EC9A0: Pre-sessional Advanced Mathematics Course

Lecture Notes: Real Analysis
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1 Preliminaries

Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the countably infinite set of *natural numbers*. For any natural number $K \in \mathbb{N}$, the K-dimensional real (Euclidean) space is the K-fold Cartesian product of $\mathbb{R}$. We denote this space by $\mathbb{R}^K$, so that $x \in \mathbb{R}^K$ is $(x_1, x_2, \ldots, x_K) = (x_i)_{i=1}^K$. Sometimes we abuse notation by letting K denote the set $\{1, 2, \ldots, K\}$. Then the typical member of $\mathbb{R}^K$ can be denoted by $(x_i)_{i\in K}$.

The *origin* of $\mathbb{R}^K$ is the vector 0 whose components $(0, 0, \ldots, 0)$ are all zero. Given any pair $a, b \in \mathbb{R}$ there are four different possible inequalities, namely: $a > b$, $a \geq b$, $a \leq b$ and $a < b$. If $a \neq b$, exactly one of these holds. But if $a = b$, then both $a \geq b$ and $a \leq b$.

Given any pair $x, y \in \mathbb{R}^K$ where $K \geq 2$, there are six different possible inequalities, namely: $x \gg y$, $x > y$, $x \geqq y$, $x \leqq y$, $x < y$, and $x \ll y$. The first three inequalities are defined so that:

1. $x \gg y$ iff $x_i > y_i$ for all $i \in K$;
2. $x > y$ iff $x \neq y$ and $x_i \geq y_i$ for all $i \in K$;
3. $x \geqq y$ iff $x_i \geq y_i$ for all $i \in K$.

Clearly $x \leqq y$ iff $y \geqq x$, etc. Given any pair $x, y \in \mathbb{R}^1$, of course, one has $x \gg y$ iff $x > y$. But in $\mathbb{R}^K$ when $K \geq 2$, none of the six inequalities may hold, as happens when $x = (1, 0)$ and $y = (0, 1)$ in $\mathbb{R}^2$.

Yet more notation: the *non-negative orthant* in $\mathbb{R}^K$ is the set $\mathbb{R}^K_+ := \{x \in \mathbb{R}^K \mid x \geqq 0\}$, whereas the *positive orthant* is $\mathbb{R}^K_{++} := \{x \in \mathbb{R}^K \mid x \gg 0\}$. When $K = 2$ these are *quadrants*, and when $K = 3$ these are *octants*. There is no special notation for the set $\mathbb{R}^K_+ \setminus \{0\} = \{x \in \mathbb{R}^K \mid x > 0\}$.

Define vector addition by $x + y = (x_1 + y_1\ x_2 + y_2 \ldots x_K + y_K)$ and for any scalar $\alpha \in \mathbb{R}$ scalar multiplication is defined by $\alpha x = (\alpha x_1\ \alpha x_2 \ldots \alpha x_K)$.

2 Vector Spaces

Much of economics, especially at the undergraduate level, relies only on vectors in a finite-dimensional Euclidean space $\mathbb{R}^K$. But this is a prominent example of the much more extensive class of real linear spaces, some of them infinite-dimensional, which arise naturally when economists consider difference and differential equations, optimal control problems, and dynamic programming.

Definition 1. *A* real linear (or vector) space *is a set L equipped with the two binary operations $(x, y) \mapsto x + y$ from $L \times L$ to L and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{R} \times L$ to L, as well as a unique* null vector *$\theta \in L$, which for all $x, y, z \in L$ and all $\lambda, \mu \in \mathbb{R}$ must satisfy:*

[1] Based on notes by Andrés Carvajal

1. $x + y = y + x$ *(addition commutes);*

2. $(x + y) + z = x + (y + z)$ *(addition is associative, so $x + y + z$ is well defined, as is any finite sum);*

3. $x + \theta = x$ *(additive identity);*

4. *for each $x \in L$, there is a unique* inverse $-x$ *such that $x + (-x) = \theta$;*

5. $\lambda(\mu x) = (\lambda\mu)x$ *(scalar multiplication is associative, so $\lambda\mu x$ is well defined);*

6. $0x = \theta$;

7. $1x = x$;

8. $(\lambda + \mu)x = \lambda x + \mu x$ *(first distributive law);*

9. $\lambda(x + y) = \lambda x + \lambda y$ *(second distributive law).*

EXERCISE 1 (Stokey and Lucas, exercise 3.2). *Show that the following are vector spaces:*

1. *any finite-dimensional Euclidean space* $\mathbb{R}^K$.

2. *the set X consisting of al infinite sequences $\{x_0, x_1, x_2, ...\}$ where $x_i \in \mathbb{R}$.*

3. *the set of all continuous functions on the interval* $[a, b]$

   Show that the following are not vector spaces:

4. *the unit circle in* $\mathbb{R}^2$

5. *the set of all integers, $Z = \{..., -1, 0, 1, ...\}$.*

6. *the set of all nonnegative functions on* $[a, b]$

3 Correspondences and Functions

DEFINITION 2. *Let X and Y be two nonempty sets. A* correspondence *f from a set X into a set Y, denoted $f : X \to Y$, is a rule that assigns to each $x \in X$ a set $f(x) \subset Y$*

For the moment, we are going to work with a particular type of correspondences where the set $f(x)$ is a singleton.

DEFINITION 3. *Let X and Y be two nonempty sets. A* function *f from a set X into a set Y, denoted $f : X \to Y$, is a rule that assigns to each $x \in X$ a unique $f(x) \in Y$*

Here, set X is said to be the *domain* of f, and Y its *target set* or *co-domain.* If $f : X \to Y$ and $A \subseteq X$, the *image of A under f*, denoted by $f[A]$, is the set

$$f[A] \equiv \{y \in Y | f(x) = y \text{ for some } x \in A\}.$$

In particular, the image $f[X]$ of the whole domain is called the *range* of f.

Definition 4. *Function $f : X \to Y$ is said to be:*

- onto*, or* surjective*, if $f[X] = Y$; it is said to be* one-to-one*, or* injective*, if*

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

- a one-to-one correspondence*, or* bijective*, if it is both onto and one-to-one.*

Example 1. *The function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $f(x_1, x_2) = x_1^2 + x_2^2$ is neither one-to-one nor onto. It is not one-to-one because $f(1, 0) = f(0, 1) = 1$. It is not onto because $[\mathbb{R}] = \mathbb{R}_+ \neq \mathbb{R}$.*

Example 2. *The function $f(x) : \mathbb{R} \setminus \{0\} \mapsto \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is one-to-one but not onto. It is one-to-one since $\frac{1}{x} = \frac{1}{y} \Leftrightarrow x = y$. Graphically, any horizontal line that intersects the graph, it does it at only one point. The function is not onto as $f[\mathbb{R}] = \mathbb{R} \setminus \{0\}$.*

Example 3. *The function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $f(x_1, x_2) = x_1 + x_2$ is onto but not one-to-one. It is onto because for any $y \in \mathbb{R}$, $f(0, y) = y$. It is not one-to-one because $f(1, 0) = f(0, 1) = 1$.*

Example 4. *The function $f(x) : \mathbb{R} \mapsto \mathbb{R}$ defined by $f(x) = x$ is one-to-one and onto.*

If $f : X \to Y$, and $B \subseteq Y$, the *inverse image of B under f*, denoted $f^{-1}[B]$, is the set

$$f^{-1}[B] \equiv \{x \in X | f(x) \in B\}.$$

If $f : X \to Y$ is a one-to-one correspondence, the *inverse function* $f^{-1} : Y \to X$ is implicitly defined by $f^{-1}(y) = f^{-1}[\{y\}]$. Notice that this would not have been be a *bona fide* definition, had we forgotten to say that f is a one-to-one correspondence (what could have gone wrong?).

The proof of the following theorem is left as an exercise.

Theorem 1. *The function $f : X \mapsto Y$ is onto iff $f^{-1}[B] \neq \varnothing$ for all non-empty $B \subseteq Y$.*

Proof: ($\Rightarrow$) Suppose $f : X \mapsto Y$ is onto. Let $B \subseteq Y$. We need to show that $f^{-1}[B] \equiv \{x \in X | f(x) \in B\} \neq \varnothing$. Let $\tilde{y} \in B$. Since f is onto, $\{y \in Y | f(x) = y \text{ for some } x \in A\} = Y$. Then, there exists $x \in X$ such that $f(x) = \tilde{y}$. Thus, $f^{-1}[B] \neq \varnothing$.

($\Leftarrow$) Suppose $f^{-1}[B] \neq \varnothing$ for all non-empty $B \subseteq Y$. We need to show that $f[X] \equiv \{y \in Y | f(x) = y \text{ for some } x \in X\} = Y$. Since $f[X] \subseteq Y$, it suffices to show that $Y \subseteq f[X]$. Let $\tilde{y} \in Y$. By hypothesis, $f^{-1}(\{\tilde{y}\}) \neq \varnothing$. Hence, there is $x \in X$ such that $f(x) = \tilde{y}$ which means that $\tilde{y} \in \{y \in Y | f(x) = y \text{ for some } x \in X\} \equiv f[X]$. Thus, $Y \subseteq f[X]$. *Q.E.D.*

From now on, we only concentrate on definitions and concepts in Euclidean spaces and maintain the assumption that $K \in \mathbb{N}$.

3.1 Distance Function

The vector space structure of $\mathbb{R}^K$ is not enough –on its own– to allow us to express geometric concepts. In R^1 the *distance* $d(a, b)$ between a and b is $\|a - b\|$. In R^2, Pythagoras' theorem implies that the *distance* $d(x, y)$ between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is the positive solution to the equation $d^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$. Even in R^3, the *distance* $d(x, y)$ between $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is the positive solution to the equation $d^2 = (x_1 -$

$y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2$. All of these three cases are covered by the single formula $d(x, y) = \sqrt{\left(\sum_{i \in K}(x_i - y_i)^2\right)}$.

More generally, any function that satisfies three intuitive properties is called a *metric* or *distance function*.

DEFINITION 5. *Given any set X, a metric or distance function on X is a function $d : X \times X \mapsto \mathbb{R}$ that satisfies three basic conditions:*

1. *$d(x, y) \geq 0$, with equality if and only if $x = y$ (positivity).*
2. *$d(x, y) = d(y, x)$ (symmetry).*
3. *$d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).*

EXAMPLE 5. *The Euclidean distance $d(x, y)$ between any two points $x, y \in \mathbb{R}^K$ is*

$$d(x, y) = \left(\sum_{i \in K}(x_i - y_i)^2\right)^{1/2}.$$

3.2 Norms

For vector spaces, metrics are usually defined in such a way that the distance between any two points is equal to the distance of their difference from the zero point. That is, since for any points x and y in a vector space X, the point $x - y$ is also in X, the metric on a vector space is usually defined in such a way that $d(x, y) = d(x - y, 0)$. To define such a metric we need the concept of a norm.

DEFINITION 6. *Given any vector space X, a norm on X is a function $\|\cdot\| : X \mapsto \mathbb{R}$ such that for all $x, y \in X$ and $\alpha \in \mathbb{R}$:*

1. *$\|x\| \geq 0$, with equality if and only if $x = \theta$ (i.e., x is the null vector);*
2. *$\|\alpha x\| = |\alpha| \, \|x\|$; and*
3. *$\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality)*

In order to measure how far from 0 an element x of $\mathbb{R}^K$ is, we use the *Euclidean norm* which is defined as[2]

$$\|x\| = \left(\sum_{k=1}^{K} x_k^2\right)^{1/2}.$$

It is obvious that when $K = 1$ the Euclidean norm corresponds to the absolute value.

[2] To avoid confusion, you can be explicit about the dimension for which the norm is being used, by adopting the notation $\|\cdot\|_K$ instead. Also, we will simplify the notation by not always writing the limits in the index of a summation, when it is obvious what these limits are; for instance, we may write $\left(\sum_k x_k^2\right)^{1/2}$ for the definition that follows.

4 Metric Spaces and Normed Vector Spaces

The space $X = \mathbb{R}^K$ equipped with its Euclidean distance $d : X \times X \to \mathbb{R}$ is a prominent example meeting the next definition.

Definition 7. *A metric space is a pair (X, d) where X is a set and $d : X \times X \mapsto \mathbb{R}$ is a metric (or distance function).*

Example 6. *Let $p \in \mathbb{R}_+$ be a positive real parameter. Define $d_p : \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ by*

$$d_p(x, y) = \sum_{i \in K} |x_i - y_i|^p.$$

Then $(\mathbb{R}^K, d_p)$ is a metric space if $p \geq 1$, but not if $0 < p < 1$

Exercise 2 (Stokey and Lucas, exercise 3.3). *Show that the following are metric spaces:*

a. *the set of integers with $d(x, y) = |x - y|$.*

b. *the set of integers with $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$.*

c. *the set of all continuous, strictly increasing functions on $[a, b]$, with $d(x, y) = max_{a \leq t \leq b}|x(t) - y(t)|$.*

d. *$\mathbb{R}$ with $d(x, y) = f(|x - y|)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, and strictly concave, with $f(0) = 0$.*

Definition 8. *A normed vector space is a pair $(X, \|\cdot\|)$ where X is a set and $\|\cdot\|) : X \mapsto \mathbb{R}$ is a norm.*

It is standard to view any normed vector space $(X, \|\cdot\|)$ as a metric space where the metric is taken to be $d(x, y) = \|x - y\|$ for all $x, y \in X$.

Exercise 3 (Stokey and Lucas, exercise 3.4). *Show that the following are normed vector spaces:*

a. *Let $X = \mathbb{R}^K$, with $\|x\| = \left[\sum_{k=1}^K x_k^2\right]^{\frac{1}{2}}$ (Euclidean Space)*

b. *Let $X = \mathbb{R}^K$, with $\|x\| = max_i|x_i|$.*

c. *Let $X = \mathbb{R}^K$, with $\|x\| = \sum_{k=1}^K |x_k|$.*

d. *Let X be the set of all bounded infinite sequences $\{x_n\}_{k=1}^{\infty}$ with $\|x\| = \sup_n |x_k|$. (This space is called l_∞)*

4.1 Sequences

A *sequence* in $\mathbb{R}^K$ is a function $f : \mathbb{N} \to \mathbb{R}^K$. If no confusion is likely, the space in which a sequence lies is omitted. Following usual notation in mathematics, we can express sequences as $(a_1, a_2, \ldots)$ or $(a_n)_{n=1}^\infty$, where $a_n = f(n)$, for $n \in \mathbb{N}$.

EXAMPLE 7. *Suppose that* $f(n) = (\sqrt{n}, 1/n, 3) \in \mathbb{R}^3$, *for all* $n \in \mathbb{N}$. *Then we can express the sequence as* $((1,1,3), (\sqrt{2}, 1/2, 3), (\sqrt{3}, 1/3, 3), \ldots)$ *or* $(\sqrt{n}, 1/n, 3)_{n=1}^\infty$.

It is very important to notice that a sequence has more structure than a set (i.e., it is more complicated). Remember that a set is completely defined by its elements, no matter how they are described. For example, the set $\{0, 3, 8, 15, 24\}$ is the same as the set $\{24, 15, 8, 3, 0\}$. However, the sequences $(0, 3, 8, 15, 24, \ldots)$ and $(24, 15, 8, 3, 0, \ldots)$ are clearly different: in a sequence, the order matters!

Using the structure that a sequence has, $(a_n)_{n=1}^\infty$ is *nondecreasing* if for all $n \in \mathbb{N}$, $a_{n+1} \geq a_n$ and *nonincreasing* if for all n, $a_{n+1} \leq a_n$. If all the inequalities in the first definition are strict, the sequence is *increasing.* while if all the inequalities in the second definition are strict, the sequence is *decreasing.*

DEFINITION 9. *The sequence* $(a_n)_{n=1}^\infty$ *in* $\mathbb{R}^K$ *is bounded above if there exists* $\bar{a} \in \mathbb{R}^K$ *such that* $a_n \leq \bar{a}$ *for all* n. *It is bounded below if there exists* $\bar{a} \in \mathbb{R}$ *such that* $a_n \geq \bar{a}$ *for all* n, *and it is bounded if it is bounded both above and below.*

Obviously, a sequence $(a_n)_{n=1}^\infty$ in $\mathbb{R}^K$ is nothing but an array of K sequences in $\mathbb{R}$: sequence $(a_{k,n})_{n=1}^\infty$ for each $k = 1, \ldots, K$. So, it should not come as no surprise that some relations exist between these objects.

THEOREM 2. $(a_n)_{n=1}^\infty$ *in* $\mathbb{R}^K$ *is bounded if and only if* $(a_{k,n})_{n=1}^\infty$ *in* $\mathbb{R}$ *is bounded for all* $k = 1, \ldots, K$.

DEFINITION 10. *Given a sequence* $(a_n)_{n=1}^\infty$ *in* $\mathbb{R}^K$, *a sequence* $(b_k)_{k=1}^\infty$ *in* $\mathbb{R}^K$ *is* a subsequence of $(a_n)_{n=1}^\infty$ *if there exists an increasing sequence* $(n_k)_{k=1}^\infty$ *such that* $n_k \in \mathbb{N}$ *and* $b_k = a_{n_k}$ *for all* $k \in \mathbb{N}$.

That is, a subsequence is a selection of some (possibly all) members of the original sequence that preserves the original order.

EXAMPLE 8. *Consider the sequence* $(1/\sqrt{n})_{n=1}^\infty$, *and note that* $(1/\sqrt{2n+5})_{n=1}^\infty$ *is a subsequence of the former. To see why, consider the sequence* $(n_m)_{m=1}^\infty = (2m+5)_{m=1}^\infty$.

EXERCISE 4. *Is* $(1/\sqrt{n})_{n=1}^\infty$ *a subsequence of* $(1/n)_{n=1}^\infty$? *How about the other way around?*

4.2 Limits of sequences

DEFINITION 11. *The point* $a \in \mathbb{R}^K$ *is a* limit *of the sequence* $(a_n)_{n=1}^\infty$ *if for every* $\varepsilon > 0$ *there exists some* $n^* \in \mathbb{N}$ *such that* $d(a_n, a) < \varepsilon$ *for all* $n \geq n^*$. *Sequence* $(a_n)_{n=1}^\infty$ *in* $\mathbb{R}^K$ *is said to be* convergent *if it has a limit* $a \in \mathbb{R}^K$.

When $(a_n)_{n=1}^\infty$ converges to a, the following notation is also sometimes used: $a_n \to a$, or $\lim_{n\to\infty} a_n = a$.

EXERCISE 5. *Does* $((-1)^n)_{n=1}^\infty$ *converge? Does* $(-1/n)_{n=1}^\infty$?

It is convenient to allow $+\infty$ and $-\infty$ to be limits of sequences. Thus, we extend the definition as follows: for a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$, we say that $\lim_{n\to\infty} a_n = \infty$ when for all $\Delta > 0$ there exists some $n^* \in \mathbb{N}$ such that $a_n > \Delta$ for all $n \geq n^*$; we also say that $\lim_{n\to\infty} a_n = -\infty$ when $\lim_{n\to\infty}(-a_n) = \infty$.

Exercise 6. *Does the sequence $(\frac{3n}{\sqrt{n}})_{n=1}^{\infty}$ have a limit? Does it converge?*

The following Theorem shows that limits are unique.

Theorem 3. *If $a_n \to x$ and $a_n \to y$, then $x = y$.*

Proof: TBA *Q.E.D.*

This Theorem simplifies the search for limits of vectors as it tell us that the limit of the vector is equal to the vector of limits of its components.

Theorem 4. *$(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^K$ converges to a if and only if $(a_{k,n})_{n=1}^{\infty}$ in $\mathbb{R}$ converges to a_k for all $k = 1, \ldots, K$.*

Proof: Let us prove sufficiency first. Given any $\epsilon > 0$, for each k there is some $n_k^* \in \mathbb{N}$ such that $|a_{k,n} - a_k| < \epsilon/\sqrt{K}$ whenever $n \geq n_k^*$. Letting $n^* = \max\{n_1^*, \ldots, n_K^*\} \in \mathbb{N}$ and $n \geq n^*$, by construction,

$$\|a_n - a\| = (\sum_k (a_{k,n} - a_k)^2)^{1/2} < (\sum_k \epsilon^2/K)^{1/2} = \epsilon.$$

For necessity, fix k and let $\epsilon > 0$. By assumption, there is $n^* \in \mathbb{N}$ after which $\|a_n - a\| < \epsilon$, which suffices to imply that $|a_{k,n} - a_k| < \epsilon$. *Q.E.D.*

Theorem 5. *A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^K$ converges to $a \in \mathbb{R}^K$ if and only if every subsequence of $(a_n)_{n=1}^{\infty}$ converges to a.*

Proof: $(\Leftarrow)$ is trivial. To show $(\Rightarrow)$, let $(b_k)_{k=1}^{\infty}$ be a subsequence of $(a_n)_{n=1}^{\infty}$. Then, there is $\{n_k\}_{k=1}^{\infty}$ such that $b_k = a_{n_k}$. Let $\varepsilon > 0$. Since $(a_n)_{n=1}^{\infty}$ converges to $a \in \mathbb{R}^K$, there exists $n^* \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$. Since $n_k \geq k$, then $k \geq n^*$ implies $n_k \geq n^*$ and, therefore, $d(b_k, a) = d(a_{n_k}, a) < \varepsilon$. *Q.E.D.*

A very useful property of limits (for both sequences and functions) is that they preserve weak inequalities.

Theorem 6. *Consider a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ and a number $a \in \mathbb{R}$. If $a_n \leq \alpha$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then $a \leq \alpha$. Similarly, if $a_n \geq \alpha$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a$, then $a \geq \alpha$.*

Proof: Let's show that $a_n \leq \alpha$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then $a \leq \alpha$. The other result follows by an analogous reasoning. Suppose $a > \alpha$. Let $\varepsilon = a - \alpha > 0$. Since $a_n \to a$, there exists N such that $|a_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N$. But then, $a_n > a - \frac{\varepsilon}{2} \geq \alpha$ for all $n \geq N$, a contradiction. *Q.E.D.*

Exercise 7. *Can we strengthen our results to say: "Consider a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ and a number $a \in \mathbb{R}$. If $a_n < \alpha$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a$, then $a < \alpha$."?*

The following theorem is also very useful:

THEOREM 7. *For sequences* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *such that* $a_n > 0$ *for all* $n \in \mathbb{N}$, *the following equivalence holds:*

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Proof: Let us prove the sufficiency statement, leaving necessity as an exercise. Suppose that $\lim_{n\to\infty}(1/a_n) = 0$ and fix $\Delta > 0$. Then, for some $n^* \in \mathbb{N}$ one has that $|1/a_n - 0| < 1/\Delta$ when $n \geq n^*$; since each $a_n > 0$, it follows that $a_n > \Delta$. *Q.E.D.*

EXERCISE 8. *Show that*

$$\lim_{n\to\infty} \left(\frac{15n^5 + 73n^4 - 118n^2 - 98}{30n^5 + 19n^3} \right) = \frac{1}{2}.$$

THEOREM 8. *Let* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be two sequences in* $\mathbb{R}$. *Suppose that for numbers* $a, b \in \mathbb{R}$, *we have that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then,*

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$;
2. $\lim_{n\to\infty}(\alpha a_n) = \alpha a$, *for all* $\alpha \in \mathbb{R}$;
3. $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$; *and*
4. *if* $b \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty}(a_n/b_n) = a/b$.

Proof: To show 1., let $\varepsilon > 0$ be arbitrary. We need to show that there exists $N \in \mathbb{N}$ such that $|a_n + b_n - a - b| < \varepsilon$ for all $n \geq N$. Since $a_n \to a$ and $b_n \to b$, there exists $N_a \in \mathbb{N}$ and $N_b \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N_a$ and $|b_n - b| < \frac{\varepsilon}{2}$ for all $n \geq N_b$. But then, by the triangle inequality:

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

THEOREM 9. *If sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is convergent, then it is bounded.*

Proof: Suppose $(a_n)_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| \leq 1$ for all $n \geq N$. Note that

$$|a_n| = |a_n - a + a| = |a_n - a| + |a| \leq 1 + |a|$$

where the last inequality follows by triangle inequality. Let

$$b = \max\left\{|a_1|, |a_2|, ..., |a_N|, 1 + |a|\right\}.$$

Hence $|a_n| \leq b$ for all $n \in \mathbb{N}$. *Q.E.D.*

THEOREM 10. *If a sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is monotone and bounded, then it is convergent.*

Proof: Suppose $(a_n)_{n=1}^{\infty}$ is monotone and bounded. Then it has a supremum, say c. Let $\varepsilon > 0$ be arbitrarily chosen. By definition of supremum, there exists $N \in \mathbb{N}$ such that $a_N > c - \varepsilon$. Since $(a_n)_{n=1}^{\infty}$ is monotone, it follows that $0 < c - a_n < c - a_N < \varepsilon$. Thus, $|a_n - c| < \varepsilon$ for all $n \geq N$. *Q.E.D.*

THEOREM 11 (Bolzano-Weierstrass). *If a sequence $(a_n)_{n=1}^{\infty} \in \mathbb{R}$ is bounded, then it has a convergent subsequence.*

An informal argument for the Bolzano-Weierstrass Theorem for sequences defined into $\mathbb{R}$ is as follows: if $(a_n)_{n=1}^{\infty}$ is bounded, then it lies in some bounded interval I_1. Slice that interval in halves. At least one of the halves will contain infinitely many terms of the sequence. Call that interval I_2, slice it in halves, and let I_3 be a half that contains infinitely many elements... By doing this indefinitely, we construct intervals $I_1, I_2, ...$ such that each I_n contains infinitely many terms of the sequence and $I_{n+1} \subseteq I_n$. By construction, we can find a subsequence $(x_{n_m})_{m=1}^{\infty}$ such that for all $m \in \mathbb{N}$, $a_{n_m} \in I_m$. This subsequence will have the property that their elements get arbitrarily close to one another as we move along the sequence (because, by construction, our "sequence" of intervals is in fact shrinking to zero diameter as m goes to ∞). Sequences with this property are said to be *Cauchy* and, in Euclidean spaces, are guaranteed to converge.

4.3 Cauchy Sequences

DEFINITION 12. *A sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}^K$ is a Cauchy sequence (or satisfies the Cauchy criterion) if for each $\varepsilon > 0$, there exists N_ε such that*

$$d(a_n, a_m) < \varepsilon, \text{ for all } n, m \geq N_\varepsilon.$$

EXERCISE 9. *Does the sequence $(1/\sqrt{n})_{n=1}^{\infty}$ have a limit? Is it Cauchy? How about $(\frac{3n}{n+\sqrt{n}})_{n=1}^{\infty}$?*

EXERCISE 10. *Repeat the last part of Exercise 9, using Theorem 7. Is it easier?*

THEOREM 12. *If a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}^K$ is convergent, then it is a Cauchy sequence.*

Proof: Suppose a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}^K$ converges to $a \in \mathbb{R}^K$. Let $\varepsilon > 0$ be arbitrary. Since the sequence converges, there exists N such that $d(a_n, a) < \frac{\varepsilon}{2}$ for all $n \geq N$. Let $m \geq n$. Then, $d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. We conclude there exists N_ε such that $d(a_n, a_m) < \varepsilon$ for any $n, m \geq N_\varepsilon$, as desired. *Q.E.D.*

THEOREM 13. *If a sequence $\{a_n\}_{n=1}^{\infty} \in \mathbb{R}$ is Cauchy, then it is bounded.*

Proof: Suppose a sequence $\{a_n\}_{n=1}^{\infty}$ is Cauchy. Let $\varepsilon = 1$. Then there exists N such that $|a_n - a_m| < 1$ for all $n, m \geq N$. Let $B = \max\{a_1, ..., a_N\}$. Now let's show that $|a_n| \leq 1 + B$ for all $n \in \mathbb{N}$. This is obvious for $n \leq N$. For $n > N$, $|a_n - a_N| < 1$. Hence,

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| \leq 1 + |a_N| \leq 1 + B$$

Q.E.D.

Definition 13. *A metric space* (X, d) *is complete if every Cauchy sequence in* X *converges to an element of* X.

Verifying the completeness of particular spaces can take some work. We take as given the following:

Fact: $\mathbb{R}$ with $d(x, y) = |x - y|$ is a complete metric space.

Exercise 11. *Show that:*

1. *the set of integers with* $d(x, y) = |x - y|$ *is a complete metric space.*
2. *The set of continuous, strictly increasing functions on* $[a, b]$*, with*

$$d(x, y) = max_{a \leq t \leq b}|x(t) - y(t)|.$$

is not a complete metric space.

Definition 14. *A complete normed vector space is called a Banach space.*

4.4 Limits of functions

Let $x \in \mathbb{R}^K$ and $\delta > 0$. The *open ball of radius* δ *around* x, denoted $B_\delta(x)$, is the set

$$B_\delta(x) = \{y \in \mathbb{R} | d(y, x) < \delta\}$$

The *punctured open ball of radius* δ *around* x, denoted $B'_\delta(x)$, is the set $B'_\delta(x) = B_\delta(x) \setminus \{x\}$.

Definition 15. *A point* $\bar{x} \in \mathbb{R}^K$ *is* a limit point of $X \subseteq \mathbb{R}^K$ *if for every* $\varepsilon > 0$, $B'_\varepsilon(\bar{x}) \cap X \neq \varnothing$

Exercise 12. *Prove the following: Let* $X \subseteq \mathbb{R}$. *A point* $\bar{x} \in \mathbb{R}$ *is a limit point of* X *iff there exists a sequence* $(x_n)_{n=1}^\infty$ *in* $X \setminus \{\bar{x}\}$ *that converges to* $\bar{x}$.

Another type of limit has to do with functions, although not directly with sequences.

Definition 16. *Consider a function* $f : X \to \mathbb{R}$, *where* $X \subseteq \mathbb{R}^K$. *Suppose that* $\bar{x} \in \mathbb{R}^K$ *is a limit point of* X *and that* $\bar{y} \in \mathbb{R}$. *We say that* $\lim_{x \to \bar{x}} f(x) = \bar{y}$ *if for every* ε *there exists* $\delta > 0$ *such that* $d(f(x), \bar{y}) < \varepsilon$ *for all* $x \in B'_\delta(\bar{x}) \cap X$.

It is important to notice that we do not require $\bar{x} \in X$ in our previous definition, so that $f(\bar{x})$ need not be defined. Also, one should notice that even if $\bar{x} \in X$, $\bar{x}$ is not always a limit point of X, in which case the definition does not apply. Finally, notice that even if $\bar{x} \in X$ and $\bar{x}$ is a limit point of X, it need not be the case that $\lim_{x \to \bar{x}} f(x) = f(\bar{x})$.

Definition 17. *Consider a function* $f : X \to \mathbb{R}$, *where* $X \subseteq \mathbb{R}^K$. *Suppose that* $\bar{x} \in \mathbb{R}^K$ *is a limit point of* X. *We say that* $\lim_{x \to \bar{x}} f(x) = \infty$ *if for every* $\Delta > 0$, *there exists* $\delta > 0$ *such that* $f(x) \geq \Delta$ *for all* $x \in B'_\delta(\bar{x}) \cap X$. *We say that* $\lim_{x \to \bar{x}} f(x) = -\infty$ *when* $\lim_{x \to \bar{x}}(-f)(x) = \infty$.

Exercise 13. *Suppose that* $X = \mathbb{R}$ *and* $f(x) = x + a$ *for some* $a \in \mathbb{R}$. *What is* $\lim_{x \to 0} f(x)$*?*

Exercise 14. *Suppose that* $X = \mathbb{R}$ *and* $f : X \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 1/x, & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

What is $\lim_{x \to 5} f(x)$*? What is* $\lim_{x \to 0} f(x)$*?*

Example 9. *Let* $X = \mathbb{R} \setminus \{0\}$ *and* $f : X \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{otherwise.} \end{cases}$$

In this case, we claim that $\lim_{x \to 0} f(x)$ *does not exist. To see why, fix* $0 < \varepsilon < 1$*, and notice that for all* $\delta > 0$*, there are* $x_1, x_2 \in B_\delta(0)$ *such that* $f(x_1) = 1$ *and* $f(x_2) = -1$*, and, hence,* $|f(x_1) - f(x_2)| = 2 > 2\varepsilon$*. Because of triangle inequality, it is thus impossible that for some* $\bar{y} \in \mathbb{R}$*, we have* $|f(x_1) - \bar{y}| < \varepsilon$ *and* $|f(x_2) - \bar{y}| < \varepsilon$*. Also, it is obvious that* $\lim_{x \to 0} f(x) = \infty$ *and* $\lim_{x \to 0} f(x) = -\infty$ *are both impossible.*

There exists a tight relationship between limits of functions and limits of sequences, which is explored in the following theorem.

Theorem 14. *Consider a function* $f : X \to \mathbb{R}$*, where* $X \subseteq \mathbb{R}^K$*. Suppose that* $\bar{x} \in \mathbb{R}^K$ *is a limit point of* X *and that* $\bar{y} \in \mathbb{R}$*. Then,* $\lim_{x \to \bar{x}} f(x) = \bar{y}$ *if and only if for every sequence* $(x_n)_{n=1}^\infty$ *in* $X \setminus \{\bar{x}\}$ *that converges to* $\bar{x}$*,* $\lim_{n \to \infty} f(x_n) = \bar{y}$*.*

Proof: ($\Leftarrow$) Suppose that for every sequence $(x_n)_{n=1}^\infty$ in $X \setminus \{\bar{x}\}$ that converges to $\bar{x}$, $\lim_{n \to \infty} f(x_n) = \bar{y}$ but $\lim_{x \to \bar{x}} f(x) \neq \bar{y}$. Then, there must exist some $\varepsilon > 0$ such that for every $\delta > 0$, there exists $x \in B'_\delta(\bar{x}) \cap X$ satisfying

$$|f(x) - \bar{y}| \geq \varepsilon.$$

Thus, for each $n \in \mathbb{N}$ there is $x_n \in B'_{1/n}(\bar{x}) \cap X$ for which $|f(x_n) - \bar{y}| \geq \varepsilon$. So, we can construct a sequence $(x_n)_{n=1}^\infty$ in $X \setminus \{\bar{x}\}$ and $\lim_{n \to \infty} x_n = \bar{x}$. But then $\lim_{n \to \infty} f(x_n) \neq \bar{y}$, which contradicts the initial hypothesis.

($\Rightarrow$) Consider any sequence $(x_n)_{n=1}^\infty$ in $X \setminus \{\bar{x}\}$ that converges to $\bar{x}$. Fix $\varepsilon > 0$. Since $\lim_{x \to \bar{x}} f(x) = \bar{y} \in \mathbb{R}$, then, there is some $\delta > 0$ such that $|f(x) - \bar{y}| < \varepsilon$ for all $x \in B'_\delta(\bar{x}) \cap X$. Since $\lim_{n \to \infty} x_n = \bar{x}$, there is $N \in \mathbb{N}$ such that $x_n \in B_\delta(\bar{x})$ for all $n \geq N$. Moreover, since each $x_n \in X \setminus \{\bar{x}\}$, we have that $x_n \in B'_\delta(\bar{x}) \cap X$ and, therefore, $|f(x_n) - \bar{y}| < \varepsilon$ for all $n \geq N$. Since $\varepsilon > 0$ was arbitrarily chosen, this implies that $\lim_{n \to \infty} f(x_n) = \bar{y}$. *Q.E.D.*

4.5 Properties of limits

Given the relationship found in Theorem 14, it comes as no surprise that a theorem analogous to Theorem 8 holds for functions.

Theorem 15. *Let* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$*. Let* $\bar{x}$ *be a limit point of* X*. Suppose that for number* $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$ *one has that* $\lim_{x \to \bar{x}} f(x) = \bar{y}_1$ *and* $\lim_{x \to \bar{x}} g(x) = \bar{y}_2$*. Then,*[3]

1. $\lim_{x \to \bar{x}} (f + g)(x) = \bar{y}_1 + \bar{y}_2$*;*

2. $\lim_{x \to \bar{x}} (\alpha f)(x) = \alpha \bar{y}_1$*, for all* $\alpha \in \mathbb{R}$*;*

3. $\lim_{x \to \bar{x}} (f.g)(x) = \bar{y}_1 . \bar{y}_2$*; and*

4. *if* $\bar{y}_2 \neq 0$*, then* $\lim_{x \to \bar{x}} (f/g)(x) = \bar{y}_1 / \bar{y}_2$*.*

[3] The following notation is introduced. We define $(f + g) : X \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$. We define $(f.g)$ and (αf), for $\alpha \in \mathbb{R}$, accordingly. Now, define $X_g^* = \{x \in X | g(x) \neq 0\}$. Then, we define $(\frac{f}{g}) : X_g^* \to \mathbb{R}$ by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$.

Proof: Let us prove only the first two statements of the theorem. The proof of the last two parts is left as an exercise. For the first statement, we have that for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(x) - \bar{y}_1| < \varepsilon/2$ for all $x \in B'_{\delta_1}(\bar{x}) \cap X$, and $|g(x) - \bar{y}_2| < \varepsilon/2$ for all $x \in B'_{\delta_2}(\bar{x}) \cap X$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, by construction, for all $x \in B'_{\delta}(\bar{x}) \cap X$ we have that $|f(x) - \bar{y}_1| < \varepsilon/2$ and $|g(x) - \bar{y}_2| < \varepsilon/2$, which implies, by triangle inequality, that

$$|(f + g)(x) - (\bar{y}_1 + \bar{y}_2)| \leq |f(x) - \bar{y}_1| + |g(x) - \bar{y}_2| < \varepsilon.$$

For the second statement, note first that if $\alpha = 0$ the proof is trivial. Then, consider $\alpha \neq 0$. Since $\lim_{x \to \bar{x}} f(x) = \bar{y}_1 \in \mathbb{R}$, then for all $\varepsilon > 0$, there is some $\delta > 0$ such that, for all $x \in B'_{\delta}(\bar{x}) \cap X$, $|f(x) - \bar{y}_1| < \varepsilon/|\alpha|$. This implies that

$$|(\alpha f)(x) - \alpha\bar{y}_1| = |\alpha(f(x) - \bar{y}_1)| = |\alpha||f(x) - \bar{y}_1| < \varepsilon,$$

and, therefore, that $\lim_{x \to \bar{x}}(\alpha f)(x) = \alpha\bar{y}_1$. *Q.E.D.*

The next result is the counterpart of Theorem 6, that "weak inequalities are preserved under limits" for sequences, for limits of functions; again, the proof is left as an exercise.

Theorem 16. *Consider $f : X \to \mathbb{R}$ and $\bar{y} \in \mathbb{R}$, and let $\bar{x} \in \mathbb{R}^K$ be a limit point of X. If $f(x) \leq \gamma$ for all $x \in X$, and $\lim_{x \to \bar{x}} f(x) = \bar{y}$, then $\bar{y} \leq \gamma$. Similarly, if $f(x) \geq \gamma$ for all $x \in X$, and $\lim_{x \to \bar{x}} f(x) = \bar{y}$, then $\bar{y} \geq \gamma$.*

Exercise 15. *The previous theorem can be proved by two different arguments. Can you give them both? (Hint: one argument is by contradiction; the other one uses Theorem 6 directly.)*

Corollary 1. *Consider $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, let $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$, and let $\bar{x} \in \mathbb{R}^K$ be a limit point of X. If $f(x) \geq g(x)$ for all $x \in X$, $\lim_{x \to \bar{x}} f(x) = \bar{y}_1$ and $\lim_{x \to \bar{x}} g(x) = \bar{y}_2$, then $\bar{y}_1 \geq \bar{y}_2$.*

5 Topology of $\mathbb{R}^K$

From now on, we deal only with subsets of $\mathbb{R}^K$, for a finite number K; that is, whenever we introduce sets X or Y, we assume that $X, Y \subseteq \mathbb{R}^K$ and use all the algebraic structure of $\mathbb{R}^K$. We also use the structure induced in $\mathbb{R}^K$ by the Euclidean norm. Whenever we take complements, they are relative to $\mathbb{R}^K$.

5.1 Open sets

The two key concepts are those of open and closed sets.

Definition 18. *Set X is* open *if for every $x \in X$, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X$.*

Example 10 (Open intervals are open sets in $\mathbb{R}$). *We define an open interval, denoted (a, b),*[4] *where $a, b \in \mathbb{R}$, as $\{x \in \mathbb{R} | a < x < b\}$. To see that these are open sets (in $\mathbb{R}$), take $x \in (a, b)$, and define $\varepsilon = \min\{x - a, b - x\}/2 > 0$. By construction, $B_\varepsilon(x) \subseteq X$. As a consequence, notice that open balls are open sets in $\mathbb{R}$. The same is true in $\mathbb{R}^K$, for any K.*

[4] Sometimes open intervals are denoted by $]a, b[$ rather that (a, b) in order to distinguish them from two-element sequences. We will, however, follow the more standard notation.

It is easy to see that if we extend the definition of the open interval (a, b) to $\{x \in \mathbb{R} | a < x < b\}$ where $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$, then it continuous to be true that open intervals are open sets. The following theorem is a specific instance of a more general principle: in any space, the empty set and the universe are open sets.

THEOREM 17. *The empty set and* $\mathbb{R}^K$ *are open.*

Proof: A set X fails to be open if one can find $x \in X$ such that for all $\varepsilon > 0$ one has that $B_\varepsilon(x) \cap X^c \neq \varnothing$. Clearly, $\varnothing$ cannot exhibit such property. The argument that $\mathbb{R}^K$ is open is left as an exercise. *Q.E.D.*

THEOREM 18. *The union of any collection of open sets is an open set. The intersection of any finite collection of open sets is an open set.*

Proof: For the first statement, suppose that Z is the union of a given collection of open sets (whether finite or infinite doesn't matter), and suppose that $x \in Z$. By definition, then, there exists a member X of the collection of sets such that $x \in X$. By assumption, X is open, so that $\exists \varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$, and it follows, then, that $B_\varepsilon(x) \subseteq Z$.

For the second part, suppose that Z is the intersection of a finite collection of open sets, say $\{X_1, X_2, \ldots, X_{n^*}\}$, and suppose that $x \in Z$. By definition, then, for each $n = 1, 2, \ldots, n^*$, it is true that $x \in X_n$. By assumption, each X_n is open, so that there exists $\varepsilon_n > 0$ such that $B_{\varepsilon_n}(x) \subseteq X_n$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n^*}\} > 0$. By construction, for each n, we have that $B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq X_n$ and therefore $B_\varepsilon(x) \subseteq Z$. *Q.E.D.*

DEFINITION 19. *We say that point x is* interior point of the set X, *if there is some $\varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$.*

The set of all the interior points of X is called *the interior of* X, and is usually denoted $\text{int}(X)$.[5] Note that $\text{int}(X) \subseteq X$.

EXERCISE 16. *Show that for every X,* $\text{int}(X)$ *is open and that X is open if and only if* $\text{int}(X) = X$.

EXERCISE 17. *Prove the following: "If $x \in int(X)$, then x is a limit point of X."*

EXERCISE 18. *Did we really need finiteness in the second part of Theorem 18? Consider the following infinite collection of open intervals: for all $n \in \mathbb{N}$, define $I_n = (-\frac{1}{n}, \frac{1}{n})$. Find the intersection of all those intervals, denoted $\cap_{n=1}^{\infty} I_n$. Is it an open set?*

5.2 Closed sets

DEFINITION 20. *Set X is* closed *if for every sequence $(x_n)_{n=1}^{\infty} \in X$ that converges to $\bar{x} \in \mathbb{R}^K$, then $\bar{x} \in X$.*

Given a set $X \subseteq \mathbb{R}^K$, we define its *closure*, denoted by $\text{cl}(X)$, as the set[6]

$$\text{cl}(X) = \{x \in \mathbb{R}^K | \forall \varepsilon > 0, B_\varepsilon(x) \cap X \neq \varnothing\}.$$

As before, the empty set and the universe are closed sets. In $\mathbb{R}^K$ these are the only two sets that have both properties, but this principle does not generalize to other spaces.

[5] Alternative, but usual, notation is X^o.
[6] Alternative notation is $\bar{X}$.

THEOREM 19. *The empty set and $\mathbb{R}^K$ are closed.*

Proof: In order for set X to fail to be closed, there has to exist $(x_n)_{n=1}^{\infty}$ satisfying that all $x_n \in X$, and that $x_n \to \bar{x}$, yet $\bar{x} \notin X$. Clearly, one cannot find such sequence if $X = \varnothing$. The argument that $\mathbb{R}^K$ is left as an exercise. *Q.E.D.*

EXERCISE 19. *If $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, $a < b$, is (a, b) closed? We define the* half-closed *interval $(a, b]$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$, $a < b$, as $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$. Similarly, we define the half-closed interval $[a, b)$ where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$, $a < b$, as $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$. Are half-closed intervals closed sets? Are they open? If $x \in \mathbb{R}^K$, is $\{x\}$ an open set, a closed set or neither?*

THEOREM 20. *A set X is closed if and only if X^c is open.*

Proof: Suppose that X^c is open, and consider any sequence $(x_n)_{n=1}^{\infty}$ satisfying that all $x_n \in X$ and converging to some $\bar{x}$; we need to show that $\bar{x} \in X$. In order to argue by contradiction, suppose that $\bar{x} \in X^c$. Since X^c is open, there is some $\varepsilon > 0$ for which $B_\varepsilon(\bar{x}) \subseteq X^c$. Since $x_n \to \bar{x}$, there is $n^* \in \mathbb{N}$ such that $\|x_n - \bar{x}\| < \varepsilon$ when $n \geq n^*$. Then, for any $n \geq n^*$, we have that $x_n \in B_\varepsilon(\bar{x}) \subseteq X^c$, which is impossible.

Suppose now that X is closed, and fix $x \in X^c$. We need to show that for some $\varepsilon > 0$ one has that $B_\varepsilon(x) \subseteq X^c$. Again, suppose not: for all $\varepsilon > 0$, it is true that $B_\varepsilon(x) \cap X \neq \varnothing$. Clearly, then, for all $n \in \mathbb{N}$ we can pick $x_n \in B_{1/n}(x) \cap X$. Construct a sequence $(x_n)_{n=1}^{\infty}$ of such elements. Since $1/n \to 0$ it follows that $x_n \to x$, and all $x_n \in X$ and X is closed, then $x \in X$, contradicting the fact that $x \in X^c$. *Q.E.D.*

THEOREM 21. *The intersection of* any *collection of closed sets is closed. The union of any finite collection of closed sets is closed.*

Proof: Left as an exercise. (Hint: do you remember DeMorgan's Laws?) *Q.E.D.*

EXERCISE 20. *Prove the following: "Given a set $X \subseteq \mathbb{R}^K$, one has $x \in cl(X)$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \to x$."*

EXERCISE 21. *Prove the following: "For every set $X \subseteq \mathbb{R}^K$, $X \subseteq cl(X)$, and X is closed if and only if $X = cl(X)$."*

EXAMPLE 11. *Closed intervals are closed sets. We define an closed interval, denoted $[a, b]$, where $a, b \in \mathbb{R}$ and $a \leq b$ as $\{x \in \mathbb{R} | a \leq x \leq b\}$. To see that these are closed sets, notice that $[a, b]^c = (-\infty, a) \cup (b, \infty)$, and conclude based on previous results.*

EXERCISE 22. *Did we really need finiteness in the second part of Theorem 21? Consider the following infinite collection of closed intervals: for all $n \in \mathbb{N}$, define $J_n = [1 + \frac{1}{n}, 3 - \frac{1}{n}]$. Find the union of all those intervals, denoted $\cup_{n=1}^{\infty} J_n$. Is it a closed set?*

EXERCISE 23. *Given $X \subseteq \mathbb{R}^K$, define* the boundary of X *as* $\mathrm{bd}(X) = \mathrm{cl}(X) \setminus \mathrm{int}(X)$. *Prove the following statements:* "X is closed if and only if $\mathrm{bd}(X) \subseteq X$. It is open if and only if $\mathrm{bd}(X) \cap X = \varnothing$." *Also, prove that*

$$\mathrm{bd}(X) = \{x \in \mathbb{R}^K | \forall \varepsilon > 0, B_\varepsilon(x) \cap X \neq \varnothing \textit{ and } B_\varepsilon(x) \cap X^c \neq \varnothing\}.$$

5.3 Compact sets

A set $X \subseteq \mathbb{R}^K$ is said to be *bounded above* if there exists $\alpha \in \mathbb{R}^K$ such that $x \leq \alpha$ for all $x \in X$; it is said to be *bounded below* if for some $\beta \in \mathbb{R}^K$ one has that $x \geq \beta$ is true for all $x \in X$; and it is said to be *bounded* if it is bounded above and below.

EXERCISE 24. *Show that a set X is bounded if and only if there exists $\alpha \in \mathbb{R}_+$ such that $\|x\| \leq \alpha$ for all $x \in X$.*

DEFINITION 21. *A set $X \subseteq \mathbb{R}^K$ is said to be* compact *if it is closed and bounded.*

EXERCISE 25. *Prove the following statement: if $(x_n)_{n=1}^{\infty}$ is a sequence defined on a compact set X, then it has a subsequence that converges to a point in X.*

6 Continuity

6.1 Continuous functions

DEFINITION 22. *Function $f : X \to \mathbb{R}$ is* continuous at *$\bar{x} \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(\bar{x})| < \varepsilon$ for all $x \in B_\delta(\bar{x}) \cap X$. It is* continuous *if it is continuous at all $\bar{x} \in X$.*

Note that continuity at $\bar{x}$ is a local concept. Second, note that $\bar{x}$ in the definition may but need not be a limit point of X. Therefore, two points are worth noticing: if $\bar{x}$ is not a limit point of X, then any $f : X \to \mathbb{R}$ is continuous at $\bar{x}$ (why?); and if, on the other hand, $\bar{x}$ is a limit point of X, then $f : X \to \mathbb{R}$ is continuous at $\bar{x}$ if and only if $\lim_{x \to \bar{x}} f(x) = f(\bar{x})$. Intuitively, this occurs when a function is such that in order to get arbitrarily close to $f(\bar{x})$ in the range, all we need to do is to get close enough to $\bar{x}$ in the domain. By Theorem 14, it follows that when $\bar{x} \in X$ is a limit point of X, f is continuous at $\bar{x}$ if and only if whenever we take a sequence of points in the domain that converges to $\bar{x}$, the sequence formed by their images converges to $f(\bar{x})$ (that in this case the concept is not vacuous follows from Exercise 12).

EXERCISE 26. *Consider the function introduced in Exercise 14. Is it continuous?*

EXERCISE 27. *Consider the function introduced in Example 9. Is it continuous? What if we change the function, slightly, as follows: $f : \mathbb{R} \to \mathbb{R}$, defined as*

$$f(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Is it continuous?

6.2 Properties and the Intermediate Value Theorem

The following properties of continuous functions are derived from the properties of limits. They are all very useful in economics.

THEOREM 22. *Suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous at $\bar{x} \in X$, and let $\alpha \in \mathbb{R}$. Then, the functions $f + g$, αf and $f \cdot g$ are continuous at $\bar{x}$. Moreover, if $g(\bar{x}) \neq 0$, then $\frac{f}{g}$ is continuous at $\bar{x}$.*

THEOREM 23. *Fucntion $f : \mathbb{R}^K \to \mathbb{R}$ is continuous if and only if, for every open set $U \subseteq \mathbb{R}$, the set $f^{-1}[U]$ is open.*

Proof: Fix $\bar{x} \in \mathbb{R}^K$ and $\varepsilon > 0$. By Example 10, we know that $B_\varepsilon(f(\bar{x}))$ is open and, therefore, so is $f^{-1}[B_\varepsilon(f(\bar{x}))]$. Since $\bar{x} \in f^{-1}[B_\varepsilon(f(\bar{x}))]$, we have that there exists some $\delta > 0$ for which $B_\delta(\bar{x}) \subseteq f^{-1}[B_\varepsilon(f(\bar{x}))]$. For such δ, the latter means that that for all $x \in B_\delta(\bar{x})$ one has that $|f(x) - f(\bar{x})| < \varepsilon$.

Now, let $U \subseteq \mathbb{R}$ be an open set, and let $\bar{x} \in f^{-1}[U]$. By definition, $f(\bar{x}) \in U$, and since U is open, there is some $\varepsilon > 0$ for which $B_\varepsilon(f(\bar{x})) \subseteq U$. Since f is continuous, there exists $\delta > 0$ such that $|f(x) - f(\bar{x})| < \varepsilon$ for all $x \in B_\delta(\bar{x})$. The latter implies $B_\delta(\bar{x}) \subseteq f^{-1}[U]$. *Q.E.D.*

THEOREM 24 (The Intermediate Value Theorem in $\mathbb{R}$). *If function $f : [a, b] \to \mathbb{R}$ is continuous, then for every number γ between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ for which $f(x) = \gamma$.*[7]

The following property will be important to prove Weirstrass Theorem.

THEOREM 25. *The image of a compact set under a continuous function is compact.*

6.3 Left- and Right-continuity

Consider a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. Suppose that $\bar{x}$ is a limit point of X, and let $\ell \in \mathbb{R}$.

DEFINITION 23. *One says that $\lim_{x \searrow \bar{x}} f(x) = \ell$, when for every $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in X \cap B_\delta(\bar{x})$ satisfying $x > \bar{x}$. In such case, function f is said to converge to ℓ as x tends to $\bar{x}$ from above. Similarly, $\lim_{x \nearrow \bar{x}} f(x) = \ell$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in X \cap B_\delta(\bar{x})$ satisfying that $x > \bar{x}$. In this case, f is said to converge to ℓ as x tends to $\bar{x}$ from below.*

DEFINITION 24. *Function $f : X \to \mathbb{R}$ is* right-continuous at $\bar{x} \in X$, *where $\bar{x}$ is a limit point of X, if $\lim_{x \searrow \bar{x}} f(x) = f(\bar{x})$. It is* right-continuous *if it is right-countinuous at every $\bar{x} \in X$ that is a limit point of X. Similarly, $f : X \to \mathbb{R}$ is* left-continuous at $\bar{x}$ *if* $\lim_{x \nearrow \bar{x}} f(x) = f(\bar{x})$, *and one says that f is left-continuous if it is left-continuous at all limit point $\bar{x} \in X$.*

EXERCISE 28. *Consider the function introduced in Exercise 27. Is it right-continuous? Left-continuous? What if, keeping the rest of the function unchanged, we redefine $f(0) = -1$? Is it left- or right-continuous? What if $f(0) = 1$.*

6.4 Derivatives

DEFINITION 25. *Let $f : \mathbb{R} \mapsto R$ be a function defined in a neighbourhood of x_0. Then f is said to be differentiable at x_0 with derivative equal to the real number $f'(x_0)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies*

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \varepsilon$$

[7] It does not matter whether $f(a) \geq f(b)$ or $f(a) < f(b)$ – we could simply have written that $\gamma \in [f(a), f(b)] \cup [f(b), f(a)]$.

Now observe that since $x - x_0 \neq 0$, we can multiply the inequality above by $|x - x_0|$ to obtain

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \varepsilon |x - x_0|$$

This new inequality admits and interesting interpretation. We think of $f(x) - f(x_0) - f'(x_0)(x - x_0)$ as the difference of two functions – the original function $f(x)$ and the function $f(x_0) + f'(x_0)(x - x_0)$. Here x_0 is thought of as a constant, as are $f(x_0)$ and $f'(x_0)$, so this second function is simply and affine function $ax+b$, where $a = f'(x_0)$ and $b = f(x_0) - x_0 f'(x_0)$.. Now the existence of the derivative of f at x_0 is a statement about the difference between the original function and the affine function $g(x) = f(x_0) + f'(x_0)(x - x_0)$ which we will think of as a statement of how well $g(x)$ approximates f. However, there are many affine functions whose graph crosses the graph of f at the point $(x_0, f(x_0))$. These functions have the form $f(x_0) + a(x - x_0)$ for any real constant a. The property that $f(x) - g(x)$ tends to 0 as $x \to x_0$ is not what distinguish $g(x)$ from other affine functions as that would have happened with any choice of a. The important difference is that $f(x) - g(x)$ goes to zero faster than for any other choice of a in the sense that it goes faster than $|x - x_0|$.

6.5 Mean Value Theorem

Theorem 26 (Mean Value Theorem). *Let f be a continuous function on $[a, b]$ that is differentiable in (a, b). Then there exists $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.*

6.6 Taylor's Theorem

Let $\mathbb{C}^n$ denote the set of functions that are n times continuously differentiable.

Theorem 27 (Taylor's Theorem). *Let f be $\mathbb{C}^n$ in a neighborhood of x_0, and let*

$$T_n(x_0, x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!} f^n(x_0)(x - x_0)^n.$$

Then for any $\varepsilon > 0$, there exists δ such that $|x - x_0| \leq \delta$ implies

$$|f(x) - T_n(x_0, x)| \leq \varepsilon |x - x_0|^n.$$

Theorem 28 (Lagrange Remainder Theorem). *Suppose f is $\mathbb{C}^{n+1}$ in a neighborhood of x_0. Then for every x in the neighbourhood there exists x_1 between x_0 and x such that*

$$f(x) = T_n(x_0, x) + \frac{1}{(n+1)!} f^{n+1}(x_1)(x - x_0)^{n+1}$$