. Our novel results warn that competing firms may patent their innovations too early, for fear of being preempted in the patent race. Contrary to usual intuitions, we have also shown that stronger patents may make firms delay equilibrium patenting. By relating our formalization to the existing literature, we have argued that it may be applied to a wide range of R&D scenarios. These scenarios include a class of two-stage patent races, in which firms compete on a 'basic' innovation with no commercial value per se, that later leads to valuable applications.

Further, we have expanded and detailed the analysis by fully solving an infinite multi-stage patent game. The analysis of this model has allowed us to derive precise and detailed predictions and to explicitly describe innovation growth. As well as a fear of preemption force which leads firms to disclose innovations too early, we have identified an intertemporal spillover force that distorts firms' stopping times in the opposite direction. Further, we have shown that stronger patents may slow down equilibrium innovation growth by delaying equilibrium patenting. We argue that our precise and detailed predictions can be taken to the data and used for policy experiments. As an illustration, we have calibrated our multi-stage game on summary statistics from the seeds-breeding industry. We have found support for weaker patent rights that could result in welfare gains of 46% relative to the status quo. These gains are achieved through the mitigation of fear of preemption. When patents are weaker, a smaller number of firms find it profitable to participate in the game. Reduced competition makes the remaining firms willing to extend their innovation disclosure times so that they are better aligned with the socially optimal disclosure times.

The most important methodological contributions of this paper is the formulation of a rich framework to study innovation disclosure timing in details. Future research could extend our framework to also analyze the invention decision. For example, the breakthrough probability could be endogeneized, and different flow costs of invention and development could be introduced. More broadly, as the study of the innovation disclosure incentives are currently underdeveloped, our framework could be extended in a number of direction and generate a rich research program.

A promising extension consists in allowing for heterogeneity in the growth processes of different firms. This may be valuable as it allows one to capture asymmetries between firms that could be due to different market shares. For example, asymmetries could represent instances in which firms have different amounts of funds that can be devoted to R&D activity, or different access to credit. Another possibility would be to study a non-stationary version of our multi-stage model, where the underlying parameters change across stages. For example, the cost of entry in the multi-stage game may change over time, so that the number of competing firms may also change. When this is the case, duplication costs would play a role in the multi-stage model, as well as in our basic model. A further possibility would be to modify our framework so that the breakthrough arrival rate of the winner is larger than the losers' arrival rates. This modification would generate a force that makes firms patent later: Firms may choose to postpone patenting for the purpose of gaining an advantage in the next race (as in Scotchmer and Green, 1990). Also, the applicability of our multi-stage game could be broadened by lifting the assumption that second-generation inventors make a take it or leave it offer to the holders of previous patents, for the case in which the patented innovations displace earlier one from the market (see the discussion on page 21). For example, it could be assumed that licensing is solved via Nash bargaining, so that the holder of previous patents appropriates of a fixed share of the new innovation expected present discounted value.[36]

When reconsidering our findings within these more involved frameworks, we expect our calibration results to be robust, because the optimal patent strength we calculated is not much

[36] More involved modifications of our set up would allow for more detailed patent policy design. Our model summarizes patent policy with a single parameter, patent strength. One possibility would be to make patent strength depend on the equilibrium stopping time, as is the case for example when patents have finite length and novel innovations displace previous innovations in the market. Within this context, it would also be interesting to study how patent strength and equilibrium stopping times interact as the license holder's bargaining power varies in the bargaining game with the license buyer. Further, one could allow patent policy to modify the growth parameters, and especially the arrival rate of the breakthrough, by modifying the length and breadth of patents.

sensitive to parameter variation.

Appendix A – Proofs omitted from section 3

Derivation of Equation (1) and of the hazard rate $h(\hat{t})$. We calculate the symmetric equilibria in which all firms adopt the same optimal stopping time T past the breakthrough arrival at time τ. We fix a firm i, suppose that all the other N firms adopt the stopping time T, and calculate i's best response. Letting $F(t)$ be the distribution of the time at which the first one of i's opponents patents its innovation, with the usual discrete-time approximation technique, the optimal value $V(t)$ at calendar time $\hat{t}$ and time t past the breakthrough arrival can be recursively written as:

$$V(t) = \max\left\{\overline{U}(t;\alpha), \quad -cdt + \frac{F(\hat{t}+dt) - F(\hat{t})}{1-F(\hat{t})}e^{-rdt}\underline{U}(T;\alpha) + \frac{1-F(\hat{t}+dt)}{1-F(\hat{t})}e^{-rdt}V(t+dt)\right\}.$$

The first term is the value $\overline{U}(t;\alpha)$ of stopping and patenting the innovation. The second term is the value for conducting research of a small period of time dt, thus paying the flow cost cdt, and then reoptimizing. With probability $[F(\hat{t}+dt) - F(\hat{t})]/[1-F(\hat{t})]$, an opponent stops the race and the firm achieves the continuation value $\underline{U}(T;\alpha)$. With complementary probability, the race continues and the firm reoptimizes, obtaining the value $V(t+dt)$.

For any t such that i weakly prefers not to patent, approximating e^{-rdt} with $1-rdt$ in the above equation, rearranging, dividing by dt both terms, and taking limits, we obtain:

$$\begin{aligned} 0 \quad = \quad & \lim_{dt\to 0}\left[-c + \frac{F(\hat{t}+dt) - F(\hat{t})}{dt}\frac{(1-rdt)\underline{U}(T;\alpha) - V(t)}{1-F(\hat{t})}\right. \\ & \left. + \frac{1-F(\hat{t}+dt)}{1-F(\hat{t})}\frac{V(t+dt)-V(t)}{dt} - r\frac{1-F(\hat{t}+dt)}{1-F(\hat{t})}V(t+dt)\right]. \end{aligned}$$

Taking the limit, and rearranging, we obtain equation (1), in which the hazard rate $h(\hat{t})$ of the event that one of i's opponents ends the race at time $\hat{t}$ is defined as $h(\hat{t}) \equiv f(\hat{t}) / [1-F(\hat{t})]$.

For any calendar time $\hat{t} \geq T$ as breakthroughs are independent and identically distributed across players,

$$F(\hat{t}) = \Pr(T + \bar{\tau} \leq \hat{t}) = \Pr(\bar{\tau} \leq \hat{t} - T) = 1 - e^{-N\lambda(\hat{t}-T)},$$

where $\bar{\tau}$ denotes the first breakthrough arrival time among i's opponents. Letting $f(\hat{t}) = N\lambda e^{-N\lambda(\hat{t}-T)}$ denote the density associated with $F(t)$, we conclude that the hazard rate $h(\hat{t}) = f(\hat{t}) / [1-F(\hat{t})]$ equals $N\lambda$, thus deriving equation (1). Because $F(\hat{t}) = 0$ and $f(\hat{t}) = 0$ for $\hat{t} < T$, the hazard rate is $h(\hat{t}) = 0$. ■

Proof of Propositions 1 and 3. Because $\lim_{T\to+\infty}\overline{U}_1(T;\alpha) = 0$ and $\lim_{T\to+\infty}\overline{U}(T;\alpha) > \lim_{T\to+\infty}\underline{U}(T;\alpha)$, it follows that $\lim_{T\to+\infty}[(N\lambda+r)\overline{U}(T;\alpha)+c(T)] > \lim_{T\to+\infty}\left[\overline{U}_1(T;\alpha) + N\lambda\underline{U}(T;\alpha)\right]$. This condition, together with the condition that $\overline{U}_1(0;\alpha) > c + r\overline{U}(0;\alpha) + N\lambda\left[\overline{U}(0;\alpha) - \underline{U}(0;\alpha)\right]$, guarantees that equation (2) has a finite, strictly positive solution. Because $\overline{U}_1(T;\alpha) > 0$, the left hand side of the equation strictly increases in T, and because $\overline{U}_{11}(T;\alpha) < 0$ and $\overline{U}_1(T;\alpha) - \underline{U}_1(T;\alpha) > 0$ the right hand side strictly decreases in T. Hence equation (2) has a unique solution. This unique solution is the stopping time associated with the unique symmetric, calendar-time independent equilibrium of our game. When the condition $\overline{U}_1(0;\alpha) > c + r\overline{U}(0;\alpha) + N\lambda\left[\overline{U}(0;\alpha) - \underline{U}(0;\alpha)\right]$ fails, the unique stopping time T equals zero. To prove Proposition 3, let

$$\phi(T,\alpha) = r\overline{U}(T;\alpha) + c - \overline{U}_1(T;\alpha) + N\lambda\left[\overline{U}(T;\alpha) - \underline{U}(T;\alpha)\right],$$

and note that, by the implicit function theorem,

$$\frac{\partial T}{\partial \alpha} \propto -\frac{\partial \phi(T,\alpha)}{\partial \alpha} = -r\overline{U}_2(T;\alpha) + \overline{U}_{12}(T;\alpha) - N\lambda\left[\overline{U}_2(T;\alpha) - \underline{U}_2(T;\alpha)\right].$$

■

Proof of Proposition 2. We first note that equation (5) has a real-valued solution because $\lim_{T^*\to+\infty} U_1^*(T^*) = 0$, and $\lim_{T^*\to+\infty} rU^*(T^*) > 0 > -(N+1)c$. Further, because $rU_1^*(T^*) > 0$ and $U_{11}^*(T^*) < 0$, equation (5) has a unique solution. This solution is strictly positive when $U_1^*(0) > (N+1)c + rU^*(0)$; otherwise it equals zero.

Further, if the term $r\left[U^*(T) - \overline{U}(T;\alpha)\right]$ is strictly positive (negative), then the left-hand side of equation (5) $rU^*(T)$ is strictly larger (smaller) than the left-hand side at the equilibrium stopping time T. Because $U^*(\cdot)$ is strictly increasing and equation (5) holds at the optimal stopping time T^*, this implies that $T^* > T$ (respectively, that $T^* < T$). ■

Appendix B – Proofs omitted from section 4

The relationship between T and g. Differentiating g with respect to T, we obtain:

$$\begin{aligned}\frac{d}{dT}E\left[\frac{x(T)}{T+\hat{\tau}}\right] &= E\left[\frac{d}{dT}\frac{x(T)}{T+\hat{\tau}}\right] = E\left[\frac{\dot{x}(T)(T+\hat{\tau}) - x(T)}{(T+\hat{\tau})^2}\right] \\ &\propto E\left[\dot{x}(T)(T+\hat{\tau}) - x(T)\right] = \dot{x}(T)T - x(T) + E[\hat{\tau}]\dot{x}(T) = \dot{x}(T)T - x(T) + \frac{x(T)}{\lambda(N+1)}.\end{aligned}$$

This quantity is negative whenever x is sufficiently concave, so that $\dot{x}(T)T - x(T)$ is sufficiently negative, and λ is not too small, so that the term is not too large. ■

Derivation of Expression (8). By definition, $Q(T)$ takes the form:

$$Q(T) = E\left[\left(-\int_0^{\hat{\tau}+T} ce^{-rs}ds + e^{-r(\hat{\tau}+T)}\frac{\alpha x(T) - c_0 + U^L(T)}{N+1} + \frac{N}{N+1}e^{-r(\hat{\tau}+T)}U^F(T)\right)\right].$$

The first term in parenthesis accounts for paying the cost c until time $\hat{\tau}+T$, where $\hat{\tau}$ is the random time of the first breakthrough arrival among the $N+1$ competitors at any given stage. Further, because the game and equilibrium are symmetric, each firm wins the stage with probability $1/(N+1)$. If so, it gets the payoff $\alpha x(T) - c_0 + U^L(T)$ at time $\hat{\tau}+T$, and this explains the second term in parenthesis; otherwise, with probability $N/(N+1)$, the firm gets the payoff $U^F(T)$ at time $\hat{\tau}+T$, and this is represented by the third term.

The expression for $Q(T)$ can be simplified as:

$$Q(T) = E\left[\left(-\frac{c}{r}\left[1-e^{-r(\hat{\tau}+T)}\right] + e^{-r(\hat{\tau}+T)}\frac{\alpha x(T)-c_0}{N+1} + e^{-r(\hat{\tau}+T)}\frac{U^L(T)}{N+1} + \frac{N}{N+1}e^{-r(\hat{\tau}+T)}U^F(T)\right)\right].$$

Using the notation ξ, we obtain:

$$Q(T) = E\left[-\frac{c}{r}\left[1-e^{-r(\hat{\tau}+T)}\right] + e^{-r(\hat{\tau}+T)}\frac{\alpha x(T)-c_0}{N+1} + e^{-r(\hat{\tau}+T)}\xi Q(T)\right],$$

which yields the expression (8), when using the definition $\beta(T) = E\left[e^{-r(\hat{\tau}+T)}\right]$. We conclude the proof by calculating $\beta(T)$. Because $\hat{\tau}$ is the earliest of $N+1$ independent breakthrough arrivals, identically exponentially distributed with hazard rate λ, we obtain:

$$E\left[e^{-r(\hat{\tau}+T)}\right] = e^{-rT}E\left[e^{-r\hat{\tau}}\right] = e^{-rT}\int_0^\infty e^{-r\hat{\tau}}(N+1)\lambda e^{-(N+1)\lambda\hat{\tau}}d\hat{\tau} = e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda+r}.$$

■

Proof of Proposition 4.

Substituting the expressions for $\overline{U}(T;\alpha)$ and $\underline{U}(T;\alpha)$ in equation (2), we obtain:

$$r\left[\alpha x(T) - c_0 + U^L(T)\right] = -c + \alpha\dot{x}(T) - N\lambda\left[\alpha x(T) - c_0 + U^L(T) - U^F(T)\right]$$

which yields equation (9), after substituting the expressions for U^L and U^F and rearranging.

It will useful for this proof to rewrite equation (9) as follows:

$$\phi(T) \equiv (r+N\lambda)\left[\alpha x(T) - c_0\right] + \left[r\rho + N\lambda(\rho-\chi)\right]Q(T) + c - \alpha\dot{x}(T) = 0. \tag{16}$$

Note that $\dot{x}(T) \to 0$ and that $Q(T) \to -c/r$ for $T \to \infty$. Because $\alpha x(T) > c/r + c_0 \geq (\rho-\chi)c/r + c_0$ for T sufficiently large, it then follows that

$$\begin{aligned} & (r+N\lambda)\left[\alpha x(T) - c_0\right] - \frac{c}{r}\left[r\rho + N\lambda(\rho-\chi)\right] + c \\ = \; & (r+N\lambda)\left[\alpha x(T) - c_0\right] + \frac{c}{r}[r(1-\rho) - N\lambda(\rho-\chi)] > 0, \end{aligned}$$

and hence that $\phi(T) > 0$, for T sufficiently large.

Further, $\alpha x(0) - c_0 \leq 0$ implies that

$$Q(0) \leq -\frac{c}{r}\frac{1-\beta(0)}{1-\xi\beta(0)} = -\frac{c}{r}\frac{1-\frac{(N+1)\lambda}{(N+1)\lambda+r}}{1-\xi\frac{(N+1)\lambda}{(N+1)\lambda+r}} = -\frac{c}{r}\frac{r}{(N+1)\lambda(1-\xi)+r},$$

and that

$$\begin{aligned} \phi(0) \;&\leq\; -\frac{c}{r}\frac{r}{(N+1)\lambda(1-\xi)+r}\left[r\rho + N\lambda(\rho-\chi)\right] + c - \alpha\dot{x}(0) \\ &=\; c\frac{r(1-\rho) + (N+1)\lambda(1-\xi) - N\lambda(\rho-\chi)}{(N+1)\lambda(1-\xi)+r} - \alpha\dot{x}(0) < 0, \end{aligned}$$

because $\alpha\dot{x}(0) > c$. Because $\phi(0) < 0$ and $\phi(T) > 0$, for T sufficiently large, it follows that there exists finite $T > 0$ such that $\phi(T) = 0$. For any such stopping time T, there is a corresponding a symmetric, stage-stationary, calendar time independent equilibrium.

In order to prove uniqueness of the symmetric, stage-stationary, calendar time independent equilibrium, we proceed as follows.

First, we reconsider the function $Q(\cdot)$, which we introduced in expression (8). Without restricting attention to equilibrium strategies, the quantity $Q(\tau)$ is the expected payoff of each firm at the beginning of each stage of the game, when every firm adopts the symmetric, stationary, calendar-time independent strategies associated with the stopping time τ. Because the strategies are symmetric, though, each firm has equal probability of winning every stage of the game. So, $Q(\tau)$ can also be interpreted as each firm's share of the joint payoff of all the firms who compete in any stage of the game, at the beginning of the stage, when they adopt the strategy associated with the stopping time τ. As a result, the stopping time τ^* which maximizes $Q(\tau)$ corresponds to the Pareto-dominant equilibrium of the auxiliary 'team problem' game in which each firm maximizes the joint expected payoff of all the firms competing in that stage of the game (see, Marschak and Radner, 1972).

In order to characterize τ^* more precisely, we now show that this team problem has a unique equilibrium. In fact, any associated equilibrium value Q must satisfy the following Bellman equation:

$$Q = \max_{\tau}\left\{-\frac{c}{r}\left[1-\beta(\tau)\right] + \left(\frac{\alpha x(\tau)/r - c_0}{N+1}\right)\beta(\tau) + \beta(\tau)\,\xi Q\right\}.$$

Because $\beta(\tau) = e^{-r\tau}\frac{(N+1)\lambda}{(N+1)\lambda + r}$ is bounded above by $\frac{(N+1)\lambda}{(N+1)\lambda + r} < 1$, the above equation has a unique fixed point Q. Further, for any value Q, there is a unique τ maximizing the above expression. Hence, the unique equilibrium value Q of the team problem is achieved by a unique equilibrium τ^*.

Now, we compare the dynamic programming equation characterizing the unique equilibrium τ^* of the team problem, with the dynamic programming equation characterizing the equilibrium stopping time T in our multistage patent race. Proceeding as we did for equation (1), we obtain:

$$rV(\tau^*) = -c + V'(\tau^*) + N\lambda\left[V(\tau^*) - V(\tau^*)\right],$$

where $V(\tau^*) = \alpha x(\tau^*) - c_0 + \xi Q(\tau^*)$, because each firm maximizes the per-capita joint payoff of all the firms that compete in the stage. Substituting and rearranging, we obtain that the unique equilibrium τ^* of the team problem is the unique solution of the function

$$\kappa(\tau) = r\left[\alpha x(\tau) - c_0\right] + c - \alpha\dot{x}(\tau) + r\xi Q(\tau).$$

Further, because τ^* maximizes $Q(\tau)$, the envelope theorem implies that $\kappa'(\tau^*) = r\alpha\dot{x}(\tau^*) - \alpha\ddot{x}(\tau^*)$; as this quantity is strictly positive, it follows that $\kappa(\tau) < 0$ if and only if $\tau < \tau^*$.

Because all equilibrium stopping times T of our multi-stage patent race solve equation (16), we can sign $\kappa(\cdot)$ at any such T by subtracting the expression for $\phi(T)$ from $\kappa(T)$:

$$\begin{aligned}\kappa(T) &= r\left[\alpha x(T) - c_0\right] + c - \alpha\dot{x}(T) + r\xi Q(T)\\ &\quad - \left\{(r+N\lambda)\left[\alpha x(T) - c_0\right] + \left[r\rho + N\lambda(\rho-\chi)\right]Q(T) + c - \alpha\dot{x}(T)\right\}\\ &= -r(\rho-\xi)Q(T) - N\lambda\left[\alpha x(T) - c_0 + (\rho-\chi)Q(T)\right] < 0.\end{aligned}$$

Because $\kappa(\tau) < 0$ if and only if $\tau < \tau^*$, this concludes that any equilibrium stopping time T of our multi-stage game is strictly smaller than τ^*.

This result implies that $Q'(T) > 0$, because, as we verify below, $Q'(\tau) > 0$ if and only if $\tau < \tau^*$. In fact, because $\beta'(\tau) = -r\beta(\tau)$,

$$\begin{aligned}
Q'(\tau) &= \frac{\beta(\tau)}{1-\xi\beta(\tau)} \frac{\alpha\dot{x}(\tau)}{N+1} + \frac{\partial}{\partial\beta}\left\{\beta(\tau)\frac{\alpha x(\tau)-c_0}{N+1} - \frac{c}{r}[1-\beta(\tau)]\right\}\frac{1}{1-\xi\beta(\tau)}\beta'(\tau) \\
&\quad + \left\{\beta(\tau)\frac{\alpha x(\tau)-c_0}{N+1} - \frac{c}{r}[1-\beta(\tau)]\right\}\frac{\partial}{\partial\beta}\left(\frac{1}{1-\xi\beta(\tau)}\right)\beta'(\tau) \\
&= \frac{\beta(\tau)}{1-\xi\beta(\tau)} \frac{\alpha\dot{x}(\tau)}{N+1} - r\beta(\tau)\left[\frac{\alpha x(\tau)-c_0}{N+1} + \frac{c}{r}\right]\frac{1}{1-\xi\beta(\tau)} \\
&\quad - r\beta(\tau)\left\{\beta(\tau)\frac{\alpha x(\tau)-c_0}{N+1} - \frac{c}{r}[1-\beta(\tau)]\right\}\frac{\xi}{[1-\xi\beta(\tau)]^2} \\
&= -\beta\frac{r[\alpha x(\tau)-c_0] + (N+1)c(1-\xi) - (1-\xi\beta)\alpha\dot{x}(\tau)}{[1-\xi\beta(\tau)]^2(N+1)} \\
&\propto -r[\alpha x(\tau)-c_0] - (N+1)c(1-\xi) + (1-\xi\beta)\alpha\dot{x}(\tau),
\end{aligned}$$

which is strictly decreasing in τ. So, because $Q'(\tau^*) = 0$, it follows that $Q'(\tau) > 0$ if and only if $\tau < \tau^*$.

Once concluded that $Q'(T) > 0$ for any T such that $\phi(T) = 0$, inspection of equation (16) shows that $\phi'(T) > 0$ at any such T, because $\dot{x}(T) > 0$ and $\ddot{x}(T) < 0$. Hence there is a unique T that solves $\phi(T) = 0$.

We have thus concluded that there is a unique equilibrium of our multistage patent race game, with symmetric, stage-stationary, calendar-time independent stopping time strategies.

The proof of 4 is completed by establishing the stated comparative statics results. This is achieved with simple but long and tedious calculus manipulations, which are omitted and available upon request. ■

Proof of Proposition 5. Because the previous uniqueness proof has shown that $\phi'(T) > 0$, the implicit function theorem implies that $dT/d\alpha$ has the same sign as

$$-\frac{\partial}{\partial\alpha}\phi(T) = -(r+N\lambda)x(T) - \frac{r\rho + N\lambda(\rho-\chi)}{N+1}\frac{\beta(T)}{1-\xi\beta(T)}x(T) + \dot{x}(T).$$

Hence, when $\rho = \xi = \chi$, there is a threshold function $\hat{\xi} : N \mapsto \xi$, such that $dT/d\alpha < 0$ if $\xi > \hat{\xi}(N)$, and $dT/d\alpha > 0$ if $\xi < \hat{\xi}(N)$. In fact, when $\rho = \xi = \chi$,

$$\frac{d}{d\xi}\left[\frac{\partial}{\partial\alpha}\phi(T)\right] = r\frac{\beta(T)x(T)}{(1-\xi\beta(T))^2(N+1)} > 0.$$

As well as determining $dT/d\alpha$, this calculus also pins down the sign of the derivative $dg/d\alpha$ because

$$\frac{dg}{d\alpha} = \frac{d}{dT}E\left[\frac{x(T)}{T+\hat{\tau}}\right]T'(\alpha),$$

and we earlier concluded that $\frac{d}{dT}E\left[\frac{x(T)}{T+\hat{\tau}}\right]$ is negative when x is sufficiently concave and λ not too small.

The relationship between the equilibrium value Q and α is determined by the decomposition:

$$\frac{dQ}{d\alpha} = \frac{\partial Q\left(T\right)}{\partial\alpha} + Q'\left(T\right)\frac{dT}{d\alpha}.$$

Finally, we know from the proof of uniqueness that $Q'\left(T\right) > 0$. Hence, $\partial Q\left(T\right)/\partial\alpha = \frac{\beta(T)x(T)}{[1-\xi\beta(T)]N+1} > 0$ implies that $dQ/d\alpha > 0$ whenever $dT/d\alpha > 0$, whereas $dQ/d\alpha$ is indeterminate if $dT/d\alpha < 0$. ■

Derivation of Expression (11) and proof of Proposition 6. We recursively set up the equation pinning down $U^S(T^*)$, and obtain:

$$\begin{aligned} U^S(T^*) &= E\left[-\int_0^{T^*+\tau}\left(N+1\right)ce^{-rs}ds + e^{-r(T^*+\tau)}\left[x\left(T^*\right)-c_0+U^S\left(T^*\right)\right]\right] \\ &= -\left[1-\beta\left(T^*\right)\right]\left(N+1\right)\frac{c}{r} + \beta\left(T^*\right)\left[x\left(T^*\right)-c_0\right] + \beta\left(T^*\right)U^S\left(T^*\right), \end{aligned}$$

where the simplification uses $E\left[e^{-r(T^*+\tau)}\right] = \beta\left(T^*\right)$. Solving for $U^S\left(T^*\right)$, we obtain expression (11).

By plugging this expression and the definition $U^*\left(T^*\right) = x\left(T^*\right) - c_0 + U^S\left(T^*\right)$ in equation (5), we obtain:

$$r\left[x\left(T^*\right)-c_0+\frac{\beta\left(T^*\right)}{1-\beta\left(T^*\right)}\left[x\left(T^*\right)-c_0\right]-\left(N+1\right)\frac{c}{r}\right] = -(N+1)c+\dot{x}\left(T^*\right),$$

which yields equation (12) with simple algebra permutations.

Letting

$$\Phi\left(T^*\right) = \frac{r}{1-\beta\left(T^*\right)}\left[x\left(T^*\right)-c_0\right]-\dot{x}\left(T^*\right),$$

it is immediate to see that $\partial\Phi\left(T^*\right)/\partial T^* > 0$ so that equation (12) has a unique finite and strictly positive solution, as long as $\Phi\left(0\right) < 0$ and $\Phi\left(T^*\right) > 0$ for T^* larger enough. The first condition is obviously satisfied because $x\left(0\right) \leq c_0$ and $\dot{x}\left(0\right) > 0$, whereas the second one is satisfied because $\dot{x}\left(T^*\right) \to 0$ for $T^* \to \infty$ and $x\left(T^*\right) > c_0$ for T^* large enough. ■

Proof of Proposition 7. To see that there is a threshold function $\bar{\xi} : N \mapsto \xi$, such that $T < T^*$ if $\xi > \xi^*(N)$ and $T > T^*$ if $\xi < \xi^*(N)$, note that T decreases in ξ by Proposition 9, whereas T^* is independent of ξ. Likewise, increasing $\rho - \chi$ decreases T relative to T^* because T^* is independent of $\rho - \chi$ and T decreases in $\rho - \chi$ by Proposition 9.

The proof of the remaining statements in Proposition 7 is tedious and not very informative, so we make it available only upon request. ■

Proof of Proposition 8. Set $\rho = \chi = \xi$. First, we plug expression (8) into the equilibrium equation (9), and simplify to obtain:

$$\left[(r+N\lambda)+\frac{1}{N+1}\frac{r\xi\beta(T)}{1-\xi\beta(T)}\right][\alpha x(T)-c_0]+\frac{1-\xi}{1-\xi\beta(T)}c=\alpha\dot{x}(T). \tag{17}$$

Plugging expression (17) into the left-hand side of inequality (10), and simplifying, we obtain that $dT/d\alpha = 0$ if and only if:

$$\frac{1-\xi}{1-\beta(T)\xi}c=\left[(r+N\lambda)+\frac{1}{N+1}\frac{r\xi\beta(T)}{1-\xi\beta(T)}\right]c_0. \tag{18}$$

Further, the proof of Proposition 6 implies that $T < T^*$ if and only if

$$\frac{r}{1-\beta(T)}[x(T)-c_0]<\dot{x}(T), \tag{19}$$

where the time T in the above inequality is the equilibrium stopping time pinned down by equation (17). Plugging equation (17) into inequality (19), simplifying, and rearranging α yields:

$$\frac{\alpha r}{1-\beta(T)}[x(T)-c_0]<\left[(r+N\lambda)+\frac{1}{N+1}\frac{r\xi\beta(T)}{1-\xi\beta(T)}\right][\alpha x(T)-c_0]+\frac{1-\xi}{1-\beta(T)\xi}c.$$

We now suppose that $dT/d\alpha = 0$, so that $\xi = \hat{\xi}(N)$. Plugging equation (18) in the above inequality, and simplifying, we obtain:

$$\frac{r}{1-\beta(T)}[x(T)-c_0]<\left[(r+N\lambda)+\frac{1}{N+1}\frac{r\xi\beta(T)}{1-\xi\beta(T)}\right]x(T).$$

We now note that left-hand side of the above inequality increases in ξ, so the worst-case scenario is when $\xi = 0$ and the above inequality takes the simpler form

$$\frac{r}{1-\beta(T)}[x(T)-c_0]<\left[(r+N\lambda)+\frac{1}{N+1}\frac{r\beta(T)}{1-\beta(T)}\right]x(T)$$

which can be rearranged as follows:

$$\left[\frac{r}{1-\beta(T)}-(r+N\lambda)-\frac{1}{N+1}\frac{r\beta(T)}{1-\beta(T)}\right][x(T)-c_0]<\left[(r+N\lambda)+\frac{1}{N+1}\frac{r\beta(T)}{1-\beta(T)}\right]c_0.$$

To show that this inequality holds, it is enough to show the negativity of the multiplier on the left-hand side,

$$\left[\frac{r}{1-\beta(T)}-(r+N\lambda)-\frac{1}{N+1}\frac{r\beta(T)}{1-\beta(T)}\right].$$

And this can be shown as follows:

$$\begin{aligned}
\frac{r}{1-\beta(T)}-\left[(r+N\lambda)+\frac{1}{N+1}\frac{r\beta(T)}{1-\beta(T)}\right] &= -N\lambda+r\frac{\beta(T)}{1-\beta(T)}\frac{N}{N+1}\\
&\propto -\lambda+r\frac{\beta(T)}{1-\beta(T)}\frac{1}{N+1}\\
&= -\lambda+r\frac{\frac{(N+1)\lambda}{(N+1)\lambda+r}e^{-rT}}{1-\frac{(N+1)\lambda}{(N+1)\lambda+r}e^{-rT}}\frac{1}{N+1}\\
&= -\lambda\left(1-e^{-rT}\right)\frac{(N+1)\lambda+r}{(N+1)\lambda(1-e^{-rT})+r}<0.
\end{aligned}$$

We have concluded that inequality (19) holds —i.e., that $T < T^*$ and hence that $\xi > \bar{\xi}(N)$, for the values of ξ such that $dT/d\alpha = 0$ —i.e., such that $\xi = \hat{\xi}(N)$. In other terms, we have concluded that $\hat{\xi}(N) > \bar{\xi}(N)$.

Now, using Propositions 5 and 7, we see the following:

- For $\xi < \min\{\bar{\xi}(N), \hat{\xi}(N)\} = \bar{\xi}(N)$, it is the case that $T > T^*$ and $dT/d\alpha > 0$.
- For $\xi > \max\{\bar{\xi}(N), \hat{\xi}(N)\} = \hat{\xi}(N)$, it is the case that $T < T^*$ and $dT/d\alpha < 0$.
- For the remaining intermediate case, $\bar{\xi}(N) < \xi < \hat{\xi}(N)$, it is the case that $T < T^*$ and $dT/d\alpha > 0$.

■

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Online Appendix

The first part of this online appendix provides the formal calculations deriving the parametric restrictions that ensure the robustness of the analysis of our model to the possibility that a firm may switch to research a new innovation keeping its current innovation secret, instead of patenting it as it is required by our multi-stage game model.

Derivation of Condition (15). Suppose that at time t since its breakthrough, firm i switches to the next innovation in the game sequence without patenting its current innovation. We allow i to patent the next innovation at any time $\bar{T}$ after a breakthrough which occurs at the random time τ after the switch. We also allow firm i to patent the current innovation at any time $\bar{\tau} \leq \tau + \bar{T}$. We let the value for secretly switching at time t be $V\left(t, \bar{\tau}, \bar{T}\right) = \rho \bar{V}\left(t, \bar{\tau}, \bar{T}\right) + (1-\rho)\left[\alpha x(t) - c_0\right]$: with probability $1-\rho$, firm i exits the game, otherwise it achieves the value $\bar{V}\left(t, \bar{\tau}, \bar{T}\right)$ which we calculate below.

We first note that $\bar{\tau} \geq \tau$ and $\bar{T} \geq T$. In fact, if patenting the current innovation after achieving the next breakthrough at time τ, firm i guarantees itself a lead in the next innovation's race. Instead, by patenting the current innovation at any time $\bar{\tau} < \tau$, firm i has no lead advantage, because breakthroughs arrivals are exponentially distributed, and the exponential distribution is memoryless.

Further, if patenting the current innovation at time $\bar{\tau} = \tau + \Delta$, firm i makes sure that no competitor will patent its innovation before time $T + \bar{\tau}$. I.e., the hazard rate of the event of being preempted in the next innovation race is zero until time $T + \Delta$ after the breakthrough time τ, and $N\lambda$ thereafter. Hence, the same calculations that lead to the determination of the equilibrium stopping time T imply that $\bar{T} = T + \Delta$: firm i patents its secretly researched innovation at time T after patenting the current one.

We now simplify notation and let $\bar{V}(t, \Delta) = \bar{V}\left(t, \bar{\tau} = \tau + \Delta, \bar{T} = T + \Delta\right)$ and $V(t, \Delta) = V\left(t, \bar{\tau} = \tau + \Delta, \bar{T} = T + \Delta\right)$. We consider the flow value $r\bar{V}(t, \Delta)$ and compare it with the equilibrium flow value $rV(t)$. First, note that, for any Δ,

$$r\bar{V}(t, \Delta) \leq -c + \lambda e^{-r\Delta}[\alpha x(t) - c_0] + \lambda\left(\bar{V}(\Delta) - \bar{V}(t, \Delta)\right) + N\lambda\left(\chi Q(T) - \bar{V}(t, \Delta)\right) \tag{20}$$

where the inequality follows because firm i can be preempted of the current innovation in the time period Δ, and where:

$$\bar{V}(\Delta) = -\frac{c}{r}\left[1 - e^{-r(T+\Delta)}\right] + e^{-r(T+\Delta)}\left(\alpha x(T+\Delta) - c_0 + \rho Q(T)\right),$$

is the value of achieving the next breakthrough and patenting the associated innovation without any risk of preemption.

We notice that, for any Δ, the value $\bar{V}(\Delta)$ is smaller than $\bar{V} = e^{-r\bar{T}}\left[\alpha x\left(\bar{T}\right) - c_0 + \rho Q(T)\right] - (1 - e^{-r\bar{T}})c/r$, the value associated with the stopping time $\bar{T}$, which we determined right before stating condition (15). Because we search a sufficient condition, we change $\bar{V}(\Delta)$ with $\bar{V}$ in inequality (20), and obtain:

$$\begin{aligned} rV(t, \Delta) \quad \leq \quad & \rho\left\{-c + \lambda e^{-r\Delta}[\alpha x(t) - c_0] + \lambda\left[\bar{V} - \bar{V}(t, \Delta)\right] + N\lambda\left[\chi Q(T) - \bar{V}(t, \Delta)\right]\right\} \\ & + r(1-\rho)\left[\alpha x(t) - c_0\right]. \end{aligned}$$

Because the equilibrium flow value $rV(t)$ satisfies the equation:

$$rV(t) = -c + V'(t) - N\lambda\left[V(t) - \chi Q(T)\right],$$

by subtracting $rV(t,\Delta)$ from $rV(t)$, we obtain:

$$\begin{aligned}
&r\left[V(t) - V(t,\Delta)\right] \geq -c + V'(t) - N\lambda\left[V(t) - \chi Q(T)\right] \\
&-\rho\left\{-c + \lambda e^{-r\Delta}[\alpha x(t) - c_0] + \lambda\left[\bar{V} - \bar{V}(t,\Delta)\right] + N\lambda\left[\chi Q(T) - \bar{V}(t,\Delta)\right]\right\} - r(1-\rho)\left[\alpha x(t) - c_0\right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&r\left[V(t) - V(t,\Delta)\right] \geq -c(1-\rho) + V'(t) + N\lambda\left[V(t,\Delta) - V(t)\right] - N\lambda(1-\rho)\left[\alpha x(t) - c_0\right] \\
&+N\lambda\chi(1-\rho)Q(T) - \lambda e^{-r\Delta}\rho[\alpha x(t) - c_0] - \lambda\rho\left[\bar{V} - \bar{V}(t,\Delta)\right] - r(1-\rho)\left[\alpha x(t) - c_0\right],
\end{aligned}$$

using the equality $V(t,\Delta) = \rho\bar{V}(t,\Delta) + (1-\rho)\left[\alpha x(t) - c_0\right]$.

Rearranging, this implies:

$$\begin{aligned}
&(r + N\lambda)\left[V(t) - V(t,\Delta)\right] \geq (1-\rho)V'(t) - c(1-\rho) - [N\lambda + r](1-\rho)\left[\alpha x(t) - c_0\right] \\
&+N\lambda\chi(1-\rho)Q(T) + \rho V'(t) - \lambda e^{-r\Delta}\rho[\alpha x(t) - c_0] - \lambda\rho\left[\bar{V} - \bar{V}(t,\Delta)\right]
\end{aligned}$$

Now, note that, in equilibrium $V'(T) = \alpha\dot{x}(T)$ and $(r+N\lambda)\left[\alpha x(T) - c_0\right] + \left[r\rho + N\lambda(\rho - \chi)\right]Q(T) = -c + \alpha\dot{x}(T)$, so that $V'(t) \geq (r + N\lambda)\left[\alpha x(t) - c_0\right] + \left[r\rho + N\lambda(\rho - \chi)\right]Q(T) + c$ for all $t \leq T$, because T is the optimal stopping time. Hence, we obtain:

$$\begin{aligned}
&(r + N\lambda)\left[V(t) - V(t,\Delta)\right] \geq (1-\rho)\left[r\rho + N\lambda(\rho - \chi)\right]Q(T) \\
&+\rho V'(t) + N\lambda\chi(1-\rho)Q(T) - \lambda e^{-r\Delta}\rho[\alpha x(t) - c_0] - \lambda\rho\left[\bar{V} - \bar{V}(t,\Delta)\right]
\end{aligned}$$

Further, we note that if $\bar{V}(t,\Delta)$ were smaller than $Q(T)$ for all Δ, then the possibility of secretly switching would be dominated by the equilibrium strategy. Hence, we proceed under the presumption that $\bar{V}(t,\Delta) \geq Q(T)$ for the relevant value of Δ and change $\bar{V}(t,\Delta)$ with $Q(T)$ in the above inequality. Then, collecting the term $Q(T)$, we obtain:

$$\begin{aligned}
&(r + N\lambda)\left[V(t) - V(t,\Delta)\right] \geq \\
&\rho\left[(1-\rho)(r + (N+1)\lambda) + \rho\lambda\right]Q(T) + \rho V'(t) - \lambda e^{-r\Delta}\rho[\alpha x(t) - c_0] - \lambda\rho\bar{V}.
\end{aligned}$$

Because $V'(t) \geq \alpha\dot{x}(t)$ for all $t \leq T$, and $\dot{x}(t)$ decreases in t while $x(t)$ increases in t, the worst case scenario for the above inequality is for $t = T$. So, it is sufficient to assume that:

$$\left\{(1-\rho)\left[r + (N+1)\lambda\right] + \rho\lambda\right\}Q(T) + \alpha\dot{x}(T) - \lambda e^{-r\Delta}[\alpha x(T) - c_0] - \lambda\bar{V} \geq 0,$$

or, simplifying and using the equilibrium equation (9), that:

$$\begin{aligned}
&\left\{(1-\rho)\left[r + (N+1)\lambda\right] + \rho\lambda\right\}Q(T) + \left[r\rho + N\lambda(\rho - \chi)\right]Q(T) + (r + N\lambda)\left[\alpha x(T) - c_0\right] \\
&+c - \lambda e^{-r\Delta}[\alpha x(T) - c_0] - \lambda\bar{V} \geq 0.
\end{aligned}$$

Because $e^{-r\Delta} < 1$, using the definition of $\bar{V}$, and collecting c/r, the above condition can be simplified as:

$$\begin{aligned}
&\lambda e^{-r\bar{T}}\left[\alpha x\left(\bar{T}\right) - c_0 + \rho Q(T)\right] \\
\leq\ &\left[r + N\lambda(1-\chi) + \lambda\right]Q(T) + \left[(N-1)\lambda + r\right]\left[\alpha x(T) - c_0\right] + \left[r + \lambda(1 - e^{-r\bar{T}})\right]c/r.
\end{aligned}$$

c	c_0	$\bar{\alpha}$	$\bar{N}+1$	$\bar{U}$	α^*	N^*+1	U^*
0.10	0.66	0.26	3	14.6	0.05	5	17.7
0.40	0.66	1	1	8.4	0.06	2	8.7
0.20	0.33	0.33	2	11.8	0.03	3	13.9
0.20	1.32	0.38	2	11.5	0.10	3	12.8

Table 4: Sensitivity: costs

γ	a_0	$\bar{\alpha}$	$\bar{N}+1$	$\bar{U}$	α^*	N^*+1	U^*
0	0.16	1	1	4.0	0.11	2	4.0
0	0.64	0.24	3	28.9	0.03	5	36.3
1	0.32	1	1	0.6	1	1	0.6

Table 5: Sensitivity: growth

Because $e^{-r\bar{T}}\rho < 1$, this condition implies

$$\begin{aligned} & \lambda e^{-r\bar{T}}\left[\alpha x\left(\bar{T}\right) - c_0\right] \\ \leq \;\; & \left[r + N\lambda(1-\chi) + \lambda\left(1-\rho\right)\right] Q\left(T\right) + \left[(N-1)\lambda + r\right]\left[\alpha x\left(T\right) - c_0\right] + \left[r + \lambda(1-e^{-r\bar{T}})\right] c/r, \end{aligned}$$

which yields condition (15), using the definition $\xi = \frac{N}{N+1}\chi + \frac{1}{N+1}\rho$, and the fact that $r + (N+1)\lambda(1-\xi) > \left[r + (N+1)\lambda\right](1-\xi)$. ■

The second part of this online appendix reports the final part of our numerical investigation. We repeat the analysis by varying the parameters of the model around the values recovered from the data. We demonstrate the robustness of the conclusions of our policy experiments on the calibrated model, and derive basic comparative statics on our variable of interest.

We begin by considering the effect of changes in the cost parameters c and c_0, as reported in Table 4. As before, $\bar{\alpha}$, $\bar{N}$ and $\bar{s}$ identify the optimal individually rational values whereas α^* and N^* identify the optimal unconstrained values. Again, the optimal values of α are extremely low, when not imposing the participation constraint. When this constraint is imposed welfare is much improved, again, by having a smaller number of firms participating in the game.[37]

Similar considerations apply when we consider changes in the growth parameters γ and a_0, as reported in Table 5, and changes in r and λ, as reported in Table 6.

[37] For example, consider the first line of Table 4. Here, we increase the flow cost c from 0.20 to 0.40, holding the other parameters fixed. We see that the optimal social welfare changes from 13.5 to 17.7, with associated optimal patent strength equal to 0.05 instead of 0.06, and with 5 competing firms, instead of 3. Further, under the participation constraint, the socially optimal welfare changes from 11.8 to 14.6, the optimal patent strength from 0.34 to 0.26 and the associated number of competing firms from 2 to 3.

r	$1/\lambda$	$\bar{\alpha}$	$\bar{N}+1$	$\bar{U}$	α^*	N^*+1	U^*
0.03	6	1	1	180.8	0.01	5	233.8
0.06	6	0.20	2	35.1	0.03	4	42.4
0.09	3	1	1	15.3	0.06	3	17.5
0.09	9	0.39	2	9.5	0.06	3	10.9

Table 6: Sensitivity: discounting and arrivals

Calculations not submitted for publication

Derivation of Condition (3). The value $U_0\left(\hat{t}\right)$ for remaining in the race at calendar time $\hat{t}$ despite not having achieved a breakthrough weakly decreases in $\hat{t}$. Participation in the race is ensured if $U_0\left(\hat{t}\right) \geq 0$, for all $\hat{t}$. For any $\hat{t} \geq T$, the value $U_0\left(\hat{t}\right)$ is constant; omitting its dependence on $\hat{t}$, it satisfies the following dynamic programming equation:

$$rU_0 = -c + N\lambda(\underline{U}\,(T;\alpha) - U_0) + \lambda(V\,(0) - U_0).$$

Solving for U_0, we obtain:

$$U_0 = \frac{\lambda(V(0) + N\underline{U}\,(T;\alpha)) - c}{r + (N+1)\lambda}.$$

The equilibrium value at any breakthrough arrival time $\tau \geq T$ is:

$$\begin{aligned} V(0) &= \left\{ e^{-rT}\overline{U}\,(t;\alpha) - \int_0^T ce^{-rs}ds \right\} \Pr(\bar{\tau} \geq \tau | \bar{\tau} + T \geq \tau) \\ &+ \int_{\tau-T}^{\tau} \left[e^{-r(T+\bar{\tau}-\tau)}\underline{U}\,(T;\alpha) - \int_0^{T+\bar{\tau}-\tau} ce^{-rs}ds \right] d\Pr(\bar{\tau}|\bar{\tau}+T \geq \tau), \end{aligned}$$

where $\bar{\tau}$ denotes the first breakthrough arrival time of any of the N opponents. Because $\Pr\left(\bar{\tau} \leq \tau\right) = 1 - e^{-N\lambda\tau}$, the above expression simplifies as:

$$V(0) = \left\{ e^{-rT}\overline{U}\,(t;\alpha) - c\frac{1-e^{-rT}}{r} \right\} e^{-N\lambda T} + \int_0^T \left[e^{-r\sigma}\underline{U}\,(T;\alpha) - c\frac{1-e^{-r\sigma}}{r} \right] N\lambda e^{-N\lambda\sigma}d\sigma,$$

with the change of variable $\sigma = T + \bar{\tau} - \tau$. Solving the integral, we obtain

$$V(0) = \overline{U}\,(T;\alpha)\,e^{-(r+N\lambda)T} + \frac{N\lambda\underline{U}\,(T;\alpha) - c}{r + N\lambda}[1 - e^{-(r+N\lambda)T}].$$

So, participation in the race is ensured when

$$\begin{aligned} & \lambda(V(0) + N\underline{U}\,(T;\alpha)) - c \\ = \; & \lambda\overline{U}\,(T;\alpha)\,e^{-(r+N\lambda)T} + \lambda\frac{\lambda N\underline{U}\,(T;\alpha) - c}{r+N\lambda}[1 - e^{-(r+N\lambda)T}] + \lambda N\underline{U}\,(T;\alpha) - c \\ = \; & \lambda\overline{U}\,(T;\alpha)\,e^{-(r+N\lambda)T} + [\lambda N\underline{U}\,(T;\alpha) - c]\left[\frac{\lambda}{r+N\lambda}[1 - e^{-(r+N\lambda)T}] + 1\right] \geq 0. \end{aligned}$$

Evidently, when $\lambda N\underline{U}\,(T;\alpha) - c \geq 0$ this inequality holds trivially. When $\lambda N\underline{U}\,(T;\alpha) < c$, we rewrite the inequality as:

$$c/\lambda \leq \frac{\overline{U}\,(T;\alpha)\,e^{-(r+N\lambda)T}}{\frac{\lambda}{r+N\lambda}[1 - e^{-(r+N\lambda)T}] + 1} + N\underline{U}\,(T;\alpha)\,,$$

which is implied by condition (3) because $[\frac{\lambda}{r+N\lambda}[1-e^{-(r+N\lambda)T}]+1]^{-1} < \left[\frac{\lambda}{r+N\lambda}+1\right]^{-1} = \frac{r+N\lambda}{r+(N+1)\lambda}$.
■

Equilibrium Analysis under the First-to-Invent rule. When the timing of invention is fully verifiable, our model can be modified to account for the 'first-to-invent' rule, as follows. Assume that, when a firm i applies for a patent at time t_i after its breakthrough, the first-mover payoff will be given to the firm j, possibly different from i, that is farther ahead in the race, i.e. to the firm with the smallest breakthrough arrival time τ_j.

We now show that there does not exist a symmetric, pure-strategy, calendar-time independent equilibrium. Consider, any candidate equilibrium stopping time T'. At any time $T'' < T'$ the value of patenting is strictly lower in this 'first-to-invent' rule game, than in our model, because an opponent may be farther ahead in the race. The value for waiting, instead is the same in the two games. At any time $T'' \geq T'$, the value for waiting is strictly larger in the 'first-to-invent' rule game, because if an opponent patents, the player will surely be awarded the patent. Instead, the value for patenting is the same in the two games.

In our model, the difference between the value for waiting and the value for patenting is strictly decreasing and continuous in T'' for all candidate equilibria T'. This difference equals zero for $T'' = T'$ only if they both equal T. This fact, together with the above arguments, conclude that there no symmetric pure-strategy calendar-time independent equilibrium in the 'first-to-invent' rule game, due to the discontinuity of the difference between the value for waiting and the value for patenting at $T'' = T'$, when they both equal T. ■

Fear of preemption and investment in research. We here briefly expand our basic game to explicitly allow investment decisions at the beginning of the race, and show how they relate to the fear of preemption we identified.

There is an infinite number of firms. Each firm pays a cost k_0 to enter the game. Those who enter choose their own arrival rate at cost $k(\lambda)$, where $k(\cdot)$ is assumed to be sufficiently convex to ensure the existence of a symmetric pure-strategy equilibrium. We focus these notes on equilibria that are symmetric in the choice of λ across players.

We first forward calculate the expected value of the game $V(0)$, as a function of the equilibrium choices T, N and λ, and of the exogenous parameters of the game. By definition, $V(0)$ takes the form:

$$V(0) = E\left[\left(-\int_0^{\hat{\tau}+T} ce^{-rs}ds + \frac{1}{N+1}\overline{U}(T;\alpha)e^{-r(\hat{\tau}+T)} + \frac{N}{N+1}\underline{U}(T;\alpha)e^{-r(\hat{\tau}+T)}\right)\right].$$

The first term in parenthesis accounts for paying the cost c until time $\hat{\tau}+T$, where $\hat{\tau}$ is the random time of the first breakthrough arrival among the $N+1$ competitors at any given stage. Further, because the game and equilibrium are symmetric, each firm wins the stage with probability $1/(N+1)$. If so, it gets the payoff $\overline{U}(T;\alpha)$ at time $\hat{\tau}+T$, and this explains the second term in parenthesis; otherwise, with probability $N/(N+1)$, the firm gets the payoff $\underline{U}(T;\alpha)$ at time $\hat{\tau}+T$, and this is represented by the third term.

The above expression can be simplified as follows:

$$V(0) = -c\frac{1-\beta(T)}{r} + \beta(T)\,\pi(T),$$

letting

$$\beta(T) \equiv E\left[e^{-r(\hat{\tau}+T)}\right] = e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda + r},$$

be the expected discount factor for the random time $\hat{\tau} + T$, and noting that the hazard rate of $\hat{\tau}$ is $(N+1)\lambda$, and letting

$$\pi(T) = \frac{1}{N+1}\overline{U}(T;\alpha) + \frac{N}{N+1}\underline{U}(T;\alpha),$$

be the average stopping payoff.

Expressing $V(0)$ as a function of T, we determine whether it increases or decreases in T:

$$\begin{aligned}
\frac{d}{dT}V(0;T) &= \left[\pi(T) + \frac{c}{r}\right]\beta'(T) + \beta(T)\,\pi'(T) \\
&= -r\beta(T)\left[\pi(T) + \frac{c}{r}\right]\beta'(T) + \beta(T)\,\pi'(T) \\
&= \beta(T)\left[\pi'(T) - r\pi(T) - c\right].
\end{aligned}$$

Hence, whenever the rate of increment $\pi'(T)$ of the average stopping payoff is larger than the flow average stopping payoff $\pi(T)$ and cost c, the expected value of the game $V(0;T)$ increases in the stopping time T, as it is intuitively the case in most applications.

Now, I argue that stronger fear of preemption will induce lower participation in the game, so that fear of preemption reduces investment in R&D at the extensive margin. To make this point clear, I introduce the stopping function $T(N,\lambda)$ which calculates the stopping time T as N and λ change. Then, I relate the participation choice N to a uniform decrease of $T(N,\lambda)$, which may be induced by a change in α or by any other exogenous change in the game. As a result of a shorter T, whenever $\pi'(T) > r\pi(T) + c$, the equilibrium game expected value $V(0)$ decreases, for any given N. To establish that participation in the game will be lower, we determine how $V(0)$ changes in N:

$$\begin{aligned}
\frac{d}{dN}V(0) &= \frac{\partial}{\partial N}V(0) + \frac{dV(0)}{dT}\frac{dT}{dN} \\
&= -\beta(T)\frac{\overline{U}(T;\alpha) - \underline{U}(T;\alpha)}{(N+1)^2} - \frac{dV(0)}{dT}\frac{\partial\phi(T,\alpha)}{\partial N} \\
&= -\beta(T)\frac{\overline{U}(T;\alpha) - \underline{U}(T;\alpha)}{(N+1)^2} - \frac{dV(0)}{dT}\lambda\left[\overline{U}(T;\alpha) - \underline{U}(T;\alpha)\right],
\end{aligned}$$

where the $dT/dN = -\partial\phi(T,\alpha)/\partial N$ by the implicit function theorem. We find that, whenever $dV(0)/dT > 0$, equilibrium game expected value $V(0)$ decreases in N. Hence, a change of T, uniform in N and λ, induced by an exogenous change in the game weakly reduces the number of firms N participating in the race.

Similarly, to make the point that fear of preemption reduces investment in R&D at the intensive margin, I need to calculate how the marginal benefit $dV(0)/d\lambda$ for investment in λ changes as equilibrium stopping time T decreases uniformly in N and λ. The optimal investment decision λ is pinned down by $dV(0;T)/d\lambda = k'(\lambda)$. Hence, by the implicit function theorem,

$$\frac{d\lambda}{dT} = -\frac{d^2V(0;T)/dT^2}{d^2V(0;T)/dTd\lambda}.$$

Using above calculations, we see that

$$\begin{aligned}
\frac{d^2}{dT^2}V(0;T) &= \beta(T)\left[\pi''(T) - r\pi'(T)\right] - r\beta(T)\left[\pi'(T) - r\pi(T) - c\right] \\
&= \beta(T)\left[\pi''(T) - 2r\pi'(T) + r^2\pi(T) + c\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2}{dTd\lambda}V(0;T) &= \left[\pi'(T) - r\pi(T) - c\right]\frac{d}{d\lambda}\beta(T) \\
&= \left[\pi'(T) - r\pi(T) - c\right]e^{-rT}\frac{(N+1)r}{\left[(N+1)\lambda + r\right]^2} > 0.
\end{aligned}$$

Hence, the equilibrium investment decreases because of stronger fear of preemption whenever $\pi''(T) - 2r\pi'(T) + r^2\pi(T) + c > 0$. A sufficient condition is that $\pi''(T) - r\pi'(T) > r[\pi'(T) - r\pi(T)]$: the time increment of the payoff net time gain $\pi'(T) - r\pi(T)$ is larger than its flow value. ■

Completion of the Proof of Proposition 4. We perform equilibrium comparative statics in the multi-stage patent race of section 4, starting with the relationship of T and Q with N. We note that

$$\frac{\partial}{\partial N}\phi(T) = \lambda\left[\alpha x(T) - c_0\right] + \left[r\rho + N\lambda(\rho - \chi)\right]\frac{\partial}{\partial N}Q(T) + \lambda(\rho - \chi)Q(T)$$

$$\begin{aligned}
\frac{\partial}{\partial N}Q(T) &= \left\{\beta(T)\frac{\alpha x(T) - c_0}{N+1}\right\}\frac{1}{1 - \xi\beta(T)}\frac{\partial}{\partial N}\beta(T) + \left\{-\beta(T)\frac{\alpha x(T) - c_0}{(N+1)^2}\right\}\frac{1}{1 - \xi\beta(T)} \\
&= \beta(T)\frac{\alpha x(T) - c_0}{N+1}\frac{1}{1 - \xi\beta(T)}\frac{1}{(N+1)}\frac{r\beta(T)}{\left[(N+1)\lambda + r\right]} \\
&\quad -\beta(T)\frac{\alpha x(T) - c_0}{(N+1)}\frac{1}{1 - \xi\beta(T)}\frac{1}{(N+1)} \\
&= -\beta(T)\frac{\alpha x(T) - c_0}{(N+1)}\frac{1}{1 - \xi\beta(T)}\frac{1}{(N+1)}\left[1 - \frac{r\beta(T)}{\left[(N+1)\lambda + r\right]}\right] < 0.
\end{aligned}$$

Hence, solving out χ, so that $r\rho + N\lambda(\rho - \chi) = r\rho + (N+1)\lambda(\rho - \xi)$, and eliminating the term

$\lambda\left(\rho-\chi\right)Q\left(T\right)$,

$$\begin{aligned}
\frac{d}{dN}\phi(T) &> \lambda\left[\alpha x\left(T\right)-c_0\right]-\left[r\rho+(N+1)\lambda(\rho-\xi)\right]\beta\left(T\right)\frac{\alpha x\left(T\right)-c_0}{(N+1)}\frac{1}{1-\xi\beta\left(T\right)}\frac{1}{(N+1)}\cdot \\
&\quad \left[1-\beta\left(T\right)\frac{r}{\left[(N+1)\lambda+r\right]}\right] \\
&= \lambda\left[\alpha x\left(T\right)-c_0\right]-e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda+r}\frac{\alpha x\left(T\right)-c_0}{(N+1)}\frac{1}{1-\xi e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda+r}}\frac{1}{(N+1)}\cdot \\
&\quad \left[1-e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda+r}\frac{r}{\left[(N+1)\lambda+r\right]}\right]\left[r\rho+(N+1)\lambda(\rho-\xi)\right] \\
&\propto 1-\frac{e^{-rT}\left(1-e^{-rT}\frac{(N+1)\lambda r}{((N+1)\lambda+r)^2}\right)}{1-\xi e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda+r}}\frac{r\rho+(N+1)\lambda(\rho-\xi)}{(N+1)\left((N+1)\lambda+r\right)}.
\end{aligned}$$

Because this expression increases in T and decreases in ρ and ξ, we take $T=0$, $\rho=1$ and $\xi=1$ and consider

$$\begin{aligned}
& 1-\frac{1-\frac{(N+1)\lambda r}{((N+1)\lambda+r)^2}}{1-\frac{(N+1)\lambda}{(N+1)\lambda+r}}\frac{r}{(N+1)\left((N+1)\lambda+r\right)} \\
= & \frac{2N^2\lambda^2+N^3\lambda^2+r\lambda+Nr^2+N\lambda^2+3Nr\lambda+2N^2r\lambda}{(N+1)\left(r+\lambda+N\lambda\right)^2}>0.
\end{aligned}$$

Thus, we obtain that $dT/dN<0$, because the uniqueness proof has shown that $\phi'\left(T\right)>0$, and hence the implicit function theorem implies that dT/dN has the same sign as $-\partial\phi(T)/\partial N$.

Further, because $dQ/dN=\partial Q\left(T\right)/\partial N+Q'\left(T\right)dT/dN$, we obtain that $dQ/dN<0$.

Turning to considering c, we have

$$\frac{\partial}{\partial c}\phi(T)=\left[r\rho+N\lambda\left(\rho-\chi\right)\right]\frac{\partial}{\partial c}Q\left(T\right)+1 \text{ and } \frac{\partial}{\partial c}Q\left(T\right)=\left\{-\frac{1}{r}\left[1-\beta\left(T\right)\right]\right\}\frac{1}{1-\xi\beta\left(T\right)}<0,$$

again substituting $r\rho+N\lambda\left(\rho-\chi\right)$ with $r\rho+(N+1)\lambda(\rho-\xi)$, we obtain

$$\frac{\partial}{\partial c}\phi(T)=-\left(r\rho+(N+1)\lambda(\rho-\xi)\right)\frac{1-e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda+r}}{r\left(1-\xi e^{-rT}\frac{(N+1)\lambda}{(N+1)\lambda+r}\right)}+1.$$

Because this expression increases in T, we consider

$$\begin{aligned}
\frac{\partial}{\partial c}\phi(T) &= -\left(r\rho+(N+1)\lambda(\rho-\xi)\right)\frac{1-\frac{(N+1)\lambda}{(N+1)\lambda+r}}{r\left(1-\xi\frac{(N+1)\lambda}{(N+1)\lambda+r}\right)}+1 \\
&= (1-\rho)\frac{r+\lambda+N\lambda}{r+(N+1)\lambda(1-\xi)}>0.
\end{aligned}$$

This concludes that $dT/dc<0$ and $dQ/dc<0$.

The comparative statics with respect to c_0 is obvious: $\partial\phi(T)/\partial c_0 < 0$ and $\partial Q(T)/\partial c_0 < 0$ imply that $dT/dc_0 > 0$, but that dU/dc_0 is ambiguous.

Turning to λ, we see that

$$\frac{\partial}{\partial\lambda}\phi(T) = N\left[\alpha x\left(T\right) - c_0\right] + N\left(\rho-\chi\right)Q\left(T\right) + \left[r\rho + N\lambda\left(\rho-\chi\right)\right]\frac{\partial}{\partial\lambda}Q > 0,$$

$$\frac{\partial}{\partial\lambda}Q = \left\{\beta\left(T\right)\frac{\alpha x\left(T\right)-c_0}{N+1} - \frac{c}{r}\left[1-\beta\left(T\right)\right]\right\}\frac{1}{1-\xi\beta\left(T\right)}\frac{\partial}{\partial\lambda}\beta\left(T\right) > 0,$$

because $\frac{\partial}{\partial\lambda}\beta\left(T\right) = re^{-Tr}\frac{N+1}{[(N+1)\lambda+r]^2} > 0$. Hence, $dT/d\lambda < 0$, but $dU/d\lambda$ is ambiguous.

Fixing ξ, the comparative statics with respect to $\rho-\chi$ is simple. Because $\xi = -\frac{N}{N+1}(\rho-\chi) + \frac{N}{N+1}\rho$, and ξ is held constant, it follows that $d\rho/d(\rho-\chi) = N/\left(N-1\right)$, so that

$$\frac{d}{d\left(\rho-\chi\right)}\phi\left(T\right) = \left(N\lambda + \frac{N}{N-1}r\right)Q\left(T\right) > 0.$$

Hence, T decreases in $\rho-\chi$ when holding ξ constant.

We conclude with the comparative statics with respect to ξ, keeping $\rho-\chi$ constant. Because $\xi = \chi + (\rho-\chi)/(N+1)$, we first note that $\partial\chi/\partial\xi = 1$ and $\partial\rho/\partial\xi = \left(N+1\right)$. So,

$$\frac{\partial}{\partial\xi}\phi(T) = r\left(N+1\right)Q\left(T\right) + \lambda N^2 Q\left(T\right) + \left[r\rho + N\lambda\left(\rho-\chi\right)\right]\frac{\partial}{\partial\xi}Q\left(T\right) > 0,$$

$$\begin{aligned}\frac{\partial}{\partial\xi}Q\left(T\right) &= \left\{\beta\left(T\right)\frac{\alpha x\left(T\right)-c_0}{N+1} - \frac{c}{r}\left[1-\beta\left(T\right)\right]\right\}\frac{\partial}{\partial\xi}\left(\frac{1}{1-\xi\beta\left(T\right)}\right)\\ &= \left\{\beta\left(T\right)\frac{\alpha x\left(T\right)-c_0}{N+1} - \frac{c}{r}\left[1-\beta\left(T\right)\right]\right\}\frac{\beta\left(T\right)}{\left[1-\xi\beta\left(T\right)\right]^2} > 0,\end{aligned}$$

and hence $dT/d\xi < 0$, whereas $dQ/d\xi$ is indeterminate. ■

Completion of the Proof of Proposition 7. Set $\alpha = 1$. Because $\phi\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ are strictly increasing, it follows that $T > T^*$ if $\phi\left(T^*\right) < 0$ and viceversa. So, because $\Phi\left(T^*\right) = 0$, we shall write:

$$\begin{aligned}\phi(T^*) &= \left(r+N\lambda\right)\left[x\left(T^*\right)-c_0\right] + \left[N\lambda\left(\rho-\chi\right)+r\rho\right]\frac{\Pi\left(T^*\right)}{1-\xi\beta\left(T^*\right)} + c - \dot{x}\left(T^*\right) - \Phi\left(T^*\right)\\ &= \left[N\lambda - r\frac{\beta\left(T^*\right)}{1-\beta\left(T^*\right)}\right]\left[x\left(T^*\right)-c_0\right] + \left[N\lambda\left(\rho-\chi\right)+r\rho\right]\frac{\Pi\left(T^*\right)}{1-\xi\beta\left(T^*\right)} + c.\end{aligned}$$

Now, we assume that $\rho = \chi = \xi$, so that

$$\phi(T^*) = \left[N\lambda - r\frac{\beta\left(T^*\right)}{1-\beta\left(T^*\right)}\right]\left[x\left(T^*\right)-c_0\right] + r\xi\frac{\Pi\left(T^*\right)}{1-\xi\beta\left(T^*\right)} + c.$$

First, we will consider extreme cases, starting with $N = 0$ and $\xi < 1$. In this case,

$$\begin{aligned}
\phi(T^*) &= -r\frac{\beta(T^*)}{1-\beta(T^*)}\left[x(T^*)-c_0\right]+r\xi\frac{\Pi(T^*)}{1-\xi\beta(T^*)}+c \\
&= -\frac{r}{1-\beta(T^*)}\left[\beta(T^*)\left[x(T^*)-c_0\right]-c\frac{1-\beta(T^*)}{r}\right]+r\xi\frac{\Pi(T^*)}{1-\xi\beta(T^*)} \\
&= \left[-\frac{r}{1-\beta(T^*)}+\frac{r\xi}{1-\xi\beta(T^*)}\right]\Pi(T^*) \\
&= -r\frac{1-\xi}{\left[1-\xi\beta(T^*)\right]\left[1-\beta(T^*)\right]}\Pi(T^*)<0
\end{aligned}$$

because $\xi < 1$ and because $\Pi(T^*) > 0$. Hence, when $N = 0$, we have that $T > T^*$ (unless $\xi = 1$, in which case, $T = T^*$).

Similarly, when $\rho = \chi = \xi = 0$, we have that $T > T^*$, because

$$\phi(T^*) = \left[N\lambda - r\frac{\beta(T^*)}{1-\beta(T^*)}\right]\left[\frac{x(T^*)}{r}-c_0\right]+c>0.$$

When $\xi = 1$, instead,

$$\begin{aligned}
\phi(T^*) &= \left[N\lambda - r\frac{\beta(T^*)}{1-\beta(T^*)}\right]\left[\frac{x(T^*)}{r}-c_0\right]+r\left\{\beta(T^*)\frac{x(T^*)/r-c_0}{N+1}-\frac{c}{r}\left[1-\beta(T^*)\right]\right\}\frac{1}{1-\beta(T^*)}+c \\
&= \left[N\lambda - r\frac{\beta(T^*)}{1-\beta(T^*)}+\frac{r}{N+1}\frac{\beta(T^*)}{1-\beta(T^*)}\right]\left[\frac{x(T^*)}{r}-c_0\right] \\
&\propto N\lambda\left[1-\beta(T^*)\right]-r\left(1-\frac{1}{N+1}\right)\beta(T^*) \\
&= N\lambda\left[1-e^{-rT^*}\frac{(N+1)\lambda}{(N+1)\lambda+r}\right]-r\left(1-\frac{1}{N+1}\right)e^{-rT^*}\frac{(N+1)\lambda}{(N+1)\lambda+r} \\
&= N\lambda\left(1-e^{-rT^*}\right)>0,
\end{aligned}$$

and so $T < T^*$, unless $N = 0$. ■

Derivation of Condition (13). We want to refine condition (3) and derive a sufficient condition so that it holds. Because, $\overline{U}(T;\alpha) = \alpha x(T) - c_0 + \rho Q(T)$, $\underline{U}(T;\alpha) = \chi Q(T)$, and Q increases in $\rho - \chi$, we consider the case in which $\chi = \rho$, so that $\xi = \rho$. With simple manipulations, we express the continuation payoffs as follows:

$$\begin{aligned}
\overline{U}(T;\alpha) &= \left[\alpha x(T)-c_0\right]\left[1+\frac{1}{N+1}\frac{\beta\beta(T)}{1-\rho\beta(T)}\right]-\frac{c}{r}\frac{\rho\left[1-\beta(T)\right]}{1-\rho\beta(T)} \\
\underline{U}(T;\alpha) &= \frac{\alpha x(T)-c_0}{N+1}\frac{\rho\beta(T)}{1-\rho\beta(T)}-\frac{c}{r}\frac{\rho\left[1-\beta(T)\right]}{1-\rho\beta(T)},
\end{aligned}$$

so that condition (3) becomes:

$$\begin{aligned}
& c/\lambda+\frac{c}{r}\frac{\rho\left[1-\beta(T)\right]}{1-\rho\beta(T)}\left[\frac{r+N\lambda}{r+(N+1)\lambda}e^{-(r+N\lambda)T}+N\right] \\
\leq\; & (\alpha x(T)-c_0)\left[\left(1+\frac{1}{N+1}\frac{\rho\beta(T)}{1-\rho\beta(T)}\right)\frac{r+N\lambda}{r+(N+1)\lambda}e^{-(r+N\lambda)T}+\frac{N}{N+1}\frac{\rho\beta(T)}{1-\rho\beta(T)}\right],
\end{aligned}$$

or, rearranging:

$$\frac{c/\lambda}{\alpha x(T)-c_0} \leq \frac{\left(1+\frac{1}{N+1}\frac{\rho\beta(T)}{1-\rho\beta(T)}\right)\frac{r+N\lambda}{r+(N+1)\lambda}e^{-(r+N\lambda)T}+\frac{N}{N+1}\frac{\rho\beta(T)}{1-\rho\beta(T)}}{1+\frac{\lambda}{r}\rho\frac{1-\beta(T)}{1-\rho\beta(T)}\left[\frac{r+N\lambda}{r+(N+1)\lambda}e^{-(r+N\lambda)T}+N\right]}.$$

Focusing on the right-hand side of the above inequality, we first see that it decreases in the term $\frac{\rho\beta(T)}{1-\rho\beta(T)}$. Thus, in order to find a sufficient condition for condition (3) to hold we change $\frac{\rho\beta(T)}{1-\rho\beta(T)}$ with $\frac{\beta(T)}{1-\rho\beta(T)}$, and we obtain the sufficient condition:

$$\frac{c/\lambda}{\alpha x(T)-c_0} \leq \left(1+\frac{1}{N+1}\frac{\rho}{1-\rho\beta(T)}\right)\frac{r+N\lambda}{r+(N+1)\lambda}e^{-(r+N\lambda)T}+\frac{N}{N+1}\frac{\rho}{1-\rho\beta(T)},$$

which can be further simplified by deleting the second term in parenthesis, to obtain condition (13). ■

An Example Showing that Our Parametric Restrictions do not Contradict Each Other. We conclude these notes by showing a simple example in which the parametric conditions we impose throughout the model do not contradict each other. The parametrization choices below are made only for ease of calculation, we do not interpret them as bearing any special substantive relevance. We first specify the function $x(\cdot)$ with the exponential decay form. We note that the specification $\alpha x(t) = a_0\frac{1-e^{-\gamma t}}{\gamma}+c_0$, so that $\alpha\dot{x}(t) = a_0 e^{-\gamma t}$, immediately satisfies all the conditions we required on the function $x(\cdot)$ in section 4, as long as $a_0 > c$.

Turning to the robustness conditions we imposed ex-post, we further assume that $c = 0$, $c_0 = 1/2$, $a_0 = r = \gamma = \lambda = 1$, $N = 3$ and $\chi = \rho = \xi = 1$, to make calculation simple. Evidently, as $c = 0$, condition (13) is satisfied. Turning to the two remaining conditions, we first calculate the equilibrium stopping time T. Plugging our $x(\cdot)$ form in expression (8), and using our parametric assumptions, we obtain: $Q(T) = \frac{1-e^{-T}}{4}\frac{\frac{4}{5}e^{-T}}{1-\frac{4}{5}e^{-T}}$. Using our $x(\cdot)$ expression and our parametric assumptions in equation (9), we obtain $4\left(1-e^{-T}\right)+Q(T) = e^{-T}$, and plugging in $Q(T)$, we obtain: $4\left(1-e^{-T}\right)+\frac{1-e^{-T}}{4}\frac{\frac{4}{5}e^{-T}}{1-\frac{4}{5}e^{-T}} = e^{-T}$. Solving out, we find: $e^{-T} = \frac{20-2\sqrt{5}}{19} \approx 0.817\,26$, and $T = -\ln\left(\frac{20-2\sqrt{5}}{19}\right) \approx 0.2018$.

Here, condition (14) is satisfied when $\left(1-e^{-\hat{T}}\right)-\left(1-e^{-T}\right) \leq \frac{1-e^{-(\hat{T}-T)}}{e^{-(\hat{T}-T)}}\frac{1-e^{-T}}{4}\frac{\frac{4}{5}e^{-T}}{1-\frac{4}{5}e^{-T}}+1/2$, where $\hat{T}$ is the unique solution of $\left(1-e^{-\hat{T}}\right)-\left(1-e^{-T}\right)-1/2+\frac{1-e^{-T}}{4}\frac{\frac{4}{5}e^{-T}}{1-\frac{4}{5}e^{-T}} = e^{-\hat{T}}$. Indeed, solving for $\hat{T}$, we find that $e^{-\hat{T}} = \frac{192\sqrt{5}+75}{608\sqrt{5}+1140} \approx 0.201\,77$ and that $T = -\ln\left(\frac{192\sqrt{5}+75}{608\sqrt{5}+1140}\right) \approx 1.1411$, so that condition (14) is verified (with slack of approximately 0.14771).

Finally, here, condition (15) is satisfied when $3\left(1-e^{-T}\right) \geq e^{-\bar{T}}\left(1-e^{-\bar{T}}\right)$, where $\bar{T}$ solves $1-e^{-\bar{T}}+\frac{1-e^{-T}}{4}\frac{\frac{4}{5}e^{-T}}{1-\frac{4}{5}e^{-T}} = e^{-\bar{T}}$. And in fact, solving yields $e^{-\bar{T}} = \frac{194\sqrt{5}+245}{304\sqrt{5}+570} \approx 0.543\,14$, so that condition (15) is verified (with slack of approximately 0.81726). ■