The Ho and Lee Term Structure Theory:

A Continuous Time Version

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1. AIMS

In their paper [H L] , Ho and Lee present an innovatory theory of the term structure of interest rates. Their theory differs from those of their predecessors (notably [V] , and [CIR]) is concentrating on the evolution of the term structure from its initial shape, rather than on an equilibrium characterisation of what the shape of the term structure should be. It works in discrete time, and with the simplifying (though apparently naive) assumption that the term structure evolves binomially, i.e. given the structure at a certain time point, then the structure at the next time point can be one of only two alternatives. However, upon this basis, and using some very reasonable arguments, they are able to characterise completely the evolution of the term structure.

Ho and Lee's assumption of binomial evolution is not as naive as it might seem; it is equivalent to the assumption that the random input to the evolution is a binomial random walk, and since a random walk is a good approximation to a Brownian Motion when the discrete time increments are small, the binomial assumption is close to the assumption that the term structure is driven by a single Brownian Motion as its random input.

Our aim is to present a continuous time analogue of the arguments and conclusions of the Ho and Lee Theory. This article is meant to stand in relation to that of Ho and Lee in the same way as the usual treatment of the Black-Scholes formula for equity options (see [BS]) stands in relation to the binomial approach to option valuation of [CRR]. Our motivation and conclusions are similar to those of [HJM], and our debt to [HJM] is clear. However, our techniques are different, being directly based on those of [HL].

A plan of this article is as follows:

Sections 2 and 3 are parallel to one another, Section 2 dealing with the original Ho and Lee Theory, and Section 3 with the continuous version. In each we present first the framework, and then the conclusions, and then the techniques of the arguments, which are based on considerations of "arbitrage across maturities", and "time homogeneity".

The work presented here has benefited greatly from discussions with my colleagues Nick Webber and Stewart Hodges, and our MBA student Chew Cheah Boon.

2. THE HO AND LEE THEORY

Here we briefly review the Ho and Lee Theory ([HL]), adapted to our notation.

2.1 Assumptions and Notation

2.1.1. The set of discrete times with which the model deals is denoted by $(t_o, t_1, t_2, ...)$. The time interval between each time is arbitrarily chosen but it should be a small quantity.

2.1.2. The model also assumes that the market clears at the times of t_o, t_1 and so on.

2.1.3. For each $t_i < t_j$, there exists a pure discount bond at time t_i which matures at time t_j. The price of this bond is denoted by $P(t_i\ t_j)$ so that if one lends £ $P(t_i\ t_j)$ at time t_i until time t_j, one gets back £1 at maturity.

The term structure at time t_i then is simply the set of prices

$$(P(t_i, t_i + 1), P(t_i, t_i+2), ...).$$

Equivalently, it may also be expressed as the set of "yields" or "rates"

$(r(t_i, t_{i+j}), r(t_i, t_{i+2}), \ldots)$.

where

$$r(t_i\ t_j) = -1/(t_i - t_j) \log P(t_i, t_j) .$$

The rate r is the interest rate provided by the bond.

The term structure may also be thought of as being the discount function $P(t_i, -)$ or yield $r(t_i, -)$.

2.1.4. The term structure evolves randomly and is assumed to evolve such that given the term structure at time t_i, that at time t_{i+1} can only be one of two alternatives namely the "upstate" and the "downstate". It is further assumed that as the model evolves over successive time increments, the effect of an "up" followed by a "down" is the same as that of a "down" followed by an "up". It follows from this that the random aspect of the evolution can be characterised simply by a random walk which might step up or down by equal amounts as time moves from t_i to t_{i+1} .

2.1.5. The proportional deviation of the discount function at time t_{i+1} from the forward discount function from time t_i to t_{i+1} (i.e. $P(t_i, -)/P(t_i, t_{i+1})$) is time homogenous and independent of the term structure. Thus, the evolution from time t_i to t_{i+1} can be represented by the functions $H(t_i, t_{i+1}, -)$ and $H^*(t_i, t_{i+1}, -)$, such that

$$P(t_{i+q}, t_q) = (P(t_i, t_q)/ P(t_i, t_{i+1}))\, H(t_i, t_{i+1}, t_q) \tag{1}$$

for all $q > i$, if the increment from t_i to t_{i+1} sees "up", and

$$P(t_{i+1}, t_q) = (P(t_i, t_q)/P(t_i, t_{i+1}))H^*(t_i, t_{i+1}, t_q) \tag{1*}$$

if the increment sees "down".

The functions H and H* are time homogeneous (i.e. $H(t_i, t_{i+q}, t_q) = H(t_{i+j}, t_{i+1+j}, t_{q+j})$ for any j, and similarly for H*) and independent of $P(t_i, -)$.

2.1.6. Finally, the model assumes that there are no taxes or transaction costs.

2.2 Conclusions

These can be summarised as:

$$H(t_i, t_{i+1}, t_q) = \frac{\exp \lambda(t_q - t_{i+1})}{\pi + (1-\pi) \exp \lambda(t_q - t_{i+1})}, \qquad (2)$$

$$H^*(t_i, t_{i+1}, t_q) = \frac{1}{\pi + (1-\pi) \exp \lambda(t_q - t_{i+1})},$$

where π and λ are parameters to be determined empirically. These parameters are difficult to interpret intuitively. The parameter λ must be negative, and is related to "volatility", $\lambda = 0$ corresponding to certainty. The parameter π is the "implied" binomial "up" probability, and we have

$$P(t_i, t_q) = [\pi P(t_{i+1}, t_q) + (1-\pi)P^*(t_{i+1}, t_q)]\, P(t_i, t_{i+1}), \qquad (3)$$

where $P(t_{i+1}\ t_q)$ corresponds to "up" and $P^*(t_{i+1}\ t_q)$ corresponds to "down" in the interval $[t_i\ t_{i+1}]$. Equation (3) tells us that the price at time t_i of the bond to mature at time t_q is the expected and discounted price of the same bond at time t_{i+1}, if π is actually the "up" probability. Thus, π would be the up probability if there were no risk premia, though the Ho and Lee Theory does not depend on there being no risk premia, and the actual "up" probability might be bigger than π .

2.3 <u>Arbitrage Across Maturities</u> (see [HL], Appendix A)

This argument is commonly used in 1-factor term structure theories (see [V], [CIR], [HL], [HJM]), and it involves constructing a hedged portfolio of two pure discount bonds of different maturities. So suppose such a portfolio contains quantities ξ_1 and ξ_2 of maturity t_{q_1} and t_{q_2} bonds. Then its value at time t_i is

$$\xi_1 P(t_i, t_{q_1}) + \xi_2 P(t_i, t_{q_2}) \tag{4}$$

and its value at time t_{i+1} is

$$\xi_1 P(t_{i+1}, t_{q_1}) + \xi_2 P(t_{i+1}, t_{q_2})$$

$$= \frac{1}{P(t_i, t_{i+1})} [\xi_1 P(t_i, t_{q_1}) \ H(t_i, t_{i+1}, q_1) + \xi_2 P(t_i, t_{q_2}) \ H(t_i, t_{i+1}, q_2)] \tag{5}$$

in the upstate, and similarly in the downstate, with H replaced by H^* and $P(t_{i+1}, -)$ replaced by $P^*(t_{i+1}, -)$.

If ξ_1 and ξ_2 are chosen such that the portfolio is hedged (i.e. the value (5) at time t_{i+1} is independent of "up" or "down"), then the return must be that of the maturity t_{i+1} bond, i.e. we have the equation.

$$\frac{\xi_1 P(t_{i+1}, t_{q_1}) + \xi_2 P(t_{i+1}, t_{q_2})}{\xi_1 P(t_i, t_{q_1}) + \xi_2 P(t_i, t_{q_2})} = \frac{1}{P(t_i, t_{i+1})} , \tag{6}$$

and a similar equation (which we will call (6*)), with $P(t_{i+1}, -)$ replaced by $P^*(t_{i+1}, -)$. From (6) and (6*) with substitutions from (5) involving H and H^*, we can determine ξ_1 and ξ_2 , and conclude that

$$\frac{1 - H^*(t_i, t_{i+1}, t_{q1})}{H(t_i, t_{i+1}, t_{q1}) - H^*(t_i, t_{i+1}, t_{q1})} = \frac{1 - H^*(t_i, t_{i+1}, t_{q2})}{H(t_i, t_{i+1}, t_{q2}) - H^*(t_i, t_{i+1}, t_{q2})} \quad . \tag{7}$$

Denoting the value of (7) by π we conclude that

$$\pi H(t_1, t_{i+1}, t_q) + (1 - \pi) H^*(t_i, t_{i+1}, t_q) = 1$$

for any t_q and hence equation (3).

It is instructive to rewrite (3) as

$$P(t_1, t_q) = P(t_1, t_{i+1}) \tilde{\mathbb{E}}[P(t_{i+1}, t_q)] \quad , \tag{8}$$

where $\tilde{\mathbb{E}}$ is the expectation corresponding to the "risk neutral probability".

2.4 Time Homogeneity (see [HL] section 2C)

Our version of [HL] equation (12) is:

$$P(t_{1+2}, t_q) = \frac{P(t_i, t_q)}{P(t_i, t_{i+2})} \; \frac{H(t_i, t_{i+1}, t_q) \; H^*(t_{i+1}, t_{i+2}, t_q)}{H(t_i, t_{i+1}, t_{i+2})} \quad . \tag{9}$$

(This is obtained by iterating (1), and gives $P(t_{i+2}, -)$ in terms of $P(t_i, -)$, if $[t_i, t_{i+1}]$ sees "up" and $[t_{i+1}, t_{i+2}]$ sees "down"). The R.H.S. of (9) is unchanged if H and H^* are swopped ("up-down" replaced by "down-up" - see Assumption 2.1.4), and so we have the following, also using $H(t_{i+1}, t_{i+2}, t_q) = H(t_i, t_{i+1}, t_{q-1})$ (time homogeneity):

$$\frac{H(t_i, t_{i+1}, t_q) \; H^*(t_i, t_{i+1}, t_{q-1})}{H(t_i, t_{i+1}, t_{i+2})} = \frac{H^*(t_i, t_{i+1}, t_q) \; H(t_i, t_{i+1}, t_{q-1})}{H(t_i, t_{i+1}, t_{i+2})} \quad . \tag{10}$$

From (10) and (3) we finally obtain formula (2) for H and H*.

3. THE HO AND LEE THEORY IN CONTINUOUS TIME

3.1 The Set Up

Now we assume that for any times $t < q$ there is a pure discount bond at time t which matures at time q, and whose value we will denote by $P(t, q)$. We take the set-up as in [HJM] and write this as

$$P(t, q) = \exp\left[-\int_{\tau=t}^{q} f(t, \tau)d\tau\right], \tag{11}$$

where the "forward rate" $f(\rho, \tau)$ (defined for $\rho < \tau$) satisfies

$$df(\rho, \tau) = \alpha(\rho, \tau)d\rho + \sigma(\rho, \tau)dB_\rho . \tag{12}$$

We assume that in equation (12) the coefficient α and σ are non-random, i.e. they do not depend on the behaviour of the term structure.

If we define the function $H(t, s, q)$ for $t \leq s \leq q$ by

$$P(s, q) = ({}^{P(t,\, q)}/_{P(t,\, s)})\, H(t, s, q) \tag{13}$$

(Cf. equation (1)), then we see that

$$H(t, s, q) = \exp\left[-\int_{\tau=s}^{q} [f(s, \tau) - f(t, \tau)]d\tau\right] \tag{14}$$

$$= \exp\left[-\int_{\tau=s}^{q}\int_{\rho=t}^{s}[\alpha(p,\tau)d\rho + \sigma(p,\tau)dB_\rho]\,d\tau\right] , \tag{15}$$

and so $H(t, s, q)$ depends on the random input $d\beta_\rho$ for $\rho \notin [t, s]$. Note that our assumption that α and σ are non-random implies that H does not depend on the previous behaviour of the term structure. We also assume that $\alpha(\rho, \tau)$ and $\sigma(\rho, \tau)$ depend only on $\tau - \rho$, and this corresponds to time homogeneity in our model. Thus, our set-up is truely a continuous time version of the Ho and Lee set-up. Our technical assumptions about $\alpha(\rho, \tau)$ and $\sigma(\rho, \tau)$ are just that they are bounded and Lipschitz right-continuous in ρ (this does allow them to jump in value as ρ increases, and with time homoegneity implies that they are left continuous in τ), and that σ is greater than zero.

3.2 Conclusions and Comparison with the Model of [HJM]

The conclusions are simply that $\sigma(\rho, \tau)$ is constant and that we have

$$\alpha(t, q) = -\gamma\sigma + \sigma^2(q - t) , \tag{16}$$

where the constant γ is the risk premium. These conclusions look the same as those of [HJM] Section 7, though we argue more fully for the constancy of σ, and we do not try to avoid the presence of the risk premium in our answer (Our γ corresponds to ϕ of [HJM] Section 7. Note that time homogeneity would force ϕ to be constant.)

As [HJM] explain, the "equivalent risk neutral probability" (Cf. [HL]) is obtained by replacing B_ρ by $\widetilde{B}_\rho$, where

$$d\widetilde{B}_\rho = dB_\rho - \gamma d\rho ,$$

and if $\widetilde{\mathbb{E}}_t$ denotes the expectation at time t with respect to B_ρ, then we can write for $t < s < q$

$$P(t, q) = \widetilde{\mathbb{E}}_t\,[(^1/B(t, s))\quad P(s, q)] \tag{17}$$

where the discount factor $B(t, s)$ is the value at time s of £1 invested at time t, at the short rate of interest. (Compare (8) and (17)).

As [HJM] also explain, the conclusions of the Ho and Lee model in continuous time are as they stand unsatisfactory, because they lead to predicting that the abort interest rate eventually becomes negative or unbounded, with high probability. (To obtain the short interest rate $r(s)$ note first that $r(s) = f(s,s)$, and hence using (12) that for $s > t$ we have

$$r(s) \equiv f(s, s) = f(t, s) + \int_{s=t}^{s} [\alpha(\rho, s)\, ds + \sigma(\rho, s) dB_\rho] \ .$$

Then substituting for α and σ using (16) yields

$$r(s) = f(t, s) - \sigma \,.\, \gamma \,.\, (s - t) + \sigma^2\!/_2\, (s - t)^2 + \sigma(B_s - B_t) \ .$$

To prevent this unsatisfactory conclusion, [HJM] allow the "volatility" σ to be attenuated when the short rate is small, but then they have to allow α to be random (see [HJM] Section 8). In [C] we present another formulation for the evolution of the term structure, which captures the features of the models of [HL] and [HJM], and in which we try to make the dependence of the coefficients on the term structure more intuitively clear.

3.3 Arbitrage Across Maturities

At time t form a portfolio of ξ_i of maturity q_i bonds ($i = 1,2$). Then its value at time t is

$$\xi_1 P(t, q_1) + \xi_2 P(t, q_2) \ , \tag{18}$$

and at time $t + \varepsilon$ is

$$\begin{aligned} &\zeta\,, P(t + \varepsilon, q_1) + \xi_2 P(t + \varepsilon, q_2) \\ &= \frac{\xi_1 P(t, q_1)}{P(t, t + \varepsilon)} H(t, t + \varepsilon, q_1) + \frac{\xi_2 P(t, q_2)}{P(t, t + \varepsilon)} H(t, t + \varepsilon, q_2) \ . \end{aligned} \tag{19}$$

Now take an approximation $\tilde{H}$ to H given by

$$\begin{aligned} &\tilde{H}(t, t + \varepsilon, q) = \\ &1 - \varepsilon \int_{\tau=t+}^{q} \alpha(t, \tau) d\tau - \Delta_t^{\varepsilon} B \int_{\tau=t+\varepsilon}^{q} \sigma(t, \tau) d\tau + \frac{1}{2} \varepsilon \left(\int_{\tau=t+\varepsilon}^{q} \sigma(t, \tau) d\tau \right)^2 \end{aligned} \tag{20}$$

(Note that $\tilde{H}$ is a Taylor expansion of H to order $(\Delta_t^{\varepsilon B})^3$ which is $o(\varepsilon)$ in expectation. By $\Delta_t^{\varepsilon} B$ we mean $B_{t+\varepsilon} - B_t$.)

If ξ_1 and ξ_2 are chosen such that

$$\xi_1 P(t, q_1) \int_{\tau=t}^{q_1} \sigma(t, \tau) d\tau + \xi_2 P(t, q_2) \int_{\tau=t}^{q_2} \sigma(t, \tau) d\tau = 0 \ , \tag{21}$$

then over the time $[t, t+\varepsilon]$ the portfolio is hedged to $o(\varepsilon)$ in expectation. To see this note that if we substitute $\widehat{H}$ for H in (19) with this choice of ξ_1 and ξ_2, then the term in $\Delta_t^\varepsilon B$ cancels. Since it is hedged, then in the absence of arbitrage opportunities, we have

$$\frac{\xi_1 P(t+\varepsilon, q_1) + \xi_2 P(t+\varepsilon, q_2)}{\xi_1 P(t,q_1) + \xi_2 P(t,q_2)} = \frac{1}{P(t, t+\varepsilon)} + o(\varepsilon) . \tag{22}$$

Substituting into (22) using the formula (20) for H and (21) for ξ_1, ξ_2, and cancelling, we obtain

$$\frac{\left\{ \int_{\tau=t}^{q_2} \sigma(t, \tau)d\,[\int_{\tau=t}^{q_1} \alpha(t, \tau)d\tau - \frac{1}{2}(\int_{\gamma=t}^{q_1} \sigma(t, \tau)d\tau)^2] - \int_{\tau=t}^{q_1} \sigma(t, \tau)d\tau[\int_{\tau=t}^{q_2} \alpha(t, \tau)d\tau - \frac{1}{2}(\int_{\tau=t}^{q_2} \sigma(t, \tau)d\tau)^2] \right\}}{\left\{ \int_{=t}^{q_2} \sigma(t, \tau)d\tau - \int_{\tau=t}^{q_1} \sigma(t, \tau)d\tau \right\}} \tag{23}$$

$$= 0 .$$

Also, since the denominator of (22) is not zero (since σ is greater than zero), we conclude that

$$[\int_{\tau=t}^{q} \alpha(t, \tau)d\tau - \frac{1}{2}(\int_{\tau=t}^{q} \sigma(t, \tau)d\tau)^2] \Big/ \int_{\tau=t}^{q} \sigma(t, \tau)d\tau \tag{24}$$

is the same for $q = q_1$ and $q = q_2$, and hence is independent of q. If we denote the value of (23) by $-\gamma$, then we have

$$\int_{\tau=t}^{q} \alpha(t, \tau)d\tau = -\gamma \left[\int_{\tau=t}^{q} \sigma(t, \tau)d\tau + \frac{1}{2}\left(\int_{\tau=t}^{q} \sigma(t, \tau)d\tau\right)^2\right] . \tag{25}$$

In the following section we show that σ is constant. For the moment note that σ constant and (25) yield (16).

3.4 Time Homogeneity

First we present this argument in an intuitive form, and then in Section 3.5 we show how to make it rigorous.

Note first that

$$\frac{H(t, t+\varepsilon, q)\ H(t+\varepsilon, t+2\varepsilon, q)}{H(t+\varepsilon, t+2\varepsilon)} = H(t, t+2\varepsilon, q) \tag{26}$$

(cf. Equation (9) above), and put

$$h_\varepsilon^u(t, q) = \varepsilon\int_{\tau=t+\varepsilon}^{q} \alpha(t, \tau)d\tau + u\int_{\tau=t+\varepsilon}^{q} \sigma(t, \tau)d\tau . \tag{27}$$

Note that $h_\varepsilon^u(t, q)$ is a Taylor expansion of $-\log H(t, t+\varepsilon, q)$ if we put $u = \Delta_t^\varepsilon B$,and the the error is order u^3, which is $o(\varepsilon)$ in expectation. Now put $d = \Delta_{t+\varepsilon}^\varepsilon B$ and substitute (27) into (26) to obtain

$$h_\varepsilon^u(t, q) + h_\varepsilon^d(t, q-\varepsilon) - h_\varepsilon^u(t, 2\varepsilon) \cong h_{2\varepsilon}^{u+d}(t, q) . \tag{28}$$

(Note that (28) is only an approximate equality, with error $o(\varepsilon)$ in expectation. The second term in (28) is obtained using time homogeneity. Of course, u and d correspond to Ho and Lee's "up" and "down".)

Now note that the R.H.S. of (28) is unchanged if we swop u and d , and put d = -u , to deduce from (28) that

$$[h_\varepsilon^u(t, q) - h_\varepsilon^{-u}(t, q)] - [h_\varepsilon^u(t, q-\varepsilon) - h_\varepsilon^{-u}(t, q-\varepsilon)] \cong [h_\varepsilon^u(t, 2\varepsilon) - h_\varepsilon^{-u}(t, 2\varepsilon)] \ . \tag{29}$$

Now substitute for h in (29) using (27) to conclude that

$$2u\int_{\tau=t+\varepsilon}^{q} \sigma(t, \tau)d\tau - 2u\int_{\tau=t+\varepsilon}^{q-\varepsilon} \sigma(t, \tau)d\tau \cong 2u\int_{\tau=t+\varepsilon}^{t+2\varepsilon} \sigma(t, \tau)d\tau \cong 2u\varepsilon\ \sigma(t, t) \tag{30}$$

and hence that

$$\frac{d}{dq}\int^{q} \sigma(t, \tau)d\tau \equiv \sigma(t, q) = \sigma(t, t) \ ,$$

i.e. $\sigma(\rho, \tau)$ is constant. (Note that in the last approximate in equality of (30) we are using the Lipschitz right - continuity of $\sigma(t, \gamma)$ in γ).

Of course this analysis is not rigorous because we have been negligent about the orders of the approxmiations, and the fact that $\Delta_t^\varepsilon B$ etc. are random. We deal with these difficulties in the final section.

3.5 Making our Arguments Technically Rigorous

Our technique for this is to got to a discrete approximation to the model of [HJM], which corresponds to the approximation in terms of which the Ito integral itself is defined. To understand this note that the Ito integral $\int_{\rho=t}^{s} g(\rho)\, dB_\rho$ is defined as the limit of the "Euler sum"

$\sum_{i=0}^{n-1} g(s_i)\, [B_{s_{i+1}} - B_{s_i}]$ as mesh (π) tender to zero, where π is the partition

$\{t = s_o < s_i < \dots < s_n = s\}$, and by mesh (π) we mean $\{(\rho_{i+1} - \rho_i) : i = 0, \dots, n-1\}$.

Thus, for small mesh the Euler sum is a good approximation to the Ito integral.

Our approximation to the model is that we work with the discrete time grid $\{0, \varepsilon, 2\varepsilon, \dots\}$, and that our integrals are replaced by the corresponding Euler sums. Thus, the times t, s, q of Section 3.1 are chosen from this time grid, and for example equations (14) and (15) are replaced by

$$H(t, s, q) = \exp\, [- \sum_{i} [f(s_i\, \tau_i) - f(t, \tau_i)]\, \varepsilon]$$

$$\{\gamma_i = s, s+\varepsilon, \dots, q-\varepsilon\}$$

$$= \exp\, [- \sum_{j} \sum_{i} [\alpha(\rho_j, \tau_i)\varepsilon + \sigma(\rho_j\, \tau_i)\Delta^{\varepsilon}_{\rho_i} B]\, \varepsilon]\, ,$$

$$\{\tau_i = s, s+\varepsilon, \dots, q-\varepsilon\}$$

$$\{\rho_j = t, t+\varepsilon, \dots, s-\varepsilon\}$$

where the terms $\Delta^{\varepsilon}_{s_j} B$ for the various j's are iid random variables, which are each normal with mean 0 and variance ε , and represent the increments $B_{\rho_{j+1}} - B_{\rho_j}$. Note that this discretised reformulation of [HJM] is not just equivalent to the Ho and Lee theory!

It is easy to recast Section 3.4 for this discretinised formulation, and the conclusion is the same, namely that σ is constant. To recast Section 3.3 is also easy, and the conclusion is an analogue of equation (25) given by

$$\left\{\begin{array}{l}\sum_i \alpha(t, \tau_i)\varepsilon \\ \{\tau_i = t, t+\varepsilon, \ldots, q-\varepsilon\}\end{array}\right\} =$$

$$\left\{\begin{array}{l}-\gamma\sum_i \sigma(t, \tau_i)\varepsilon \\ \{\tau_i = t, t+\varepsilon, \ldots, q-\varepsilon\}\end{array}\right\} + \left\{\begin{array}{l}\frac{1}{2}(\sum_i \sigma(t, \tau_i)\varepsilon)^2 \\ \{\tau_i = t, t+\varepsilon, \ldots, q-\varepsilon\}\end{array}\right\}$$

$$\equiv \gamma\sigma(q - t) + \frac{1}{2}\sigma^2(q - t)^2 \tag{31}$$

(Since σ is constant).

On "differencing" equation (31) (i.e. subtracting from it the alteration of itself with $q - \varepsilon$ substituted for q), we obtain

$$\alpha(t, q) = -\gamma\sigma + \sigma^2(q - t - \tfrac{\varepsilon}{2}). \tag{32}$$

Our conclusion for the discretised system is that σ being constant and α being as in (32) uniquely give time homogeneity and prevent arbitrage across maturities.

To draw the corresponding conclusion for the continuous system (with (16) instead of (32)) note that for small ε, (32) is a good approximation to (16), and the Euler sums are good approximations to the Ito integrals; therefore for the continuous system the conclusion holds as closely as we please. But the continuous system does not involve ε, therefore the conclusion must hold without approximation.

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