PH210 Problem Set 8

(1) LS 2.5.15 a), b) (prenex normal form)

(2) Recall that the *minimal closure* of a set X under a function f is the smallest set X^*_f such that $X \subseteq X^*_f$ and for all $x \in X^*_f$, $f(x) \in X^*_f$. Show that the notion of minimal closure is **not** first-order definable in the following sense: Let $\mathcal{L}$ contain the single unary predicate symbol P and the unary function symbol f. Show that there is *no* set of sentences Γ over any language $\mathcal{L}' \supseteq \mathcal{L}$ such that there is a $\mathcal{L}'$-formula $\theta(x)$ such that for all $\mathcal{L}'$-models $\mathcal{M}$,

$$(\star) \qquad \text{for all } a \in |\mathcal{M}|,\ \mathcal{M} \models \theta(\overline{a}) \text{ if and only if } a \in [\![P]\!]^*_{f^{\mathcal{M}}}$$

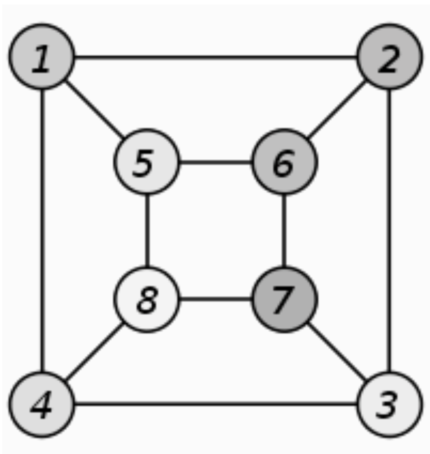
Here $[\![P]\!]^*_{f^{\mathcal{M}}}$ denotes the minimal closure of the denotation of P in $\mathcal{M}$ under $f^{\mathcal{M}}$. [Hint: i) Note that an alternative characterization of the minimal closure of X under f is $X^*_f = \{y \mid y \in X \vee \exists x[x \in X \wedge y = f^n(x)$ for some $n \in \mathbb{N}\}$ where $f^n(y)$ denotes the result of applying f to y n times; ii) use the compactness theorem.]

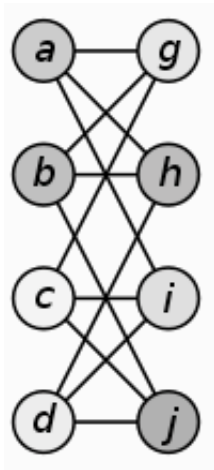
(3) LS 3.2.8

(4) LS 3.2.10

(5) LS 3.2.13

(6) Construct an isomorphism between the following two graphs (G_1 and G_2):





(7) $\star$ Recall that the following set of axioms Γ characterizes dense linear orders without endpoints:
i) $\forall x[\neg x < x]$ (non-reflexity)
ii) $\forall x \forall y \forall z[(x < y \wedge y < z) \rightarrow x < z]$ (transitivity)
iii) $\forall y \forall y \forall z[x < y \vee y < x \vee x = y]$ (linearity)
iv) $\forall x \exists y[y < x]$ (no left end point)
v) $\forall x \exists y[x < y]$ (no right end point)
vi) $\forall x \forall y[x < y \rightarrow \exists z[x < z \wedge z < x]]$ (density)

Show that $\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ consisting of the rationals and normal order on $\mathbb{Q}$ satisfies Γ. Show that $\mathcal{Q}$ is the *unique* countable model of Γ up to isomorphism – i.e. that for every countable model $\mathcal{M}$, if $\mathcal{M} \models \Gamma$, then $\mathcal{M} \cong \mathcal{Q}$.