Asymmetric Monetary Policy and the Yield Curve

Vineer Bhansali\textsuperscript{a}, Matthew P. Dorsten\textsuperscript{a}, and Mark B. Wise\textsuperscript{b}

\textsuperscript{(a)} PIMCO, 840 Newport Center Drive, Suite 300
Newport Beach, CA 92660

\textsuperscript{(b)} California Institute of Technology, Pasadena, CA 91125

Abstract

We discuss the Taylor rule near low inflation and interest rates. Using an additional option-like term in the Federal Reserve’s loss function (i.e., the “deflation put”) we extend the classic Taylor rule to one with an asymmetric response that is more accommodative when the inflation rate is very low. Once calibrated, this payoff profile gives an exact, and easily communicable prescription for Federal Reserve policy under regimes of low inflation. Simple models of central bank behavior can produce highly complex yield curve shapes. Using the usual Taylor rule and our proposed extension as building blocks, we construct a robust framework for generating realistic yield curves and the evolution of the economy. Our main focus is the impact on the yield curve and the economy of the “deflation put”. We find that for economies like the U.S. the deflation put reduces yields for all maturities. We also find that in highly leveraged economies (such as Japan) the consequence of an asymmetric deflation fighting policy may result in improved economic conditions, but also raises the possibility of higher long term yields as a consequence.
1 Introduction

Traditional models for the yield curve are based on essentially an arbitrage-free statistical approach. After assuming reasonable stochastic processes for the key underlying variables, such models are calibrated using a combination of historical and cross-sectional fits to observed yield curves. While these models provide an elegant and computationally effective way to explain the yield curve, as well as tools to exploit relative value opportunities, they lack in deeper economic underpinnings. On the other hand, pure macroeconomic models attempt to explain the behavior of the yield curve based on macroeconomic aggregates such as GDP, inflation etc. In doing so, they do not use the information implicit in the prices of traded securities that make up the yield curve, and the forecasts that they make for individual yields are not related by arbitrage constraints.

In this paper, we combine the macroeconomic approach and the arbitrage-free approach by going back to the fundamental building block - the short rate, and the behavior of the Central Bank, which drives the short rate via the so-called Taylor rule. In this setting, the short rate is driven by a set of policy rules that link macroeconomic variables and the short rate together. By stitching together the short rate for future periods, we are then able to compute the yield for any maturity. The power of our approach is in allowing us to run numerous experiments by changing the monetary policy rule based on economic factors, and evaluating the impact of the change on the whole yield curve. The focus of our paper is to explore the behavior of the yield curve as a consequence of possible asymmetries in the monetary policy rule.

At Jackson Hole in 2003, Federal Reserve Governor Janet Yellen elegantly stated the need for pre-emptive, asymmetric response when nominal rates and inflation are very low [29]. She argued for a non-linear policy that would call for a central bank to lower its interest rate more rapidly toward zero and hold it at a low level for longer than the classic Taylor rule [25] would suggest:

The research shows that it is important to have a “cushion” in the inflation target to minimize the deterioration in macroeconomic performance due to the “zero bound” problem. For the United States, research suggests that the cushion of at least 1 percent (on top of measurement bias) is needed to avoid significant deterioration in macroeconomic performance, while a 2 percent cushion virtually eradicates economic problems relating to the influence of the zero bound. A larger inflation buffer becomes especially desirable if there is good reason to think that a “neutral real fed funds rate” is particularly low, as it might be in a post bubble economy, so that the odds of hitting the zero-bound are high. (Janet Yellen, Jackson Hole Meeting, 2003.)

It has also been pointed out that while linear response rules such as the Taylor rule have worked very well, the Federal Reserve’s “risk-management” approach was essential to countering large, negative shocks posing serious asymmetric risks [29]. It is generally accepted that the Federal Reserve has done a very good job of dynamically updating its targets for the structural constants in the Taylor rule - such as the equilibrium real rate and the target inflation rate. But most empirical estimates assume the symmetry of the Taylor
rule, and use the simple linear Taylor rule that is obtained by minimizing a quadratic loss function of inflation and output gaps and an inertial term in interest rate changes.

Nominal funds rates and inflation in the U.S. have recently approached the zero bound. Real rates have also fallen below zero on more than one occasion. With very low levels of interest rates recently seen in Japan, Switzerland and the U.S., there has been debate on policy actions when the nominal rate is close to the zero lower bound. As Federal Reserve Chairman Ben Bernanke has remarked “...policymakers are well advised to act preemptively and aggressively to avoid facing the complications raised by the zero lower bound.” [4].

Putting the views of Janet Yellen and Ben Bernanke together, it appears logical to explore the deflation avoidance problem with a modification of the Taylor rule that is both preemptive and asymmetric. At Jackson Hole in August of 2005, Alan Greenspan explicitly admitted the risk-management nature of policy under his Federal Reserve leadership [12]:

In effect, we strive to construct a spectrum of forecasts from which, at least conceptually, specific policy action is determined through the tradeoffs implied by a loss-function. In the summer of 2003, for example, the Federal Open Market Committee viewed as very small the probability that the then-gradual decline in inflation would accelerate into a more consequential deflation. But because the implications for the economy were so dire should that scenario play out, we chose to counter it with unusually low interest rates.

The product of a low-probability event and a potentially severe outcome was judged a more serious threat to economic performance than the higher inflation that might ensue in the more probable scenario. Moreover, the risk of a sizable jump in inflation seemed limited at the time, largely because increased productivity growth was resulting in only modest advances in unit labor costs and because heightened competition, driven by globalization, was limiting employers' ability to pass through those cost increases into prices. Given the potentially severe consequences of deflation, the expected benefits of the unusual policy action were judged to outweigh its expected costs.

(Alan Greenspan, Jackson Hole Meeting, 2005.)

Under Alan Greenspan’s oft-quoted approach of “risk-management” for monetary policy, we are tempted to cast the problem in terms of a monetary policy “put option” (the put option being one where the underlying is inflation rate, and the strike the threshold at which policy becomes asymmetric, e.g. the 1% level of inflation in Janet Yellen’s speech). Options markets are well known to provide insurance for risk-managers who are willing to part with some premium to avert the consequences of disastrous yet severe low probability events.

In this paper we show that such a dynamic extension of the Taylor rule can be derived from first principles using an intuitively appealing modification of the usual quadratic central bank loss function. The modified loss function contains an additional term analogous to the price of a put option on future inflation and it is derived using some simple assumptions in Section 3 of this paper. Numerous variations of the Taylor rule have been proposed in the literature to account for country specific monetary policy dynamics [16, 23, 19]). Also, various mechanisms for escaping the zero bound have been proposed [9, 24, 3]). Econometric estimates of [5, 14] have demonstrated that structural shifts in the coefficients of the
Taylor rule have occurred in the past, coinciding with the change in the Federal Reserve chairmanship. More recently, [17], it has argued that because of the so called “Greenspan put”, market participants can benefit disproportionately by availing themselves of cheap insurance embedded in the yield curve.

A major goal of this paper is to explore the consequences of our proposed modification of the Taylor rule on the yield curve and the development of the economy. Considerable effort has recently been devoted to including macroeconomic information in term structure modeling. Much of this work has focused on improving our understanding of the economic determinants of the level and shape of the yield curve. Whereas some research has included the effects of macro variables without including macroeconomic structure (see for example, Refs. [1, 2, 28, 6, 15, 18, 8, 10, 22, 7]), we take a more macroeconomical approach and introduce a simple model of the economy involving the short-term interest rate, inflation, and output gap. This approach is similar to Refs. [20, 13].

We us a model for the economy where output gap and inflation evolve forward in time via coupled random walks roughly corresponding to the Phillips curve and the IS equations. Regressions on historical data for inflation, output gap, and the Fed funds rate are used to determine the parameters of these equations. We assume a Federal Reserve policy that chooses the short rate in terms of inflation and the output gap. The solution to the resulting set of equations provides the stochastic evolution of the interest rate for all future times and may be used to compute risk-free yields for any maturity by computing expectations.

We consider two different Federal Reserve policies. One is the standard Taylor rule, which gives a simple linear relation between the Fed funds rate, output gap, and inflation. The parameters in the standard Taylor rule are fit to historical data. We are able in this case to derive an approximate analytic formula for Treasury yields for all maturities that agrees well with our numerical results. This Treasury yield model is similar in spirit to some of the recent literature (e.g., Ref. [22]). The other is our proposed modification of the Taylor rule, which has the Federal Reserve responding more aggressively when inflation is unusually low. We find that for the U.S. economy this “deflationary put” decreases longer-term treasury yields for most choices of parameters (note that it always decreases the short rate, given the same inflation and output gap). We also consider an economy where inflation and output gap are more sensitive to the policy rate (i.e., a “leveraged” economy). In such a case the deflationary put term can increase long-term yields. Furthermore, numerical simulations indicate that during periods in which the standard Taylor rule would lead to negative inflation and output gap, the deflationary put term increases these quantities. The difference between the yield curves with the same underlying dynamics points out the importance of estimating the form and correct coefficients for the Taylor rule used by a policy setter.

The methods we develop can be extended to the case where the Federal Reserve also responds more aggressively than the Taylor rule to unusually large inflation, i.e., “an inflation call”. We derive a loss function that gives rise to a modified Taylor rule with an inflation call and briefly explore its impact on treasury yields.

In the next section, we detail our model for the economy and review the derivation of the standard Taylor rule by minimizing a loss function that is quadratic in inflation and output gaps and contains an inertial term in interest rate changes. Some of the details are presented in Appendix 1. Appendix 2 discusses an analytic approximation to the Treasury
yield based on the model discussed in Section 2. The modified Taylor rule is introduced in Section 3. Section 4 details the fits to the parameters from historical data, and in Section 5 we use these results to calculate the yield on Treasury bonds, both for the standard and modified Taylor rules. Section 6 describes the highly leveraged economy and demonstrates the benefits of the deflationary put. In Section 7 we return to the U.S. economy and discuss the impact of an inflation call on treasury yields. Brief concluding remarks are given in Section 8.

2 Loss function minimization and the Taylor rule

The economy is described by two stochastic differential equations that give the evolution of inflation $\pi(t)$ and output gap $y(t)$ with time $t$ (loosely the Phillips curve and IS equations),

\[
\begin{align*}
\frac{d\pi(t)}{dt} &= \mu_\pi dt + \alpha_1 \pi(t) dt + \alpha_2 y(t) dt + \sigma_\pi dw_\pi, \\
\frac{dy(t)}{dt} &= \mu_y dt + \beta_1 y(t) dt - \beta_2 (i(t) - \pi(t)) dt + \sigma_y dw_y.
\end{align*}
\] (1)

Here $w_\pi$ and $w_y$ are standard Brownian motions and $i(t)$ is the short rate adjusted by the federal reserve at discrete times $t_0$ separated by intervals $\Delta t$. Hence if $t_0$ denotes the present time then $i(t) = i_0$, for $t_0 < t < t_1 = t_0 + \Delta t$, $i(t) = i_1$, for $t_1 < t < t_2 = t_1 + \Delta t$, etc. The coefficients $\alpha_1$ and $\beta_1$ are negative so the random walks for inflation and output gap are mean reverting to values determined in part by the drifts $\mu_\pi$ and $\mu_y$. The coefficient $\alpha_2$ is positive since a positive output gap is a driver of inflation and similarly the coefficient $\beta_2$ is positive since a Federal Reserve funds rate less than the inflation rate increases the output gap. Using the evolution determined by Eq. (1) the inflation and output gap at any time $t > t_0$ are determined by the initial values $\pi(t_0), y(t_0)$, the Fed funds rate $i(t)$ for $t_0 < t < t_1$, the constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_\pi, \mu_y$ and the standard deviations $\sigma_\pi$ and $\sigma_y$ of inflation and output gap respectively. A discrete formulation for the evolution of inflation and output gap very similar to Eqs. (1) has been used in Ref. [27].

At time $t_0$ the Federal Reserve adjusts the short rate $i_0$, and the possible future values of the short rate $i_j$, $j = 1, \ldots$, in an attempt to minimize the loss function,

\[
L_{\text{Taylor}} = \frac{1}{2} E_{t_0} \left[ \sum_{n=0}^{\infty} \gamma e^{-d' n \Delta t} (i_{n+1} - i_n)^2 + \int_{t_0}^{\infty} dt e^{-d'(t-t_0)} ((\pi(t) - \pi^*)^2 + \lambda y(t)^2) \right]. \quad (2)
\]

The first term in the loss function $L$ is an inertia term which expresses the Fed's aversion to making changes in the short rate. The last two terms reflect the Fed's desire to simultaneously have inflation at the targeted value $\pi^*$ and the output gap minimized. The constant $d'$ is a measure of how much the Federal Reserve discounts the impact of events in the future. For example, it cares more about inflation next year than twenty years from now. With $d' = 0$ the integral over time in the loss function converges only if $\alpha_1 \beta_1 - \alpha_2 \beta_2 > 0$. If this condition is not satisfied a large enough discount factor constant, $d'$, will make the time integration converge.

This coupled set of stochastic differential equations can be integrated formally without knowing the functional form of $i(t)$. To simplify notation we use a two-element column vector $\mathbf{v}(t)$ with first element equal to inflation $v_1(t) = \pi(t)$ and second element equal
to output gap $v_2(t) = y(t)$. We also introduce the two-by-two matrix $M$ with elements $M_{11} = \alpha_1$, $M_{12} = \alpha_2$, $M_{21} = \beta_2$ and $M_{22} = \beta_1$, and the vector $ds(\tau)$ with components $ds_1(\tau) = \sigma_\pi dw_\pi(\tau)$ and $ds_2(\tau) = \sigma_y dw_y(\tau)$. In terms of these quantities the solution is the vector

$$
v(t) = e^{M(t-t_0)}v(t_0) + \left(e^{M(t-t_0)} - 1\right)M^{-1}\mu - \beta_2 \int_{t_0}^t \mu(\tau) e^{M(t-\tau)} n \, d\tau + \int_{t_0}^t e^{M(t-\tau)} ds(\tau), \quad (3)
$$

where $n$ is the unit vector with components $n_1 = 0$ and $n_2 = 1$, and $\mu$ is a vector with components $\mu_1 = \mu_\pi$ and $\mu_2 = \mu_y$. For the remainder of this paper we set $t_0 = 0$.

The values of $i_j$ are determined by minimizing the loss function with respect to them, i.e., by solving the equations $dL/di_k = 0$. We find this solution in Appendix 1. It has the form,

$$
\sum_{j=0}^\infty X_{kj}i_j + Y_k^\pi \pi(0) + Y_k^y y(0) + Z_k = 0
$$

where the quantities $X_{kj}$, $Y_k^\pi$, $Y_k^y$, and $Z_k$ are independent of $i_0$, $\pi(0)$ and $y(0)$ and explicit expressions for them are given in Appendix 1. Multiplying by the inverse of the matrix $X$ and looking at the equation for $i_0$ gives the Taylor rule

$$
i_0 + \sum_{k=0}^\infty X_{0k}^{-1} Y_k^\pi \pi(0) + \sum_{k=0}^\infty X_{0k}^{-1} Y_k^y y(0) + \sum_{k=0}^\infty X_{0k}^{-1} Z_k = 0
$$

(4)

For this paper explicit expressions for the inverse of $X$, for $Y$ and for $Z$ are not needed. However it is interesting to note that in the limit of very large $\gamma$ simple explicit expressions for the coefficients in the Taylor can be derived in terms of the parameters of the economic model in Eq. (1) and the loss function. In the limit $\gamma \to \infty$ all the future $i_j$’s to equal $i_0$ in order to minimize the loss function. In other words the Federal Reserve is choosing the short rate that if held constant minimizes the above loss function. Of course, even with this simplifying assumption, the actual Federal Reserve funds rate does not stay constant since at each meeting they must recompute the above loss function with new input data for inflation and output gap which may give rise to a different $i_0$ that minimizes the loss function.

We can rewrite Eq. (5) in the more familiar form,

$$
i_0 = r^* + \theta_1(\pi(0) - \pi^*) + \theta_2 y(0) + \pi(0) = \hat{c} + \hat{\theta}_1 \pi(0) + \theta_2 y(0),
$$

where

$$
\hat{c} = r^* - \theta_1 \pi^*, \quad \hat{\theta}_1 = \theta_1 + 1.
$$

(7)

Typically, Eq. (6) is used as the starting point for empirical analyses to which we will return in a later section. Since the Federal Reserve funds rate cannot be negative if the solution to Eq. (5) gives a negative $i_0$ then the appropriate Federal Reserve funds rate is actually zero. Hence the Taylor rule, which gives the short rate at any time $t$ in terms of inflation and output gap at that time, is,

$$
i(t) = \text{Max}[\hat{c} + \hat{\theta}_1 \pi(t) + \theta_2 y(t), 0].
$$

(8)
The treasury yield $Y(T)$ is determined by compounding this short rate. Explicitly,

$$\text{Exp}(-TY(T)) = E_0 \left[ \text{Exp} \left( - \int_0^T \text{d}\tau [i(\tau) + i_r(\tau)] \right) \right].$$

(9)

In Eq. (34) we have introduced a risk premium factor, $i_r(t)$. We take it to obey a simple deterministic evolution, $di_r = \mu_r \text{d}t$, which, together with the initial condition $i_r(0) = 0$, gives the simple linear form $i_r(t) = \mu_r t$. It is straightforward to use a more complicated evolution for the risk premium including effects such as its correlation with inflation. Note that introducing the risk premium in this way does not allow arbitrage opportunities in portfolios constructed from investments in Treasury bonds of different maturities.

Neglecting the max function\(^1\) in the Taylor rule (see, Eq. (8)) and making the approximation that the Federal Reserve resets the short rate continuously in time allows us to derive relatively simple analytic formulas for the yield curve that provide intuition without the need for numerical simulations. This analytic expression for the yield curve is derived in Appendix 2. It has the same form as the two-factor version of the Vasicek model [26].

### 3 A risk management strategy for a precommitted deflation fighting Federal Reserve

We propose that in order to fight deflation the Federal Reserve acts more aggressively than the classic Taylor rule of Eq. (6) when inflation is less than some critical value. In other words, the Federal Reserve’s loss function now has an additional risk-management term which penalizes deviations below the critical inflation rate $\pi_{\text{min}}$. This is modeled by having the Federal Reserve minimize a new loss function,

$$L = L_{\text{Taylor}} + L_{\text{Deflation}},$$

(10)

where $L_{\text{Deflation}}$ is the term that reflects the Federal Reserve’s aversion to dangerously low inflation rates.

$$L_{\text{Deflation}} = E_0 \left[ \frac{1}{\epsilon} \text{Max}[\pi_{\text{min}} - \pi_{\text{avg}}, 0] \right].$$

(11)

To make the model more realistic, we assume that the Federal Reserve is managing the risk of sustained deflation and responds not to one low inflation print, but to a time-average $\pi_{\text{avg}}$:

$$\pi_{\text{avg}} = d \int_0^\infty \text{d}t \ e^{-dt} \pi(t).$$

(12)

The constant $1/d$ characterizes the time interval over which inflation is averaged. Note that we do not insist that $d = d'$. The parameter $\epsilon$ characterizes how aggressive the Federal Reserve will be to deflation. For small $\epsilon$ the Federal Reserve is very averse to situations where inflation (averaged over roughly a time period $1/d$) falls significantly below the value

\(^1\)In a subsequent section of this paper we show that this approximation is quite good for reasonable volatilities.
\( \pi_{\text{min}} \). \( \pi_{\text{avg}} \) is a normal random variable and formulas like Eq. (11) are familiar from option pricing as the payoff for a put option. We will call it the Federal Reserve’s deflationary put.

The loss function \( L_{\text{Taylor}} = E_0[\mathcal{L}_{\text{Taylor}}] \) where,

\[
\mathcal{L}_{\text{Taylor}} = \frac{1}{2} \left( \sum_{n=0}^{\infty} \gamma e^{-\Delta t (i(t_{n+1}) - i(t_n))^2} + \int_0^\infty d\tau e^{-d\tau \left( \left( \pi(t) - \pi^* \right)^2 + \lambda y(t)^2 \right)} \right). \tag{13}
\]

There is nothing unique about \( \mathcal{L}_{\text{Taylor}} \). However, it is the simplest possible choice and gives a linear relationship between \( i_0 \) and \( \pi(0) \) and \( y(0) \). Adding higher powers of inflation and output gap to \( \mathcal{L}_{\text{Taylor}} \) would lead to a non-linear relationship which would presumably be harder to implement by a pragmatic Federal Reserve. Furthermore the additional terms would be less important when output gap is near zero and inflation is near its target value. Similarly for a Federal Reserve that responds more aggressively when expected future inflation averaged over some time horizon is unusually low, we want to add a term \( L_{\text{Deflation}} = E_0[\mathcal{L}_{\text{Deflation}}(\pi_{\text{avg}})] \), where \( \mathcal{L}_{\text{Deflation}}(\pi_{\text{avg}}) \) is non zero only for \( \pi_{\text{avg}} < \pi_{\text{min}} \). Hence \( \mathcal{L}_{\text{Deflation}}(\pi_{\text{avg}}) \) must be proportional to \( \theta(\pi_{\text{min}} - \pi_{\text{avg}}) \), where \( \theta(x) \) is equal to one if \( x > 0 \) and is zero for \( x < 0 \). The loss function in Eq. (11), with

\[
\mathcal{L}_{\text{Deflation}}(\pi_{\text{avg}}) = \frac{1}{\epsilon} \max[\pi_{\text{min}} - \pi_{\text{avg}}, 0], \tag{14}
\]

is the simplest one of this form that is continuous in \( \pi_{\text{avg}} \). One could multiply this by an arbitrary function of \( f(\pi_{\text{avg}}) \). However if the Federal Reserve is successful in preventing deflation then typically \( \pi_{\text{avg}} \) will not drop much below \( \pi_{\text{min}} \) and it is a reasonable approximation to replace this function by the constant \( f(\pi_{\text{min}}) \) which is then absorbed into the value of \( \epsilon \).

The first derivative of \( \mathcal{L}_{\text{Deflation}}(\pi_{\text{avg}}) \) with respect to \( \pi_{\text{avg}} \) is discontinuous at \( \pi_{\text{min}} \). For small \( \sigma_{\pi_{\text{avg}}} \), this discontinuity results in a modification of the Taylor rule in Eq. (6) so that the value of \( r^\ast \) drops sharply as \( \pi_{\text{avg}} \) falls below \( \pi_{\text{min}} \). Note that \( \pi_{\text{avg}} \) depends on both the initial value of inflation \( \pi(0) \) and output gap \( y(0) \). For longer averaging times, \( 1/d \), it is situations where both these are small that are more likely to give rise to dangerously low values of \( \pi_{\text{avg}} \).

Since \( \pi_{\text{avg}} \) is normally distributed, we can evaluate \( L_{\text{Deflation}} \) in terms of \( \pi_{\text{min}} \) and the mean and variance of \( \pi_{\text{avg}} \). The result for \( L_{\text{Deflation}} \) is

\[
\frac{\sigma_{\pi_{\text{avg}}}}{\sqrt{2\pi}} \exp \left( -\frac{(E_0[\pi_{\text{avg}}] - \pi_{\text{min}})^2}{2\sigma_{\pi_{\text{avg}}}^2} \right) - \left( \frac{E_0[\pi_{\text{avg}}] - \pi_{\text{min}}}{2} \right) \text{erfc} \left( \frac{E_0[\pi_{\text{avg}}] - \pi_{\text{min}}}{\sqrt{2\sigma_{\pi_{\text{avg}}}^2}} \right), \tag{15}
\]

where the complementary error function is defined by

\[
\text{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty dx e^{-x^2}. \tag{16}
\]

The formula in Eq. (3) allows us to write

\[
\sigma_{\pi_{\text{avg}}}^2 = \frac{d \left[ \sigma_y^2 (\beta_1 - d)^2 + \sigma_y^2 \sigma_\epsilon^2 \right]}{2 D^2} \tag{17}
\]
and
\[ E_0[\pi_{avg}] = \frac{-\alpha_2 \beta_2 i_{avg} + d(d - \beta_1)\pi(0) + d\alpha_2 y(0) + (d - \beta_1)\mu_\pi + \alpha_2 \mu_y}{D}, \tag{18} \]
where \( i_{avg} \) is defined as \( \pi_{avg} \) is above. Note the right-hand side of the last equation depends on \( i_{avg} \) and not some other function of \( i(t) \) because of the exponential averaging defined in Eq. (12). The factor in the denominator is \( D = \det(M - d) = (\alpha_1 - d)(\beta_1 - d) - \alpha_2 \beta_2 \).

The standard Taylor rule of Eq. (6) is generally applicable when inflation is near its target value and output gap is near zero; this will occur most of the time under the regime of a responsive Federal Reserve. Similarly the modification of the Taylor rule implied by adding to the loss function the expression in Eq. (11) is a general result based on a simple power series expansion under the further assumption that the Federal Reserve acts more aggressively when expected average inflation drops below a critical value. It is a modification of the structure of the Taylor rule away from target inflation and output gap regimes.

The Fed sets the short rate today, \( i_0 \), by minimizing the total loss function with respect to the \( i_k \) (see section two). The contribution of the deflationary put to this minimization can now be calculated from Eq. (15):
\[ \frac{dL_{Deflation}}{di_k} = \frac{\alpha_2 \beta_2}{D \epsilon} \left( \frac{di_{avg}}{di_k} \right) \mathcal{P}(\pi_{avg} < \pi_{min}) = \frac{\alpha_2 \beta_2}{2D \epsilon} \left( \frac{di_{avg}}{di_k} \right) \text{erfc} \left( \frac{E_{l_0}[\pi_{avg}] - \pi_{min}}{\sqrt{2\sigma^2_{\pi_{avg}}}} \right), \tag{19} \]
where \( \mathcal{P}(\pi_{avg} < \pi_{min}) \) denotes the probability that \( \pi_{avg} \) is less than \( \pi_{min} \). Hence, the deflationary put modifies the Taylor rule to
\[ i_0 = \hat{c} + \theta_1 \pi(t_0) + \theta_2 y(t_0) + \frac{\alpha}{2} \text{erfc} \left( \frac{E_{l_0}[\pi_{avg}] - \pi_{min}}{\sqrt{2\sigma^2_{\pi_{avg}}}} \right), \tag{20} \]
where,
\[ \alpha = -\sum_{k=0}^{\infty} X^{-1}_{0k} \left( \frac{\alpha_2 \beta_2}{D \epsilon} \right) \left( \frac{di_{avg}}{di_k} \right), \tag{21} \]
is negative.

In the limit \( d \to \infty, i_{avg} \to i_0 \) and so at order \( 1/d^2 \) the mean of \( \pi_{avg} \) has \( i_{avg} \) replaced by \( i_0 \). We find this to be an excellent approximation for the values of the model parameters we use, even for \( d \) as low as 2 yr\(^{-1}\). Using this fact, we will replace \( i_{avg} \) with \( i_0 \) in the expression for \( \pi_{avg} \).

The modified Taylor rule in Eq. (20) is a transcendental equation for \( i_0 \), and if its solution is negative, the short rate is set to zero. In practice, the coefficient of \( i_0 \) in \( E_0[\pi_{avg}] \) is small enough that we can set it to zero without significantly changing the value of \( i_0 \) implied by Eq. (20). For instance, using our estimates of the parameters from the next section, \( \alpha_1 = -0.54, \alpha_2 = 0.27, \beta_1 = -0.27, \) and \( \beta_2 = 0.19 \), the coefficient of \( i_{avg} \) is 0.01, while the coefficient of \( \pi(t_0) \) is 0.8. The other coefficients are significantly larger than 0.01 as well. We make this simplification in simulating the yield curves below, but we estimate
the value of $\alpha$ from historical data using the full form of Eq. (20) since the transcendental equation poses no difficulty when the short rate is given by data.

Neglecting the $i_0$ dependence of $E_0[\pi_{\text{avg}}]$, the partial derivative of $i_0$ with respect to $\pi(0)$ becomes,

$$\frac{\partial i_0}{\partial \pi(0)} = \hat{\theta}_1 - \frac{\alpha}{\sqrt{2\pi \sigma^2_{\pi_{\text{avg}}}}} e^{-\frac{(\pi(0) - \pi_{\text{min}})^2}{2\sigma^2_{\pi_{\text{avg}}}}}.$$  

(22)

When $E_0[\pi_{\text{avg}}]$ is near $\pi_{\text{min}}$ (i.e., $\pi(0) - \pi_{\text{min}} \approx 0$) the exponential is near unity. The modification in the Taylor rule is important when $E_0[\pi_{\text{avg}}]$ is near $\pi_{\text{min}}$ and it increases $\partial i_0/\partial \pi(0)$ which corresponds to the Federal Reserve reacting more aggressively to changes in interest rates. Thus the deflation put increases the magnitude of the response function asymmetrically, near the critical threshold, as dictated by the “delta” of the option. When $\sigma_{\pi_{\text{avg}}}$ is large, i.e. there is uncertainty in the value of inflation (which could be due to external shocks such as energy prices, or due to the commitment of the Federal Reserve to its mandates), the magnitude of the correction term decreases and the deflation fighting term becomes less effective.

4 Parameter estimates from historical data

The parameters $\mu_{\pi,y}$, $\alpha_{1,2}$, and $\beta_{1,2}$ of the economic model described in Eqs. (1) are estimated from historical inflation, effective Federal Reserve funds rate, and GDP data from 1954Q3 to 2005Q4 using ordinary least squares. The inflation and GDP figures are quarterly, and the Federal Reserve funds data are monthly figures averaged quarterly. Inflation rates are computed as the annualized quarterly percent change in the seasonally adjusted
PCE chain-type price index published by the BEA. The effective Federal Reserve funds rates are taken from the Federal Reserve H.15 release. The output gap is constructed as the fractional deviation of real GDP (at a seasonally adjusted annual rate, from the BEA) from real potential GDP as estimated by the CBO. The parameters are estimated from the data by regressing $\pi(t + \Delta t)$ against $\pi(t)$ and $y(t)$ and $y(t + \Delta t)$ against $y(t)$ and $[i(t) - \pi(t)]$. The results are shown in Table 1. Note that we use annualized units throughout this paper.

<table>
<thead>
<tr>
<th>Period</th>
<th>$\mu_\pi$ (in%)</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$r^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1954Q3-2005Q4</td>
<td>2.1[1.1, 3.0]</td>
<td>-0.54[-0.8, -0.3]</td>
<td>0.27[0.04, 0.5]</td>
<td>0.76</td>
</tr>
</tbody>
</table>

Table 1: Macroeconomic model parameters estimated from historical data. The 90% confidence intervals are shown in brackets.

The Taylor rule parameters of Eq. (5) are estimated from recent data, 1987Q3 to 2005Q4. We do not use the entire postwar sample because it is likely the parameters of the Taylor rule have changed over that period. Using data from just the Greenspan era should be more consistent. For a discussion of this issue, see Refs. [11, 14, 5, 21]. For this fit we define inflation as the year-on-year change in the PCE deflator. This is less volatile than the annualized quarterly change and is perhaps a better approximation to what the Federal Reserve watches. This introduces a slight inconsistency into our approach, since this definition of inflation is not identical to that used in the macroeconomic model, but it improves the fits and should not affect the results significantly. The results are shown in Table 2, and the predicted response of our model is compared with the data in Figure 1.

<table>
<thead>
<tr>
<th>Period</th>
<th>$\hat{c}$ (in%)</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$r^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1987Q3-2005Q4</td>
<td>1.9[1.3, 2.5]</td>
<td>1.3[1.1, 1.5]</td>
<td>0.86[0.72, 0.99]</td>
<td>0.75</td>
</tr>
</tbody>
</table>

Table 2: Estimates of the parameters of the standard Taylor rule. The 90% confidence intervals of the estimates are shown in brackets.

The parameter $\alpha$ of Eq. (20) is fit from very recent data, 2000Q1 to 2005Q4, with the standard Taylor rule parameters fixed. This fit depends on the values of all the other parameters in our model since the put term does. We take $\sigma_\pi = \sigma_y = 1.5\%$ and $d = 2$, and set the parameters in Tables 1 and 2 to their central values, except for $\mu_\pi$ and $\mu_y$. We fix the inflation drift parameter at the lower end of its 90% confidence interval, $\mu_\pi = 1.1\%$, since its estimate is biased by the high inflation of the 1970s and 80s. A smaller value is more appropriate today. The output gap drift parameter is taken to be 0.48\% because this value causes $y(t)$ to mean revert to zero, instead of the slightly negative value implied by the best-fit value of $\mu_y$.

The fit results are shown in Table 3. Note the small $r^2$ values of the fits. In part these reflect the difficulty of estimating $\alpha$ from data in which inflation is rarely below $\pi_{\min}$. But they are artificially low because of our fitting method. We regress $i - \hat{c} - \theta_1 \pi - \theta_2 y$ against
Table 3: Estimates of put term $\alpha$ for two different values of $\pi_{\text{min}}$. The 90% confidence intervals are shown in brackets. The parameters of the standard Taylor rule are fixed to their central values in Table 2. The macroeconomic model parameters are fixed to their central values as well, with the exception of $\mu_\pi = 1.1\%$ and $\mu_y = 0.48\%$. The volatilities are chosen to be $\sigma_\pi = \sigma_y = 1.5\%$ and the averaging parameter is $d = 2$.

The erfc function in Eq. (20). This removes most of the correlation between the left-hand and right-hand sides of Eq. (20), and the $r^2$ values are naturally lowered.

5 Sample yield curves

In this section, we compare yield curves generated with the evolution specified by Eqs. (1) and the standard Taylor rule with ones assuming the modified Taylor rule. The evolution equations are solved and the yields calculated numerically with the max function included in the Taylor rules (so that the Federal Reserve funds rate does not become negative). The evolution equations are solved using the explicit Euler method, and the interest rate is integrated to get the yield using the extended trapezoid rule. Our step size is one quarter, $dt = 0.25$. The expectation in the yield formula is calculated by averaging over $10^5$ interest rate paths. The values of the parameters used in the yield curve plots are their central values, with the exception of $\mu_\pi$, $\mu_y$, and $\alpha$. The parameters $\mu_\pi$ and $\mu_y$ are taken to be 1.1% and 0.48%, for reasons described above. These parameter choices cause the expected values of inflation, output gap, and the short rate to tend toward 2%, 0%, and 4.5%, respectively. Because the coefficient of the deflationary put is so poorly determined in our fits, we choose it to be $\alpha = -3\%$. We also choose $\pi_{\text{min}} = 1\%$ and $d = 2$.

The plots in Figure 2 each show two sample interest rate paths for given volatilities under the evolution given by Eqs. (1) and the standard Taylor rule. The relatively large volatilities make it impossible to extract quantitative information from single paths, but the plots illustrate some of the variety of possibilities in our model. In particular, the interest rate occasionally reaches the zero bound and gets stuck for some period of time. This happens more when the volatilities are large.

Yield curves for various starting values of inflation, $\pi_0$, and output gap, $y_0$, are shown in Figure 3 with the risk premium $\mu_r$ set to zero. The initial Federal Reserve funds rate, $i_0$, is calculated from these assuming either the standard or modified Taylor rule. Note that this means the initial interest rate for the standard Taylor rule evolution is greater than that of the modified rule evolution since the put term always decreases the rate.

For the evolution here, the modified Taylor rule of Eq. (20) is simplified as described above: since the coefficient of $i_0$ in $E_{t_0}[\pi_{\text{avg}}]$ is small relative to the other terms, we can set it to zero without significant effect. This simplifies the evolution considerably, since there is no longer a transcendental equation to solve. Note that in the yield curves in the last
Figure 2: Sample interest rate evolutions for initial Fed funds 4.5% and initial inflation 3.4%. The two interest rate paths in the left plot have $\sigma_\pi = \sigma_y = 1.0\%$, whereas the two paths in the right plot have $\sigma_\pi = \sigma_y = 1.5\%$. The drift parameters are $\mu_\pi = 1.1\%$ and $\mu_y = 0.48\%$, but all other parameters are held fixed at their central values.

row of Figure 3 inflation starts below $\pi_{\text{min}}$. This results in the modified yield curve starting significantly below the one generated with the standard Taylor rule.

One must be careful in interpreting the results of our model for longer-term bonds. The drift terms in our evolution equations contribute more to these yields than to those of shorter-term bonds. This is why all the yield curves plotted in Fig. 3 tend toward 4.5%. This is the value $i(t)$ mean reverts to for the drift values chosen.

Figure 4 shows the effect of changing $d$ and $\alpha$. Note that making $d$ large increases the sensitivity of the deflationary put to temporarily low values of inflation. In the limit $d \to \infty$, the put responds to the instantaneous value of inflation. In practice $d = 4$ is large. The plot for $d = 30$, for instance, would be very similar to the plot for $d = 4$. For small values, $d \lesssim 0.5$, corresponding roughly to averaging over two years or more, the modified yield curve is very similar to the one generated with the standard Taylor rule. This is because inflation does not stay below $\pi_{\text{min}}$ enough to make $\pi_{\text{avg}}$ small. The effect of the put coefficient $\alpha$ is intuitive: making it larger in magnitude results in a proportionately larger modification of the yield curve.

Figure 5 shows the effect of a nonzero risk premium $\mu_r = 0.05\%$. This value increases the yield on the 10-year bond by 25 basis points. Figure 6 shows a plot of the volatility of the Treasury yield as given in Eq. (40) for two different values of the volatilities in the evolution equations. Note this plot uses the analytic approximation. The yield volatility could be extracted from our numerical computations, but the results would be very similar.

6 Highly leveraged economy at risk of deflation

In this section we move our model parameters away from their fit estimates to explore a hypothetical economy in which $\alpha_2$ and $\beta_2$ are bigger, that is, an economy that responds more dramatically to interest rate movements. Tentative estimates of our model parameters
Figure 3: Yield curves for various values of initial inflation and output gap. In the left column $\sigma_\pi = \sigma_y = 1.0\%$, corresponding to $\sigma_{\pi_{\text{avg}}} = 0.4\%$. The plots in the right column are the same but with $\sigma_\pi = \sigma_y = 1.5\%$, corresponding to $\sigma_{\pi_{\text{avg}}} = 0.6\%$. In all plots, the solid line gives the analytic approximation to the standard Taylor rule evolution. The dashed line is the simulation result including the max function in the standard Taylor rule, and the dotted line is the simulation result for the modified Taylor rule with $\pi_{\text{min}} = 1\%$ and $d = 2$. All parameters are taken to be their central fit values, except for $\mu_\pi = 1.1\%$, $\mu_y = 0.48\%$, and $\alpha = -3\%$. The risk premium is set to zero here, $\mu_r = 0$. 

$\pi_0 = 3.4\%, y_0 = -2.1\%, \sigma_\pi = \sigma_y = 1.0\%$

$\pi_0 = 1.2\%, y_0 = 1.2\%, \sigma_\pi = \sigma_y = 1.0\%$

$\pi_0 = 0.5\%, y_0 = 0.5\%, \sigma_\pi = \sigma_y = 1.0\%$
Figure 4: The effect of varying \( d \) and \( \alpha \) on the bottom-right graph of Figure 3. In the left graph \( \alpha \) is held constant at \(-3.0\%\), and in the right graph \( d \) is held constant at \(2.0\). Note that the scale in these graphs is not the same as in Figure 3.

Figure 5: The effect of a risk premium on the upper-right graph of Fig. 3. The two plots are identical with initial inflation \(3.4\%\) and initial output gap \(-2.1\%\), except that the yield curves in the left plot have \(\mu_r = 0\%\) and those in the right have \(\mu_r = 0.05\%\). The line styles and the values of the other parameters are as in Figure 3, but the scale of the plots is different.

for Japan in the last five years suggest this scenario might be applicable there. The primary motivation for this is to see whether a central bank policy using the deflationary put can lead to increased long-term yields. We find this is indeed possible.

The model in Eq. (1) is stable long term only if \(\det(M) = \alpha_1\beta_1 - \alpha_2\beta_2\) is positive. This can be seen in the formulas in the Appendix, since if this does not hold \(\exp(M)\) contains exponentials of positive quantities diverging as \(t \to \infty\). (This would not be the case without the zero bound in the Taylor rule. Without it, the model is stable as long as \(\alpha_1, \beta_1 < 0\) and \(\theta_1, \theta_2 > 0\), conditions that have strong macroeconomic motivation. With these assumptions, \(\exp(M)\) contains only exponentials of negative or imaginary quantities.) Since we want to consider a model in which \(\alpha_2\) and \(\beta_2\) are large enough to make \(\det(M) < 0\), we must stabilize
the model somehow. We do this by adding nonlinear mean reversion terms:

\[
\begin{align*}
    d\pi(t) &= \mu_\pi dt + \alpha_1 \pi(t) dt - \tilde{\alpha}_1^2 \pi(t)^3 dt + \alpha_2 y(t) dt + \sigma_\pi dw_\pi, \\
    dy(t) &= \mu_y dt + \beta_1 y(t) dt - \tilde{\beta}_1^2 y(t)^3 dt - \beta_2 [i(t) - \pi(t)] dt + \sigma_y dw_y.
\end{align*}
\]  

(23)

These terms prevent inflation and output gap from random walking away to negative infinity. A modified Taylor rule can still be derived in the presence of these terms in the large \(d\) limit. In this limit, the nonlinear terms do not contribute, and the modified Taylor rule is just Eq. (20) with \(E_{t_0}[\pi_{avg}]\) replaced by \(\pi_0\). For simplicity, we still use the expression for \(\sigma_{\pi_{avg}}\) given in Eq. (17), even though it too could be expanded in powers of \(1/d\). This makes little numerical difference for the parameter values we use.

Figure 7 shows two sample paths for this economy. Each column shows one path. Inflation is shown first, then output gap, and then the short rate at the bottom. We choose \(\alpha_2 = 0.57\) and \(\beta_2 = 0.8\). The nonlinear coefficients are taken to be \(\tilde{\alpha}_1 = 0.2\%^{-1}\) and \(\tilde{\beta}_2 = 0.1\%^{-1}\), and the other model parameters are the same as in the previous section. With these coefficients, the long-term trends of expected inflation, output gap, and short rate are approximately 0.75%, -1.25%, and 1.75%, respectively. This small trend value of inflation—below \(\pi_{\text{min}}\)—puts the economy in danger of deflation. Initial inflation is taken to be 0.5% and initial output gap 1.2%. Note this starting value of inflation is below \(\pi_{\text{min}}\); this enhances the effect of the put. These plots demonstrate the effectiveness of the modified Taylor rule. In the first column the effect is more dramatic, but in both sample evolutions, inflation and output gap using the modified rule (the dotted lines) are well above those using the standard Taylor rule (the solid lines). The inflation path in the upper left shows how the modified rule can prevent the economy from getting stuck in a long run of deflation. While the solid line goes negative and stays there for almost twenty years, the dotted line stays slightly above zero. The output gap plot below this shows the same effect. The modified output gap stays closer to zero than the output gap from the standard rule. The interest rate path in the bottom-left plot shows how this is achieved. The interest rate using the modified rule drops faster, allowing it to come back up sooner and higher. The interest rate
Figure 7: Comparison between standard and modified Taylor rule evolution in two sample paths of the highly leveraged economy. Initial inflation is 0.5%, and initial output gap is 1.2%. Each column represents one path of the economy. Inflation is shown first, with the standard evolution the solid line and modified evolution the dotted. In the middle is output gap, and at the bottom is the interest rate evolution. The volatilities are $\sigma_\pi = \sigma_y = 1.0\%$, corresponding to $\sigma_{\pi_{avg}} = 0.4\%$. The model parameters are taken to be their best-fit values, except for $\mu_\pi = 1.1\%$, $\mu_y = 0.48\%$, $\alpha_2 = 0.57$, $\beta_2 = 0.8$, $\tilde{\alpha}_1 = 0.2\%^{-1}$, and $\tilde{\beta}_1 = 0.1\%^{-1}$. The modified rule parameters are $\alpha = -3\%$, $d = 2$, and $\pi_{min} = 1\%$. 
Figure 8: Yield curves for the highly leveraged economy with initial inflation 0.5\% and initial output gap 1.2\%. The dashed lines give yield curves for standard Taylor rule evolution, and the dotted lines give yield curves for modified evolution. The analytic approximation to the standard evolution without the nonlinear terms is shown as the solid lines for comparison. The left plot has $\sigma_\pi = \sigma_y = 1.0\%$, corresponding to $\sigma_{\pi_{\text{avg}}} = 0.4\%$, and the right plot has $\sigma_\pi = \sigma_y = 1.5\%$, corresponding to $\sigma_{\pi_{\text{avg}}} = 0.7\%$. The model parameters are the same as in Fig. 7, and the risk premium, $\mu_r$, is set to zero.

from the standard evolution goes down too slowly, and by the time it hits zero the economy is already in deflation and the central bank can do nothing more. The modified rule can have the effect of bringing the interest rate to zero more than would be the case with the standard rule, but as these plots show the effect on the economy is beneficial.

Figure 8 compares yield curves for this economy assuming the standard Taylor rule with those assuming the modified rule. The analytic approximation without the nonlinear mean reversion terms is shown as well to illustrate that these terms do not unduly affect the results. The left plot has smaller volatilities than the right plot. Notice that inflation starts below $\pi_{\text{min}}$ in these plots. These plots demonstrate that it is possible for the modified Taylor rule to increase long-term yields. The bottom row of Fig. 7 shows why this happens. Under the modified Taylor rule evolution for these initial conditions, the interest rate starts out lower than under the standard evolution, but because the modified rule is more effective at stimulating the economy, the interest rate can jump back up more often and go higher when it does. This builds up over time and has the effect seen in the yield curves. Note the yield curves cross earlier in the plot with lower volatilities. This is because the modified Taylor rule is more effective in this case since the argument of the erfc function in Eq. (20) is larger when the volatilities are lower.

7 Modified Taylor rule with an inflation call

Within the economic model we are considering the Taylor rule ensures that the likelihood of persistent high inflation is low. However, it is still interesting to see what the impact of an aggressively anti-inflation bias of the Fed, as expressed by an inflation "call-option" would do to the shape of the yield curve. Suppose market participants expect that the rate
at which the Fed would fight inflation deviations from a preannounced range will become increasingly aggressive as the limits of the range are violated. The new policy rule is now effectively like an option "collar" position, with both a put on the downside (say at 1% inflation), and a call on the upside (say at 3% inflation). Then, the new loss function is

\[ L = L_{\text{Taylor}} + L_{\text{Deflation}} + L_{\text{Inflation}}, \]

where the new contribution to the loss function is

\[ L_{\text{Inflation}} = E_0 \left[ \frac{1}{\epsilon'} \max[\pi_{\text{avg}} - \pi_{\text{max}}, 0] \right], \]

and \( \pi_{\text{avg}} \) is defined in Eq. (12). For simplicity we use the same time averaging for \( \pi_{\text{avg}} \) in the inflation call as we did in the deflation put. Eq (25) is the same form as the payoff for a call option on average inflation \( \pi_{\text{avg}} \) with "strike price" \( \pi_{\text{max}} \).

Using the same methods as in section 3 the modified Taylor rule becomes,

\[ i_0 = \hat{c} + \hat{\theta}_1 \pi(t_0) + \theta_2 y(t_0) + \frac{\alpha}{2} \text{erfc} \left( \frac{E_0[\pi_{\text{avg}}] - \pi_{\text{min}}}{\sqrt{2\sigma^2_{\pi_{\text{avg}}}}} \right) - \frac{\alpha'}{2} \text{erfc} \left( \frac{\pi_{\text{max}} - E_0[\pi_{\text{avg}}]}{\sqrt{2\sigma^2_{\pi_{\text{avg}}}}} \right), \]

where \( \alpha' = (\epsilon/\epsilon')\alpha \).

We consider the impact of the inflation call term in the modified Taylor rule on the yield curve using the same methods and parameters as in Section 5. Figure 9 is the same as Figure 3, except that a new dot-dashed curve is presented that shows the yield curve resulting from the modified Taylor rule in Eq. (26). For simplicity we have set \( \alpha' = \alpha \). Comparing the dotted curve which follows from the modified Taylor rule in Eq. (20) with the dashed-dot one that follows from the modified Taylor rule that contains the inflation call in Eq. (26) we see that when the initial inflation is high \( \pi_0 = 3.4\% \) the inflation call lowers the slope for maturities less than five years. When inflation is very low initially, \( \pi_0 = 0.5\% \) it has the opposite effect raising the slope. In all cases the inflation call increases the yield for the same fixed initial economic conditions.

8 Concluding remarks

We derived a modification of the standard Taylor rule appropriate to a deflation averse Federal Reserve by adding to the loss function an additional term that resembles the payoff on a put option. The modification of the Taylor rule corresponds to a Federal Reserve that is more accommodative when inflation drops below a critically low value than what the standard Taylor rule would imply. Under the assumption that the usual quadratic loss function is supplemented by a term that vanishes for inflation below a critical value we gave a general argument for its form.

Using a simple model of the economy and Taylor-style policy rules for the short rate we calculated Treasury yields. Our macroeconomic model includes the inflationary pressure of a positive output gap and the tendency of positive real short rates to decrease output gap. These standard macroeconomic facts combined with stabilizing mean reversion terms are the basic ingredients of our economy. Taylor rules giving the short rate in terms of
Figure 9: Yield curves for various values of initial inflation and output gap. In the left column $\sigma_{\pi} = \sigma_y = 1.0\%$, corresponding to $\sigma_{\pi\text{avg}} = 0.4\%$. The plots in the right column are the same but with $\sigma_{\pi} = \sigma_y = 1.5\%$, corresponding to $\sigma_{\pi\text{avg}} = 0.6\%$. In all plots, the solid line gives the analytic approximation to the standard Taylor rule evolution. The dashed line is the simulation result including the max function in the standard Taylor rule, and the dotted line is the simulation result for the Taylor rule with deflation put for $\pi_{\text{min}} = 1\%$ and $d = 2$. The dot-dashed line is the result for the Taylor rule with both deflation put and inflation call with $\pi_{\text{max}} = 3\%$. All parameters are taken to be their central fit values, except for $\mu_{\pi} = 1.1\%$, $\mu_y = 0.48\%$, and $\alpha = \alpha' = -3\%$. 

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output gap and inflation then lead to a complete model for the short rate. The parameters of this model are estimated from historical data from 1954 to 2005. We derived an analytic formula for the yield curve using the reasonable approximation that interest rates rarely hit zero. Numerical simulations of our model confirm these results for a wide range of initial conditions and evolution volatilities.

The main purpose of this paper is to explore the implications of an alternative Taylor rule that includes the effect of a “deflationary put option” to manage the risk associated with sustained low inflation. We consider its effects in two different economies, one similar to the current U.S. economy and one that responds even more quickly to central bank interest rate changes and has a high risk of deflation. In the first economy, we show that the effect of this modified rule on yields can be significant, depending on the current values of inflation and output gap. For some reasonable initial conditions, the modified yield curve can be as much as 50 basis points below the standard one.

For the second type of economy, we show that the benefits of the modified Taylor rule can be significant. It does a much better job than the standard rule of warding off deflation, and it keeps output gap closer to zero than it otherwise would be. We also show that in this economy the modified rule can push long-term yields above what the standard Taylor rule would give. This is simply the result of the effectiveness of the modified Taylor rule: it stimulates the economy and therefore allows the future short rate to be higher than with the standard rule.

We would like to thank Vadim Yasnov for help with the simulations and Richard Clarida and Paul McCulley for numerous enlightening discussions. We would also like to thank Brian Sack for helpful comments.
Appendix 1: Derivation of the Taylor rule

The Taylor rule can be derived by finding the values of \( i_j \) that the Federal Reserve chooses to minimize the loss function in Eq. (2). They are given by the solution to \( dL/di_k = 0 \). The derivatives of \( \pi(t) \) and \( y(t) \) with respect to \( i_k \) are

\[
\frac{dv(t)}{di_k} = \beta_2 \theta(t - k\Delta t)e^{Mt}M^{-1}\left(e^{-M\text{Min}[t,(k+1)\Delta t]} - e^{-Mk\Delta t}\right)n, \tag{27}
\]

where the step function, \( \theta(x) \), is zero for \( x < 0 \) and one for \( x > 0 \). The derivatives with respect to \( i_k \) of inflation rate and output gap do not involve the fluctuating pieces and hence the short rates \( i_k \) are determined by solving the system of linear equations

\[
\frac{dL_{\text{Taylor}}}{di_k} = \gamma \left( (i_k - i_{k+1})e^{-dk\Delta t} + (1 - \delta_{0k})(i_k - i_{k-1})e^{-d(k-1)\Delta t} \right) + \int_0^\infty dt \, e^{-dt} \left( (E_0[\pi(t)] - \pi^*) \frac{d\pi(t)}{di_k} + \lambda E_0[y(t)] \frac{dy(t)}{di_k} \right) = 0, \tag{28}
\]

where \( \delta_{jk} \) denotes the Kronecker delta (It is one if \( j = k \) and zero otherwise.). More explicitly

\[
\sum_{j=0}^\infty X_{kj}i_j + Y^\pi_k \pi(0) + Y^y_k y(0) + Z_k = 0, \tag{29}
\]

where

\[
X_{kj} = \gamma \left[ \delta_{kj} \left( e^{-d'k\Delta t} + (1 - \delta_{0k})e^{-d'(k-1)\Delta t} \right) - \delta_{(j+1)k}e^{-d'j\Delta t} - \delta_{(k+1)j}e^{-d'k\Delta t} \right] \\
+ \int_0^\infty dt e^{-d't} \left( \frac{dv(t)}{di_k} \right)^T \Lambda \left( \frac{dv(t)}{di_j} \right), \tag{30}
\]

\[
Y^\pi_k = \int_0^\infty dt e^{-d't} n^Te^{Mt} \Lambda \left( \frac{dv(t)}{di_k} \right), \quad Y^y_k = \int_0^\infty dt e^{-d't} n^T e^{Mt} \Lambda \left( \frac{dv(t)}{di_k} \right), \tag{31}
\]

and

\[
Z_k = \int_0^\infty dt e^{-d't} \left[ \mu^T (M^{-1})^T \left( e^{Mt} - 1 \right) \Lambda \left( \frac{dv(t)}{di_k} \right) - \pi^* n^T \left( \frac{dv(t)}{di_k} \right) \right]. \tag{32}
\]

Here a superscript \( T \) denotes the transpose, \( n' \) is the two component column vector with components \( n'_1 = 1, n'_2 = 0 \) and \( \Lambda \) is a diagonal \( 2 \times 2 \) matrix that has non zero components \( \Lambda_{11} = 1 \) and \( \Lambda_{22} = \lambda \).

Multiplying by the inverse of the matrix \( X \) and looking at the equation for \( i_0 \) gives the Taylor rule

\[
i_0 + \sum_{k=0}^\infty X_{0k}^{-1} Y^\pi_k \pi(0) + \sum_{k=0}^\infty X_{0k}^{-1} Y^y_k y(0) + \sum_{k=0}^\infty X_{0k}^{-1} Z_k = 0, \tag{33}
\]

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where the quantities $X_{kj}$, $Y^\tau_k$, $Y^\nu_k$, and $Z_k$ are independent of $i_0$, $\pi(0)$ and $y(0)$.

**Appendix 2: Analytic approximation for the Treasury yield**

In this section, we neglect the max function in the Taylor rule (see, Eq. (8)) because it is a good approximation except for large volatilities, and because it allows us to derive relatively simple analytic formulas that provide intuition without the need for numerical simulations. In a subsequent section of this paper we show that this approximation is quite good for reasonable volatilities.

Neglecting the max function in the Taylor rule (see, Eq. (8)) the Treasury yield $Y(T)$ for a maturity $T$ is given by

$$\text{Exp}(-TY(T)) = E_0 \left[ \text{Exp} \left(- \int_0^T d\tau [i(\tau) + i_r(\tau)] \right) \right]$$ (34)

As remarked in the text we take the risk premium to obey a simple deterministic evolution, $di_r = \mu_r dt$, which, together with the initial condition $i_r(0) = 0$, gives the simple linear form $i_r(t) = \mu_r t$.

To compute the expectation value above we further assume that the Federal Reserve adjusts the short rate according to the Taylor rule continuously in time. With this approximation the resetting of the Federal Reserve funds rate via the Taylor rule gives rise to the following equations for the evolution of the actual inflation and output gap,

$$
\begin{align*}
\frac{d\pi(t)}{dt} &= \mu_\pi dt + \alpha_1 \pi(t) dt + \alpha_2 y(t) dt + \sigma_\pi dw_\pi, \\
\frac{dy(t)}{dt} &= \mu_y dt + \beta_1 y(t) dt + \beta_2 \pi(t) dt + \sigma_y dw_y.
\end{align*}
$$ (35)

Here $\hat{\beta}_1 = \beta_1 - \theta_2 \beta_2$, $\hat{\beta}_2 = \beta_2(1 - \hat{\theta}_1)$, and $\hat{\mu}_y = \mu_y - \beta_2 \hat{c}$. The solution to these equations is

$$
\mathbf{v}(t) = e^{\hat{M}t} \mathbf{v}(0) + \left( e^{\hat{M}t} - 1 \right) \hat{M}^{-1} \hat{\mu} + \int_0^t e^{\hat{M}(t-\tau)} ds(\tau),
$$ (36)

where $\hat{M}$ is the two-by-two matrix with elements $\hat{M}_{11} = \alpha_1$, $\hat{M}_{12} = \alpha_2$, $\hat{M}_{21} = \hat{\beta}_2$ and $\hat{M}_{22} = \hat{\beta}_1$. The vector $\hat{\mu}$ has elements $\hat{\mu}_1 = \mu_\pi$ and $\hat{\mu}_2 = \hat{\mu}_y$. The vectors $\mathbf{v}(t)$ and $ds(\tau)$ are defined as in the previous section.

It is convenient to introduce the vector $\Theta$ with components $\Theta_1 = \hat{\theta}_1$ and $\Theta_2 = \theta_2$, so that the Taylor rule is $i(t) = \hat{c} + \hat{\theta}_1 \pi(t) + \theta_2 y(t) = \hat{c} + \Theta^T \mathbf{v}$. Inflation and output gap are normal random variables and so writing $\mathbf{v} = \tilde{\mathbf{v}} + \bar{\mathbf{v}}$, where $E_0[\tilde{\mathbf{v}}] = \tilde{\mathbf{v}}$ and $E_0[\bar{\mathbf{v}}] = 0$, the yield is

$$
Y(T) = \frac{1}{2} \mu_\pi T + \hat{c} + \frac{1}{T} \int_0^T dt \Theta^T \tilde{\mathbf{v}}(t) - \frac{1}{2T} \Theta_a \Theta_b \int_0^T dt \int_0^T dt' E_0[\tilde{\mathbf{v}}(t) a \tilde{\mathbf{v}}(t') b].
$$ (37)

We are implicitly summing over all repeated indices. Note that if the yield and $\mathbf{v}$ are expressed in percent, the final term above must be divided by 100. Using Eq. (36) this can
be simplified substantially. The integral without the expectation is simple to do, but the other is not. By interchanging the integrals in the expression for $\hat{v}$ above with the integrals over $t$ and $t'$, it can be written as

$$-rac{1}{2T}\Theta_d \Theta_b \int_0^T \!\! d\tau \int_\tau^T \!\! dt \int_\tau^T \!\! dt' \left( e^{\hat{M}(t-\tau)} \right)_{ac} \Sigma_{cd} \left( e^{\hat{M}(t'-\tau)} \right)_{bd},$$

(38)

where the diagonal matrix $\Sigma$ has elements $\Sigma_{11} = \sigma_x^2$ and $\Sigma_{22} = \sigma_y^2$. The integrals over $t$ and $t'$ are now easily done, and the final result for the yield is

$$Y(T) = \frac{1}{2} \mu_r T + \hat{c} - \Theta^T \hat{M}^{-1} \hat{\mu} + \frac{1}{T} \Theta^T \hat{M}^{-1} \left( e^{\hat{M}T} - 1 \right) \left( v(0) + \hat{M}^{-1} \hat{\mu} \right)$$

$$- \frac{1}{2} \Theta^T \hat{M}^{-1} \Sigma \left( \hat{M}^{-1} \right)^T \Theta + \frac{1}{T} \Theta^T \hat{M}^{-1} \Sigma \left( \hat{M}^T - 1 \right) \left( \hat{M}^{-2} \right)^T \Theta$$

$$- \frac{1}{2T} \int_0^T \!\! d\tau \Theta^T \hat{M}^{-1} e^{\hat{M}(T-\tau)} \Sigma e^{\hat{M}^T(T-\tau)} \left( \hat{M}^{-1} \right)^T \Theta.$$

(39)

The symbol $\top$ denotes the transpose. The remaining integral can be computed with the formulas given at the end of this appendix.

In this model the volatility of the Treasury yield $\sigma_{Y(T)}$ is related to the inflation and output gap volatilities. It describes how much the yield varies under changes of initial inflation and output gap. We find that

$$\sigma_{Y(T)}^2 = \frac{1}{T^2} \Theta^T \hat{M}^{-1} \left( e^{\hat{M}T} - 1 \right) \Sigma \left( e^{\hat{M}^T} - 1 \right) \left( \hat{M}^{-1} \right)^T \Theta.$$

(40)

This is plotted below in Fig. 6 for sample values of $\sigma_x$ and $\sigma_y$.

The yield can also be calculated by diagonalizing $\hat{M}$ and converting Eq. (35) to a two-factor version of the Vasicek model [26]:

$$dx(t) = k_x [\theta_x - x(t)]dt + \sigma_x dw_x,$$

$$dz(t) = k_z [\theta_z - z(t)]dt + \sigma_z dw_z.$$

(41)

The correlation between the Brownian motions $dw_x$ and $dw_z$ is denoted $\rho$, and the short rate is $i(t) = \hat{c} + x(t) + z(t) + \mu_r t$, including the risk premium as above. The Treasury yield in this model is

$$Y(T) = \frac{1}{2} \mu_r T + \hat{c} + \theta_x + \theta_z + \frac{1 - e^{-k_x T}}{k_x T} [x(0) - \theta_x] + \frac{1 - e^{-k_z T}}{k_z T} [z(0) - \theta_z]$$

$$- \frac{\sigma_x^2}{2k_x^2} \left( 1 - 2 \frac{1 - e^{-k_x T}}{k_x T} + \frac{1 - e^{-2k_x T}}{2k_x T} \right) - \frac{\sigma_z^2}{2k_z^2} \left( 1 - 2 \frac{1 - e^{-k_z T}}{k_z T} + \frac{1 - e^{-2k_z T}}{2k_z T} \right)$$

$$- \rho \sigma_x \sigma_z \frac{k_x}{k_z} \left[ 1 - \frac{1 - e^{-k_x T}}{k_x T} - \frac{1 - e^{-k_z T}}{k_z T} + \frac{1 - e^{-(k_x + k_z) T}}{(k_x + k_z) T} \right].$$

(42)

The values of the Vasicek parameters as functions of our model parameters are given in terms of the eigenvalues of $\hat{M}$:

$$\lambda_{\pm} = \frac{\alpha_1 + \hat{\beta}_1 \pm \sqrt{\left( \alpha_1 - \hat{\beta}_1 \right)^2 + 4 \alpha_2 \hat{\beta}_2}}{2}.$$

(43)
Using these formulas, in our model the parameters in Eq. (42) have the values

\[
x(0) = \frac{\theta_2(\lambda_+ - \alpha_1) + \alpha_2 \hat{\theta}_1}{\alpha_2(\lambda_- - \lambda_+)} [\pi(0)(\lambda_- - \alpha_1) - y(0)\alpha_2],
\]

(44)

\[
z(0) = \frac{\theta_2(\lambda_- - \alpha_1) + \alpha_2 \hat{\theta}_1}{\alpha_2(\lambda_- - \lambda_+)} [-\pi(0)(\lambda_+ - \alpha_1) + y(0)\alpha_2],
\]

(45)

\[
k_{x,z} = -\lambda_+,
\]

(46)

\[
\theta_x = \frac{\theta_2(\lambda_+ - \alpha_1) + \alpha_2 \hat{\theta}_1}{\lambda_+ \alpha_2(\lambda_- - \lambda_+)} [-\mu_x(\lambda_- - \alpha_1) + \hat{\mu}_y\alpha_2],
\]

(47)

\[
\theta_z = \frac{\theta_2(\lambda_- - \alpha_1) + \alpha_2 \hat{\theta}_1}{\lambda_- \alpha_2(\lambda_- - \lambda_+)} [\mu_x(\lambda_+ - \alpha_1) - \hat{\mu}_y\alpha_2],
\]

(48)

\[
\sigma_x = \frac{\theta_2(\lambda_+ - \alpha_1) + \alpha_2 \hat{\theta}_1}{\alpha_2(\lambda_- - \lambda_+)} \sqrt{\sigma_\eta^2(\lambda_- - \alpha_1)^2 + \sigma_y^2\alpha_2^2},
\]

(49)

\[
\sigma_z = \frac{\theta_2(\lambda_- - \alpha_1) + \alpha_2 \hat{\theta}_1}{\alpha_2(\lambda_- - \lambda_+)} \sqrt{\sigma_\eta^2(\lambda_+ - \alpha_1)^2 + \sigma_y^2\alpha_2^2},
\]

(50)

\[
\rho = \frac{\sigma_\eta \alpha_2 \hat{\beta}_2 - \sigma_y \alpha_2^2}{\sqrt{\sigma_\eta^2(\lambda_- - \alpha_1)^2 + \sigma_y^2\alpha_2^2} \sqrt{\sigma_\eta^2(\lambda_+ - \alpha_1)^2 + \sigma_y^2\alpha_2^2}}.
\]

(51)

Note the two-factor model given by Eq. (35) is not identical to the two-factor Vasicek model. Many real values of our model parameters result in complex Vasicek parameters. (Of course the yield is still real.) In fact, our historical estimates of the parameters, given below, are an example of this.

Finally we give the formulas needed to compute the remaining integral in the analytic expression for the yield curve in Eq. (39). It is convenient to introduce the combination of coefficients

\[
\delta = \sqrt{(\alpha_1 + \hat{\beta}_1)^2 + 4\alpha_2 \hat{\beta}_2 - 4\alpha_1 \hat{\beta}_1}.
\]

(52)

Explicit expressions for the exponential of the evolution matrix \( M \) multiplied by some number \( p \) are

\[
(e^{\tilde{M}p})_{11} = \left( \frac{\alpha_1 + \hat{\beta}_1 + \delta}{2\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 + \delta)p/2} - \left( \frac{\alpha_1 + \hat{\beta}_1 - \delta}{2\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 - \delta)p/2},
\]

(53)

\[
(e^{\tilde{M}p})_{12} = \left( \frac{\alpha_2}{\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 + \delta)p/2} - \left( \frac{\alpha_2}{\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 - \delta)p/2},
\]

(54)

\[
(e^{\tilde{M}p})_{21} = \left( \frac{\hat{\beta}_2}{\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 + \delta)p/2} - \left( \frac{\hat{\beta}_2}{\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 - \delta)p/2},
\]

(55)

and

\[
(e^{\tilde{M}p})_{22} = \left( \frac{-\alpha_1 + \hat{\beta}_1 + \delta}{2\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 + \delta)p/2} - \left( \frac{-\alpha_1 + \hat{\beta}_1 - \delta}{2\delta} \right) e^{(\alpha_1 + \hat{\beta}_1 - \delta)p/2}.
\]

(56)
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