PH210 Problem Set 5 Solutions

(1) Specify a first-order language $\mathcal{L}_k$ for describing kinship relationships which reflects as closely as possible the kinship terms we have in English (e.g. "child", "parent", "brother", "sister", "mother", "father", "grandmother", "grandfather", "aunt", "uncle", "nephew","cousin", etc.). Make sure to distinguish which of these terms express relations and which express functions.

Solution: Our language $\mathcal{L}_k$ will consist of:
Predicate Symbols: $Parent(x,y), Child(x,y), Son(x,y), Daughter(x,y),$
$Brother(x,y), Sister(x,y), Grandparent(x,y), Grandmother(x,y),$
$Grandfather(x,y), Grandchild(x,y), Grandson(x,y), Granddaughther(x,y),$
$Uncle(x,y), Aunt(x,y), Nephew(x,y), Cousin(x,y).$
Function Symbols: $mother(x), father(x), wife(x), husband(x)$
Constant Symbols: $\bar{p}, \bar{e}, \bar{c}, \bar{b}, \bar{w}, \bar{h}, \bar{d}$

i) What is the similarity type of your language? $\langle 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2; 1,1,1,1; 7\rangle$

ii) Use $\mathcal{L}_k$ to formalize the following English sentences:
   a) Everyone has a parent. $\forall x \exists y Parent(x,y)$
   b) Everyone has exactly two parents. $\forall x \exists y \exists z[Parent(x,y) \wedge Parent(x,z) \wedge y \neq z \wedge \forall u[Parent(x,u) \rightarrow (u = y \vee u = z))]$
   c) No one is there own brother. $\forall x[\neg Brother(x,x)]$
   d) The father of your father is your grandfather. $\forall x[Grandfather(father(father(x)), x)]$
   e) Your aunt's children are your cousins. $\forall x \forall y \forall z[(Aunt(x,y) \wedge Child(y,z)) \rightarrow Cousin(x,y)]$

iii) Construct a model of your language based on the royal family tree depicted here `http://en.wikipedia.org/wiki/British_Royal_Family}`

iv) We shall use the part of the family free which starts at Queen Elizabeth and Prince Philip and ends with Princes William and Harry for our model $\mathcal{M}$. We interpret constants as:
$\bar{p}^{\mathcal{M}}$ = Prince Philip
$\bar{e}^{\mathcal{M}}$ = Queen Elizabeth
$\bar{c}^{\mathcal{M}}$ = Prince Charles
$\bar{b}^{\mathcal{M}}$ = Camilla Parker Bowles
$\bar{d}^{\mathcal{M}}$ = Princess Dianna
$\bar{w}^{\mathcal{M}}$ = Prince William
$\bar{h}^{\mathcal{M}}$ = Prince Harry
The predicate and function symbols are interpreted as depicted in the family tree – e.g. $Parent^{\mathcal{M}} = \{\langle \bar{c}^{\mathcal{M}}, \bar{e}^{\mathcal{M}}\rangle, \langle \bar{c}^{\mathcal{M}}, \bar{p}^{\mathcal{M}}\rangle, \langle \overline{w}^{\mathcal{M}}, \bar{c}^{\mathcal{M}}\rangle, \langle \overline{w}^{\mathcal{M}}, \overline{d}^{\mathcal{M}}\rangle, \langle \overline{h}^{\mathcal{M}}, \bar{c}^{\mathcal{M}}\rangle, \langle \overline{h}^{\mathcal{M}}, \overline{d}^{\mathcal{M}}\rangle\}$

v) Note that some relations expressed by predicates in $\mathcal{L}_k$ are definable in terms of one another (e.g. x is y's child iff y is x's parent) and these relationships can be formalized as biconditional statements of $\mathcal{L}_k$ (e.g. $\forall x \forall y[Child(x,y) \leftrightarrow Parent(y,x)]$). Identify and formalize as many of these equivalences as possible.
E.g. $\forall x \forall y[Child(x,y) \leftrightarrow Parent(y,x)]$
$\forall x \forall y[Grandchild(x,y) \leftrightarrow Grandparent(y,x)]$
$\forall x \forall y[Cousin(x,y) \leftrightarrow \exists z(Parent(z,x) \wedge (Aunt(z,y) \vee Uncle(x,y)))]$

vi) Is it possible to define the relation expressed by "x is y's ancestor" in terms of the relations of $\mathcal{L}_k$? If so, provide an appropriate biconditional. If not, say why.

vii) We say that 'x is y's ancestor" if there is a chain of descent going from y to x down the family tree. The relation which express such a descent are ones such as Child and Grandchild. The problem with expressing the relation of ancestor with such relations is that the chain of descent may be arbitrarily long and since our expression needs to be finite we hit a problem. This can be illustrated by supposing that we only have the relation $Parent(x,y)$ which holds just in case y is x's parent. Then we could express "x is y's ancestor" via a disjunction of the form

$$Parent(x,y) \vee \exists x_1[Parent(x,x_1) \wedge Parent(x_1,y) \vee$$
$$\exists x_1 \exists x_2[Parent(x,x_1) \wedge Parent(x_1,x_2) \wedge Parent(x_2,y)] \vee \ldots$$
$$\exists x_1 \ldots \exists x_n[Parent(x,x_1) \wedge \ldots \wedge Parent(x_{n-1},x_n) \wedge Parent(x_n,y)] \vee \ldots$$

But unless there is a bound on the number of generations, such an expression will have infinite length and thus not be a sentence of $\mathcal{L}_k$.

viii) Suppose we introduce to $\mathcal{L}_k$ the predicates $Male(x)$ and $Female(x)$. Is it possible to state any additional equivalences?

ix) E.g. $\forall x \forall y[Grandfather(x,y) \leftrightarrow (Grandparent(x,y) \wedge Male(x))]$
$\forall x \forall y[Grandmother(x,y) \leftrightarrow (Grandparent(x,y) \wedge Female(x))]$
$\forall x \forall y[Son(x,y) \leftrightarrow (Child(x,y) \wedge Male(x))]$
$\forall x \forall y[Daughter(x,y) \leftrightarrow (Child(x,y) \wedge Female(x))]$
$\forall x \forall y[Brother(x,y) \leftrightarrow (Male(x) \wedge \forall z(Parent(z,x) \leftrightarrow Parent(z,y)))]$
$\forall x \forall y[Nephew(x,y) \leftrightarrow (Male(x) \wedge (Aunt(y,x) \vee Uncle(z,y)))]$

(2) 2.4.1 *Solution:* Our language $\mathcal{L}$ will consist of:
Predicate symbols: $=$
Function symbols: P, M, σ
Constant symbol: $\bar{0}$.

We shall interpret as follows: $P(t_1,t_2)^{\mathcal{N}} = t_1^{\mathcal{N}} + t_2^{\mathcal{N}}, M(t_1,t_2)^{\mathcal{N}} = t_1^{\mathcal{N}} \cdot t_2^{\mathcal{N}}$,
$\sigma(t_1)^{\mathcal{N}} = t_1^{\mathcal{N}} + 1, \bar{0}^{\mathcal{N}} = 0$

i) $\sigma(\sigma(\sigma(\sigma(\sigma(\bar{0})))))^{\mathcal{N}} = P(\sigma(\sigma(\sigma(\bar{0}))), \sigma(\sigma(\bar{0})))^{\mathcal{N}} = 5$

ii) We proceed by induction on n. By definition $\bar{0}^{\mathcal{N}} = 0$, so we have the base case. Now, suppose that we have a term t such that $t^{\mathcal{N}} = n$, then $\sigma(t)^{\mathcal{N}} = t^{\mathcal{N}} + 1 = n + 1$ and so by induction we are done.

iii) From (ii) we know that there is a term t such that $t^{\mathcal{N}} = n$. Now, let $t_1 = P(t, \bar{0})$ and $t_{k+1} = P(t_k, \bar{0})$ then $t_1^{\mathcal{N}} = P(t,\bar{0})^{\mathcal{N}} = n + 0 = n$ and assuming that $t_k^{\mathcal{N}} = n$ we have that $t_{k+1}^{\mathcal{N}} = P(t_k, \bar{0})^{\mathcal{N}} = t_k^{\mathcal{N}} + 0 = n + 0 = n$. So by induction $t_k^{\mathcal{N}} = 0$ for all k. Clearly then there are infinitly many terms t such that $t^{\mathcal{N}} = n$.

(3) 2.4.2 *Solution:*

i) $(\bar{1} \rightarrow \bar{0})^{\mathcal{U}} = 0$, $\neg\bar{0}^{\mathcal{U}} = 1$ so $(\bar{1} \rightarrow \bar{0}) \rightarrow \neg\bar{0}^{\mathcal{U}} = 1$.
$((\neg\bar{0}) \rightarrow (\bar{1} \rightarrow \bar{0}))^{\mathcal{U}} = 0$, so $(((\bar{1} \rightarrow \bar{0}) \rightarrow \neg\bar{0}) \wedge (\neg\bar{0} \rightarrow (\bar{1} \rightarrow \bar{0}))^{\mathcal{U}} = 0$

ii) Note that the symbol $\leftarrow$ is **NOT** a connective in the language. So this question may contain a typo. However, if we interpret $\varphi \leftarrow \psi$ as $\psi \rightarrow \varphi$ we see that $(\bar{1} \leftarrow \neg(\neg\bar{0} \vee \bar{1}))^{\mathcal{U}} = 1$.

(4) 2.5.2

i) We want to show $\models (\forall x\varphi(x) \rightarrow \psi) \leftrightarrow \exists x(\varphi(x) \rightarrow \psi)$ given that $x \notin FV(\psi)$. Let $\mathcal{M}$ be any suitable structure. If we have that $\mathcal{M} \models \forall x\varphi(x) \rightarrow \psi$ then for every $a \in |\mathcal{M}|$ we have $\mathcal{M}_a^{\overline{a}} \models (\varphi(x) \rightarrow \psi)[\bar{a}/x]$ and hence $\mathcal{M}_a^{\overline{a}} \models \varphi[\bar{a}/x] \rightarrow \psi$ since $x \notin FV(\psi)$. Now note that either 1) $\mathcal{M} \models \exists x\varphi(x)$ or 2) $\mathcal{M} \not\models \exists x\varphi(x)$. In case 1), there is some $b \in |\mathcal{M}|$, $\mathcal{M}_b^{\overline{b}} \models \varphi(\overline{x})[\overline{b}/x]$. It then follows that $\mathcal{M} \models \psi$ (since $\mathcal{M}_a^{\overline{a}} \models \varphi[\bar{a}/x] \rightarrow \psi$ for every $a \in |\mathcal{M}|$) and hence $\mathcal{M}_b^{\overline{b}} \models (\varphi(x) \rightarrow \psi)[\overline{b}/x]$ since $x \notin FV(\psi)$. But then $\mathcal{M} \models \exists x(\varphi(x) \rightarrow \psi)$. In case 2), there is $a \in |\mathcal{M}|$ such that $\mathcal{M}_a^{\overline{a}} \not\models \varphi(x)[\overline{a}/x]$. Hence $\mathcal{M}_a^{\overline{a}} \models (\varphi(x) \rightarrow \psi)[\overline{a}/x]$ since the antecedent is false in $\mathcal{M}$. Thus again $\mathcal{M} \models \exists x(\varphi(x) \rightarrow \psi)$. Conversely, suppose $\mathcal{M} \models \exists x(\varphi(x) \rightarrow \psi)$. Then there is some $b \in |\mathcal{M}|$ such that $\mathcal{M}_b^{\overline{b}} \models \varphi[\overline{b}/x] \rightarrow \psi$. Now suppose that $\mathcal{M}_b^{\overline{b}} \models \forall x\varphi(x)$. Then, in particular, $\mathcal{M}_b^{\overline{b}} \models \varphi[\overline{b}/x]$, and hence $\mathcal{M} \models \psi$. Hence $\mathcal{M} \models \forall x\varphi(x) \rightarrow \psi$.

ii) We want to show $\models (\exists x\varphi(x) \rightarrow \psi) \leftrightarrow \forall x(\varphi(x) \rightarrow \psi)$ where $x \notin FV(\psi)$. Suppose, $\mathcal{M} \models \exists x\varphi(x) \rightarrow \psi$ and let $a \in |\mathcal{M}|$ be arbitrary. We have either 1) $\mathcal{M}_a^{\overline{a}} \models \varphi(x)[\overline{a}/x]$ or 2) $\mathcal{M}_a^{\overline{a}} \not\models \varphi(x)[\overline{a}/x]$. In case 1), $\mathcal{M} \models \exists x\varphi(x)$ and hence $\mathcal{M} \models \psi$. Thus $\mathcal{M}_a^{\overline{a}} \models (\varphi(x) \rightarrow \psi)[\overline{a}/x]$ since $x \notin FV(\psi)$. In case 2), we also have $\mathcal{M}_a^{\overline{a}} \models (\varphi(x) \rightarrow \psi)[\overline{a}/x]$ since after the substitution the antecedent is false in $\mathcal{M}$ and $\psi[\overline{a}/x] = \psi$ since, again, $x \notin FV(\psi)$. Since a was arbitrary, it hence follows that $\mathcal{M}_a^{\overline{a}} \models \forall x(\varphi(x) \rightarrow \psi)$. Conversely, if $\mathcal{M} \models \forall x(\varphi(x) \rightarrow \psi)$, then $\mathcal{M}_a^{\overline{a}} \models (\varphi(x) \rightarrow \psi)[\overline{a}/x]$ for every $a \in |\mathcal{M}|$. But then $\mathcal{M}_a^{\overline{a}} \models \varphi(x)[\overline{a}/x] \rightarrow \psi$ for every $a \in |\mathcal{M}|$ since $x \notin FV(\psi)$. Now suppose that $\mathcal{M} \models \exists x\varphi(x)$. It then follows that $\mathcal{M}_b^{\overline{b}} \models \exists x\varphi(\overline{b})$ for some $b \in |\mathcal{M}|$. But then

$\mathcal{M} \models \psi$. Hence $\mathcal{M} \models \exists x\varphi(x) \rightarrow \psi$.

iii) We want to show $\models (\psi \rightarrow \exists x\varphi(x)) \leftrightarrow \exists x(\psi \rightarrow \varphi(x))$ where $x \notin FV(\psi)$. If $\mathcal{M} \models (\psi \rightarrow \exists x\varphi(x))$, then $\mathcal{M} \not\models \psi$ or $\mathcal{M}_b^{\bar{b}} \models \varphi(x)[\bar{b}/x]$ for some $a \in |\mathcal{M}|$. In the first case, $\mathcal{M}_a^{\overline{a}} \models \psi \rightarrow \varphi(x)[\overline{a}/x]$ for any $a \in |\mathcal{M}|$, in which case $\mathcal{M} \models \exists x(\psi \rightarrow \varphi(x))$ since $x \notin FV(\psi)$ and $|\mathcal{M}| \neq \emptyset$. In the second case, $\mathcal{M}_b^{\bar{b}} \models (\psi \rightarrow \varphi(x))[\bar{b}/x])$ since the consequent is true after the substitution. Thus $\mathcal{M} \models \exists x(\psi \rightarrow \varphi(x))$. Conversely, suppose that $\mathcal{M} \models \exists x(\psi \rightarrow \varphi(x))$. Then there is $b \in |\mathcal{M}|$ such that $\mathcal{M}_b^{\bar{b}} \models (\psi \rightarrow \varphi(x))[\bar{b}/x]$. In this case, either $\mathcal{M} \not\models \psi$ or $\mathcal{M}_b^{\bar{b}} \models \varphi(x)[\bar{b}/x]$ in which case $\mathcal{M} \models \exists x\varphi(x)$. Hence in either case $\mathcal{M} \models \psi \rightarrow \exists x\varphi(x)$.

iv) Similar to iii).

(5) 2.5.6 Show $\not\models \exists x\varphi(x) \rightarrow \forall x\varphi(x)$
Let $\varphi(x) \equiv P(x)$ and consider any model $\mathcal{M}$ in which 1) $P^{\mathcal{M}} \neq \emptyset$ and 2) $P^{\mathcal{M}} \neq |\mathcal{M}|$. Then $\mathcal{M} \models \exists x\varphi(x)$ by 1), but $\mathcal{M} \not\models \forall x\varphi(x)$ by 2). Hence $\mathcal{M} \not\models \exists x\varphi(x) \rightarrow \forall x\varphi(x)$. (Concrete example: $|\mathcal{M}| = \{a, b\}$, $\mathcal{P}^{\mathcal{M}} = \{a\}$.)

(6) 2.5.12 Show that $\not\models \neg\exists y\forall x[S(y,x) \leftrightarrow \neg S(x,x)]$.
Suppose for a contradiction that $\mathcal{M} \models \exists y\forall x[S(y,x) \leftrightarrow \neg S(x,x)$ and let $b \in |\mathcal{M}|$ be such that $\mathcal{M} \models \forall x[S(y,x) \leftrightarrow \neg S(x,x)][b/y]$. Then we would have that for all $a \in |\mathcal{M}|$, $\mathcal{M} \models S(y,x) \leftrightarrow \neg S(x,x)][b/y][a/x]$ and thus taking $a = b$, $\mathcal{M} \models S(y,x) \leftrightarrow \neg S(x,x)][b/y][b/x]$. Hence $\mathcal{M}_b^{\bar{b}} \models S(\bar{b},\bar{b}) \leftrightarrow \neg S(\bar{b},\bar{b})$. But note that this cannot happen since $S(\bar{b},\bar{b})$ and $\neg S(\bar{b},\bar{b})$ must differ in truth value in every model. Hence we have derived a contradiction from the hypothesis that $\exists y\forall x[S(y,x) \leftrightarrow \neg S(x,x)$ is true in $\mathcal{M}$. Since every sentence is either true or false in $\mathcal{M}$, we may therefore conclude that $\mathcal{M} \models \neg\exists y\forall x[S(y,x) \leftrightarrow \neg S(x,x)$.

NB: The hint points out that the validity of $\neg\exists y\forall x[S(y,x) \leftrightarrow \neg S(x,x)]$ (or equivalently that $\exists y\forall x[S(y,x) \leftrightarrow \neg S(x,x)]$ is false in all models) is akin to Russell's Barber Paradox (taking $S(x,y)$ to formalize x *shaves* y the sentence formalizes "There is no barber who shaves only those people who do not shave themselves"). Taking $S(x,y)$ to represent the membership relation form set theory (i.e. $x \in y$), the same reasoning show that there cannot be a set containing exactly those sets which are not members of themselves.

(7) 2.5.15. Show$\models \exists x(\varphi(x) \rightarrow \forall y\varphi(y))$.
Let $\mathcal{M}$ be a suitable structure. We have two cases. On the one hand, suppose that $\mathcal{M}_a^{\overline{a}} \models \varphi(\bar{a})$ for every $a \in |\mathcal{M}|$. Then $\mathcal{M} \models \forall y\varphi(y)$ and hence $\mathcal{M} \models (\varphi(x) \rightarrow \forall y\varphi(y))[\bar{b}/x]$ for some $b \in |\mathcal{M}|$ since the consequent is true in $\mathcal{M}$ and $|\mathcal{M}| \neq \emptyset$. On the other hand, suppose that there is some $b \in |\mathcal{M}|$ such that $\mathcal{M}_b^{\bar{b}} \models \neg\varphi(\bar{b})$. Then note that $\mathcal{M}_b^{\bar{b}} \models \varphi(\bar{b}) \rightarrow \forall y\psi(x)$, since the antecedent will be false in $\mathcal{M}$. In both cases we hence have $\mathcal{M} \models \exists x(\varphi(x) \rightarrow \forall y\psi(y))$.