# Interest Rate Volatility and the Term Structure of Interest Rates

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## Abstract

Interest rate volatility is a key element when valuing fixed income securities. This paper looks at how the traditional models of interest rate movements handle the uncertainty of interest rates, and analyse recent attempts to incorporate more realistic interest rate movements into models for pricing interest rate dependent securities.

We find that the earlier, single factor, models of the term structure place quite severe restrictions on the shape and form of possible yield curves, due to a large extent to their assumptions about interest rate volatility. Recent attempts to model the term structure have followed two main approaches. The first approach has been to incorporate interest rate volatility as a second stochastic factor in equilibrium type models of the term structure with the advantage that prices are analytically tractable, but with the disadvantages that the resulting term structures belong to a limited family which will generally be inconsistent with the observed term structures. The second approach focuses on building models that use information contained in the prices of traded securities about the whole term structure of both spot or forward rates and rate volatilities. We show that this latter structure may be constrained to evolve over time in an unstable way that was not originally intended by the user.

The link between the uncertainty of interest rates and the value of differing maturity bonds, suggests a relationship between the level of interest rate volatility and the shape of the term structure of interest rates. We analyse this relationship and develop a method to empirically test it using the prices of traded securities.

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# Interest Rate Volatility and The Term Structure of Interest Rates

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## I/ Introduction

Interest rate volatility is a key element when valuing fixed income securities which have options or any kind of contingent payoff attached. Many bonds have been issued which are callable by the issuer if interest rates drop significantly, allowing the re-issuance of a bond with a much lower coupon. Other issues are puttable by the investor, allowing her to put the bond back to the issuer if interest rates rise, using the proceeds to purchase another bond with a higher coupon. These option features can significantly affect the price of the bond to which they are attached.

Many investors are also aware that the uncertainty affecting interest rate moves, as well as the level of the rates, affects the relative attractiveness of bonds which differ in their time to maturity and the level of their coupon payments. It is not surprising therefore that interest rate volatility plays an important role in the academic literature on the pricing of interest rate derivative securities, a role that is becoming increasingly important, as witnessed by a number of recently published articles. In this paper we look at how the traditional models of interest rate movements handle the uncertainty of interest rates, and analyse recent attempts to incorporate more realistic interest rate movements into models for pricing interest rate dependent securities. We find that the earlier, single factor, models of the term structure place quite severe restrictions on the shape and form of possible yield curves, due to a large extent to their assumptions about interest rate volatility.

Recent attempts to model the term structure have followed two main approaches. The first approach has been to incorporate interest rate volatility as a second stochastic factor in equilibrium type models of the term structure. This approach has the advantage that pure discount bond prices, and in some cases pure discount bond option prices are analytically tractable, but with the disadvantages that the resulting term structures still belong to a limited family which will generally be inconsistent with the observed term structure, and that the user has no freedom to control the term structure of interest rate volatility. The second approach focuses on building models that use information contained in the prices of traded securities about the whole term structure of both spot or forward rates and rate volatilities. We show that this latter structure may be constrained to evolve over time in a way that was not originally intended by the user.

The link between the uncertainty of interest rates and the value of differing maturity bonds, with or without embedded options, suggests a relationship between the level of interest rate volatility and the shape of the term structure of interest rates. We analyse the relationship between the shape of the yield curve, in particular its curvature, and the volatility of interest rates. Our analysis allows us to gain a measure of implied volatility from the 'underlying' instrument, i.e. the spot yield curve, rather than from derivatives traded on the yield curve. We develop a method to empirically test this relationship using the prices of traded securities.

The plan of this paper is as follows.

In section II we analyse the role played by interest rate volatility in one-factor models which have been used to price interest rate dependent securities. We extend our analysis in section III to two multi-factor models, namely Fong and Vasicek [1992a, 1992b], and Longstaff and Schwartz [1992a, 1992b], where the volatility of interest rates is itself assumed to follow a stochastic process. Section IV looks at models which are constructed to use market information about both the term structure of interest rates and the term structure of interest rate volatilities, and in section V we show how these volatility functions are constrained to evolve through time for two of the models developed by Hull and White [1990].

In section VI we explore the relationship between interest rate volatility and the shape of the term structure, proposing a number of empirical tests which are carried out in section VII. Finally, section VIII contains our conclusions.

## II/ Interest Rate Volatility and the Pricing of Interest Rate Derivative Securities.

To see why bond prices depend on volatility we can look at the negative exponential relationship between interest rates and discount factors. Jensen's inequality holds for this convex function implying the bond price, taken as the expectation of the bonds payoffs discounted under the risk adjusted measure, is higher than the risk adjusted path of the expected short rates, the difference arising from the uncertainty of future interest rate moves.

We now turn our attention to models that are frequently used to price interest rate derivatives, focusing on their treatment of interest rate volatility. One of the most widely used models to value bond options is the Black Scholes (1973) model which was originally developed to price options on stocks. The price of the discount bond is an input into the model which assumes that the bond follows a lognormal diffusion process:

$$dP(r,t,s) = \mu P(r,t,s)dt + \sigma P(r,t,s)dz \qquad (1)$$

where

$P(r,t,s)$ = Price at time $t$, of a pure discount bond to mature at time $s$, if the short rate is $r$ at time $t$.

$dP(r,t,s)$ = change in the bond price $P(r,t,s)$.

$\mu$ = instantaneous return on the bond

$\sigma$ = instantaneous volatility of bond return

$dt$ = small increment in time

$dz$ = increment in the Weiner process

Embedded within the model is the potentially serious problem that the volatility of the bond price is assumed to be a constant $\sigma$. However, as the price of the bond is known with certainty at the maturity date, the bond's price volatility must tend to zero as it approaches that date. The assumption may not be unreasonable for short dated options on long dated bonds, but becomes significant when the two maturities are closer together. The problem is usually overcome by making the forward bond price the underlying instrument and using Black (1976).

Prices of interest rate dependent securities do not always depend on the price of discount bonds and so in many traditional models of the term structure it is the process followed by the interest rate, and not the bond, that is important. This process, plus information on risk preferences is sufficient in the absence of arbitrage opportunities, to determine the evolution of the yield curve

Perhaps the simplest equilibrium model of the term structure assumes that the short rate follows a random walk with both the drift and the variance of the process constant[1].

$$dr = \mu dt + \sigma dz \tag{2}$$

The pricing of zero coupon bonds is then given by the equation

$$P(r,t,s) = \exp(-r(s-t) - \tfrac{1}{2}\mu(s-t)^2 + \tfrac{1}{6}\sigma^2(s-t)^3) \tag{3}$$

This simplest case illustrates the reliance of the bond pricing formula, and so also the bonds return, not only on the current level of the short rate, but also the volatility associated with the short rate.

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1 This model is proposed by Merton [1973] and Ingersoll [1987].

If we relax the constant drift and volatility assumption of the short rate dynamics represented by equation (2), to allow them to be functions of the short rate itself and time, we obtain the following Ito equation for the short rate:

$$dr = \mu(r,t)dt + \sigma(r,t)dz \quad (4)$$

This general process contains three of the best known models of short term interest rate movements as special cases, and which were proposed by Vasicek (1977), Cox-Ingersoll-Ross (1985), and Dothan (1978). In all three of these models the instantaneous drift of the process, $\mu(r,t) = \alpha(\beta - r)$, represents a force that keeps pulling the process back towards its long term mean $\beta$ with a force $\alpha$ which is proportional to the deviation of the process from the mean. Also the short rate is assumed to be the single source of uncertainty.

$$\begin{aligned} &Vasicek: && \sigma(r,t) = \sigma \\ &CIR\ \ : && \sigma(r,t) = \sigma\sqrt{r} \\ &Dothan: && \sigma(r,t) = \sigma r \end{aligned} \quad (5)$$

Under the Ornstein-Uhlenbeck diffusion process in the Vasicek paper the volatility of the process is represented by a constant $\sigma$. The constant volatility assumption has the major advantage that the conditional distribution of $r$ at any point of time in the future is normal, but with the disadvantages that interest rates can become negative with positive probability and that rates of different maturities have the same variability at all times. This last point is frequently violated in practice where we see long spot rates less volatile that short rates. In the Cox-Ingersoll-Ross paper the volatility of the short rate increases with the square root of the rate itself, precluding the existence of negative interest rates, and allowing more variability at times of high interest rates, and less variability when rates are low. Dothan models the interest rate process as a geometric random walk with the volatility increasing with the level of the rate.

Carverhill and Strickland [1992a] study the dynamics of money market short-term interest rates implied from the prices of interest rate cap agreements to see if they are consistent with the models whose dynamics are given by equation (4). They work with the discretised version of the general model that contains the processes described by equation (4) as a special case and which is represented by the following equation:

$$\Delta r_t = \alpha(\beta - r_{t-1})\Delta t + u_t \tag{6}$$

$u_t$ is the random increment in the series, assumed not to be autocorrelated. Vasicek's model proposes that $u_t$ is homoscedastic, whereas the models of Cox-Ingersoll-Ross and Dothan propose that $u_t$ is not homoscedastic. Carverhill and Strickland's tests indicate that the variance of the short rate for their data is not constant although the technique of scaling the test equations and re-estimating the residuals is unable to distinguish between the form of the variance being related to the level of the interest rate or the square root of the level.

These tests, and others performed by the same authors showing that long term interest rates have different properties to short term interest rates, suggest that these single factor models of the term structure, although often providing analytical tractability, do not provide a realistic representation of reality.

Investors and academics associated with fixed income markets understand that the volatility of interest rates is not a constant quantity. At times the bond markets are very volatilie, with rates changing drastically from day to day. At other times the bond markets are quiet with only very small changes occurring. Figure 1 shows the implied volatility[2] of LIFFE short sterling futures options over the period 5

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2 Imputed from Black [1976].

November 1987 to 5 March 1990. It is apparent that this quantity is not constant over time.

The evidence of 'volatile volatility' has led a number of authors to propose models of the term structure that incorporate the volatility of the instantaneous interest rate as a second stochastic factor. We now go on to present an analysis of two of these models.

## III/ Interest Rate Volatility as a Stochastic Factor

We have suggested that the possible term structures resulting from a 1 factor model cannot be too varied in their shape, and also we have seen that the variability of interest rates is itself variable. This suggests that models which allow interest rate volatility to follow a stochastic process will allow a richer variety of possible yield curves than the one factor equilibrium models and hence a more reasonable representation for the term structure of interest rates. Two papers, that are currently enjoying a great deal of analysis, start from the traditional term structure theory - i.e., a specification of the stochastic factors that explain the movement of interest rates. The difference between these papers and the earlier papers of section II is their explicit recognition of interest rate uncertainty as a second stochastic factor.

Longstaff and Schwartz (1992a) draw upon the same general equilibrium framework as Cox-Ingersoll-Ross (1985), beginning with the specification of the dynamics of two economic state variables that affect the returns on physical investment and the preferences of a representative investor. The processes are necessarily chosen for their analytic tractability, and are given as:

$$dx = (\gamma - \delta x)dt + \sqrt{x}dz_1$$
$$dy = (\eta - \nu y)dt + \sqrt{y}dz_2 \quad (7)$$

$\gamma$, $\delta$, $\eta$, and $\nu$ are parameters of the risk adjusted process for the uncorrelated state variables.

The fundamental partial differential equation for all default-free interest rate contingent claims, $H(x,y)$, given the processes in equation (7) can be shown to be:

$$\frac{1}{2}xH_{xx} + \frac{1}{2}yH_{yy} + (\gamma - \delta x)H_x + (\eta - \nu y)H_y - (\alpha x + \beta y)H = H_\tau \quad (8)$$

where subscripts denote partial derivatives and $\tau = s - t$. The solution to (8) subject to the appropriate boundary conditions determines the value of any interest rate contingent claim.

The equilibrium instantaneous interest rate and the variance of changes in this rate are given in this framework as a weighted sum of the state variables, $x$ and $y$, where the weights relate to parameters of the return process for physical investment.

$$r = \alpha x + \beta y$$
$$v = \alpha^2 x + \beta^2 y \quad (9)$$

The form of $r$ and $v$ allows the authors to express their results in terms of $r$ and $v$ as the state variables, although expressing the results in terms of the original $x$ and $y$ is computationally easier.

The resulting joint process for the dynamics of the short rate and the volatility of the short rate (or of the original state variables) allow us to write down closed form solutions for the prices of pure discount bonds and options on pure discount bonds. Bond prices, when expressed in terms of $r$ and $v$, take the form,

$$P(r,v,t,s) = \exp(G(\tau)+C(\tau)r+D(\tau)v) \tag{10}$$

where $\tau = s - t$. The price is a function of the state variables and the risk adjusted parameters of these variables. The functions of time $G$, $C$, and $D$ are tractable and easy to compute.

Call options on discount bonds satisfy the partial differential equation (8) with the appropriate boundary condition. The value of a call option with exercise price $K$ and maturity $T$, on a discount bond with maturity $s$ is given as,

$$\begin{aligned} C(r,v,\tau,K,T,s) = P(r,v,s)\psi(\theta_1,\theta_2;4\gamma,4\eta,\omega_1\omega_2) \\ -KP(r,v,T)\psi(\theta_3,\theta_4;4\gamma,4\eta,\omega_3\omega_4) \end{aligned} \tag{11}$$

The cumulative distribution functions are bivariate noncentral chi-squared. The solution of these functions, as presented by Longstaff and Schwartz, involves a double integration across the product of two univariate noncentral chi-squared densities. This operation can be so cumbersome that even the authors admit that numerically solving the partial differential equation, with the appropriate boundary condition, is as easy as evaluating equation (8) directly[3]. In another paper (Longstaff and Schwartz [1992b]) the authors solve for derivative payoffs by integrating the payoff across the resulting bivariate non-central chi-squared density[4].

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3 Longstaff and Schwartz [1992a], Footnote 15, pp 1271.

4 This density has closed form involving the evaluation of modified Bessel functions.

Chen and Scott [1992] in a recent paper analyse the pricing of interest rate options in the same special case of the two-factor Cox-Ingersoll-Ross model as Longstaff and Schwartz. The authors show that the multivariate integrals of the bond option pricing formula (equation (11)), can be reduced to univariate numerical integrations, reducing substantially the computation time required by this model.

Apart from the computational difficulty of the cumulative distribution functions of the Longstaff and Schwartz model, there are a number of problems which stem from the transformations of equation (9). The form of the transformations implies a high degree of correlation between $r$ and $v$. Also, in order for $x$ and $y$ to be non-negative we need $v$ to be bounded by $\alpha r$ and $\beta r$.

Figure 2 illustrates the empirical correlation between the 3 month Libor rate and the implied volatility from short sterling options for the period 5 November 1987 - 5 March 1990.

One final restriction that we need, for the model to be non-degenerate, is $\gamma > \frac{1}{2}$ and $\eta > \frac{1}{2}$. The degenerate case, if this restriction is violated is the single factor square root process of Cox-Ingersoll-Ross.

Fong and Vasicek (1992a, 1992b) start with a specification of the short rate and the variance of the short rate as their state variables . The dynamics of the two factors are more clearly interpretable than the resulting processes for these two elements in the Longstaff and Schwartz model.

$$\begin{aligned} dr &= \alpha(\bar{r} - r)dt + \sqrt{v}dz_1 \\ dv &= \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2 \end{aligned} \qquad (12)$$

The two Weiner processes, $dz_1$ and $dz_2$, are assumed to have a correlation of $\rho$. Both processes tend to revert to a long term mean value, with the strength of the reversion being proportional to the variables current deviation from the mean. The magnitude of the random component of the short rate is governed by the volatility of the short rate, whilst for the volatility of the short rate it is proportional to the level of volatility. Under the above specification for the state variables, and an assumption about the market prices of risk for *r* and *v*, Fong and Vasicek (1992a) derive a partial differential equation that determines the price of a pure discount bond. The solution to this equation has closed form:

$$P(t,r,v) = \exp[-rD(t) + vF(t) + G(t)] \tag{13}$$

The functions of time *D*, *F*, and *G*, are obtained as the solution of ordinary differential equations to which the partial differential equation reduces. *D*(*t*) is found to be the duration measure of the Vasicek [1977] paper, whilst *F*(*t*) and *G*(*t*) are computed to be complicated expressions involving the confluent hypergeometric function. Although these functions are difficult to evaluate - the solution proposed by Fong and Vasicek requires complex (as opposed to real) algebra - we have developed extremely efficient series solutions which allow us very easily to compute pure discount bond prices.

One of the problems with the specification of the state variables given in (12) is that interest rates can become negative. The process for the short rate is essentially Vasicek [1977]. The extra uncertainty added by allowing the variance of the short rate itself to follow a stochastic process implies that the probability of observing negative rates is higher in the two factor model than its one-factor equivalent.

Although the addition of an extra factor allows a greater degree of flexibility in a two factor setting, the resulting possible yield curves will still generally be inconsistent with the observed market term structure. As well as determining the initial yield curve, in both the one factor and the two factor models discussed so far the user does not have the freedom to control the term structure of volatility. The mathematical relationships in the models produce a term structure of volatility that depends on the level of the state variables and the risk-adjusted parameters used.

As an illustrative example we consider three of the one factor models described in section II. These are the Ingersoll, Vasicek and Cox-Ingersoll-Ross models and they belong to a general class of one-factor equilibrium type models which Brown and Schaefer [1992] refer to as 'affine yield class' models, so called because they produce zero-coupon yields which are affine (linear) in the short rate of interest. These models start from an Ito diffusion of the form of equation (4) and are characterised by the resulting pure discount bond prices having the functional form:

$$P(r,t,s) = A(t,s)\exp(-rB(t,s)) \tag{14}$$

where $A(t,s)$ and $B(t,s)$ are positive constants for each $t \leq s$. For the three models mentioned above we have:

*Ingersoll* $\quad B(t,s) = (s-t)$

*Vasicek* $\quad B(t,s) = \frac{1}{\alpha}\left(1 - e^{-\alpha(s-t)}\right)$

*CIR* $\quad B(t,s) = \left( \dfrac{e^{(\phi_2(s-t))} - 1}{\phi_2\left(e^{(\phi_2(s-t))} - 1\right) + \phi_1} \right)$

where $\phi_1$, and $\phi_2$ are functions of the parameters of the risk-adjusted short rate process.

The yields on pure discount bonds, $R(r,t,s)$, are given by:

$$R(r,t,s) = -\frac{\hat{A}(t,s)}{(s-t)} + \frac{B(t,s)}{(s-t)} r \tag{15}$$

where $\hat{A}(t,s) = \ln A(t,s)$. Applying Ito's lemma to equation (15) we obtain the term structure of yield volatility as:

$$\sigma_R(t,s) = \frac{B(t,s)}{(s-t)} \sigma(r,t) \tag{16}$$

In Figure 3 we plot the term structure of spot rate volatilities for one of the two-factor models, the Fong and Vasicek model, whilst Figure 4 plots the empirical structure, for yields implied by US Dollar swap rates, over a 2.5 year period beginning November 1987[5]. We can see from the differences between the figures that not being able to control for the term structure of volatility within the model could result in biased contingent claim values.

## IV/ Term Structure Consistent Models

Although the approach outlined in the previous two sections has the important advantage that all derivatives are valued on a common basis it has the severe disadvantage that the term structures, of both yields and yield volatilities, provide

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5 A full description of the data is given in section VII.

a limited family which may not correctly price many traded bonds. This can be seen in the implied shape of the forward rate curve. Once the prices of as many bonds as there are state variables are specified, the remainder of the bond prices (and hence the forward rate curve) are fixed. This implies that the forward rate curve is too simplistic and does not match the observed term structure. By valuing interest rate derivatives with reference to a theoretical yield curve rather than the actually observed curve, equilibrium models produce contingent claims prices that disregard key market information affecting the valuation of any interest rate derivative security.

The most obvious market data that could be used to price interest rate derivatives is the term structure of (spot or forward) interest rates and the term structure of interest rate volatilities. Interest rate term structures can be obtained from the prices of pure discount bonds (if they exist) or implied from the prices of coupon paying bonds. Volatility functions can be estimated statistically from historical term structure movements (see for example Heath, Jarrow and Morton [1990a]), or implied from market derivative prices (such as interest rate caps).

Many models that have appeared in the literature since 1986 are what we term 'whole yield curve' models[6]. Ho and Lee (1986) were the first authors to build a model that set out to model the dynamics of the entire term structure in a way that was automatically consistent with the initial (observed) term structure of interest rates. In much the same way that the Black-Scholes model, for pricing options on stocks, can be inverted to obtain the implied volatility of stock prices consistent with the option price, Ho and Lee reasoned that the same principle could be applied to the pricing of bonds.

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6 A deeper analysis of whole yield curve models can be found in Hodges, et al [1992]. I concentrate in this paper on the assumptions made by the models with respect to interest rate volatility.

The Ho and Lee model is developed in the form of a binomial tree relating future movements of the yield curve explicitly to it's initial state. Although the authors do not discuss the issue of convergence, a number of other authors (e.g. Dybvig (1988) and Jamshidian (1988)) show that the continuous time limit can be characterized by the short rate process:

$$dr = \theta(t)dt + \sigma dz \tag{17}$$

where $\theta(t)$, the drift during the short time interval $dt$, is a function of time in order to make the model consistent with the initial term structure of interest rates. The model describes the whole volatility structure by a single parameter $\sigma$, which is independent of the level of the short rate, and implies that spot rates and forward rates that differ in their maturity are all equally variable, all future spot rates are normally distributed, and all possible yield curves at a future time are parallel to each other. A further difficulty of the model is that it incorporates no mean reversion, and as a result there is a positive probability that future interest rates will become negative.

Heath, Jarrow, and Morton (1992) develop an arbitrage pricing model for valuing interest rate contingent claims under a stochastic term structure of interest rates, generalising the Ho-Lee model to continuous time with multiple stochastic factors. The stochastic structure of the model is exogenously imposed upon the forward rate curve, due to technical considerations about the volatilities of zero-coupon bonds as they approach maturity. In its most general form, with $n$ independent Brownian motions, the assumed stochastic process for changes in the entire forward rate curve is given by

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T)dz_i \tag{18}$$

As with the well-known Black-Scholes analysis, the drift term does not affect derivative prices, serving only a technical role. The volatility function(s) are chosen at the discretion of the user. Choosing the function to be constant, or as a function of time and the forward rate, leads to path-independent models with analytical solutions such as the continuous time Ho-Lee, Vasicek or Cox-Ingersoll-Ross. Choosing the volatility functions to best fit historical term structure movements or market options data generally leads to the evolution of the term structure being path-dependent, with a considerable increase in computation times due to the exponential growth in the tree of the discrete time approximation.

Black, Derman and Toy (1990) present a discrete time model, the continuous time limit of which is given by the stochastic differential equation:

$$d\log r = (\theta(t) - \phi(t)\log r)dt + \sigma(t)dz \tag{19}$$

with

$$\sigma(t) = e^{-\int \phi(t)dt} \tag{20}$$

The function of time in the drift term, $\theta(t)$, allows the model to be fitted to the initial spot rate curve as in the Ho and Lee model. The function $\sigma(t)$ defines the future short rate volatility and is determined entirely by the function $\phi(t)$ which is chosen to fit the current term structure of spot rate volatilities. Taking natural logarithms in (20) and differentiating with respect to $t$ we obtain:

$$\phi(t) = \frac{\sigma'(t)}{\sigma(t)} \tag{21}$$

If the future short rate volatility declines over time therefore, Black-Derman-Toy implies that $r$ is mean-fleeing.

Finally, in this theme, Hull and White (1990, 1991, 1992) present a number of articles which try to reconcile existing short rate processes with observed term structures. In it's most general form their work can be represented by the equation:

$$dr = [\theta(t) - \phi(t)r]dt + \sigma(t)r^{\beta}dz \qquad (22)$$

The three functions of time allow the model to fit the observed term structures of the spot rate and spot rate volatilities and also the volatility of the short rate over all times. The form of equation (22) allows a number of well-known models to be made consistent with the observed term structures.

In the next section we discuss a number of special cases of equation (22) and show that the behaviour of the volatility structure that these initial term structure consistent models entail is completely constrained by the model, and under certain conditions may change over time.

## V/ Constraints on the Evolution of Volatility Functions in the Hull-White Analysis[7]

Hull and White [1990] discuss in detail two special cases of equation (22), specifically the cases when $\beta=0$ and $\beta=1/2$. Their proposed models can be seen, and are presented as extensions to the Vasicek and Cox-Ingersoll-Ross models of section II due to the similarity of the nature of the short rate processes to those presented in the original papers:

7 The original ideas underlying this section came from a theorem in Carverhill [1992]. Carverhill has since written a working paper (Carverhill [1992]) outlining many ideas similar to those contained here.

$$dr = [\theta(t) - \phi(t)r]dt + \sigma(t)dz \qquad (23)$$

$$dr = [\theta(t) - \phi(t)r]dt + \sigma(t)\sqrt{r}dz \qquad (24)$$

Both of the models are of the affine yield class, i.e. the solutions to pure discount bond prices and discount bond yields are of the form of equations (14) and (15) respectively. Also the term structure of yield volatility is given by equation (16).

By substituting partial derivatives of equation (14) into the fundamental differential equation satisfied by the pure discount bond price and equating powers of the short rate, Hull-White derive a partial differential equations that must be satisfied by the function $B(t,s)$. For the extended Vasicek model this equation is given by:

$$B_t(t,s)B_s(t,s) - B(t,s)B_{ts}(t,s) + B_t(t,s) = 0 \qquad (25)$$

Solving equation (25) under the appropriate boundary conditions Hull-White obtain a closed form solution for the generalised duration measure:

$$B(t,s) = \frac{B(0,s) - B(0,t)}{B_t(0,t)} \qquad (26)$$

Combining (16) and (26) we find that the spot rate yield volatilities at time $t$, $\sigma_R(t,s)$, are constrained to be related to the initial, time 0, spot rate volatility curve in the following way:

$$\sigma_R(t,s) = \frac{\sigma(t)}{(s-t)} \frac{\left[s\sigma_R(0,s) - t\sigma_R(0,t)\right]}{\left[t\sigma_R'(0,t) + \sigma_R(0,t)\right]} \qquad (27)$$

where prime denotes partial differentiation with respect to $t$ in order to avoid a conflict of notation. If the initial volatility curve is as in Vasicek then its evolution remains stable, i.e. it doesn't change shape. If, however, the initial volatility function is different from that of Vasicek, then it may evolve in an unstable way (i.e. it doesn't retain the same shape). The subsequent evolution of the volatility curve may therefore be different from that that was originally intended by the user.

For the extended Cox-Ingersoll-Ross model there does not seem to be closed form solution to the partial differential equation governing the evolution of $B(t,s)$:

$$B_t(t,s)B_s(t,s) - B(t,s)B_{ts}(t,s) + B_t(t,s) + \frac{\sigma(r,t)^2}{2} B(t,s)B_s(t,s) = 0 \qquad (28)$$

However, the spot rate volatility curve will still evolve as a combination of equations (28) and (16). Numerical methods can be used to show that the same problems with the curve's evolution that we saw with the extended Vasicek model will exist for the extended Cox-Ingersoll-Ross model.

Attractive though whole yield curve models may be, as we have seen they are not without their dangers. At any date we must estimate functions for the term structure of interest rates, the term structure of volatility and the time path of volatility. However, the volatility functions may be constrained to evolve in an undesirable way and, just as with implied volatility in a conventional options model, when we look at market prices at a later date, there is no guarantee that they will be consistent with the previously estimated functions. In particular, choosing volatility functions to some extent independently of the term structure of interest rates poses the danger that we may ignore the information that the shape of the term structure contains about the anticipated volatility of interest rates. Convexity considerations or other more formal analysis lead very easily to relationships between the concavity of spot rates with respect to maturity and the prospective volatility of interest rates.

## VI The Relationship Between the Shape of the Term Structure and Interest Rate Volatility

In the first part of this section we study the question of the relationship between the shape of the term structure and interest rate volatility in the context of Equilibrium Models as represented by the papers Vasicek and Cox-Ingersoll-Ross. This issue has been studied empirically by Litterman, Scheinkman and Weiss [1991], and by Brown and Dybvig [1990], and theoretically by Brown and Schaefer [1991].

The paper Brown-Dybvig is empirical, and their principle motivation is to test the single factor Cox-Ingersoll-Ross equilibrium model of the term structure. This model describes the term structure at any time in terms of the short rate and 3 other parameters, which are functions of parameters of the interest rate process assumed by the model. They estimate these 4 parameters so as to fit bond prices on a month by month basis and identify the long rate and the term structure volatility in terms of these parameters. The results concerning the long rate and the stability of the extracted parameters suggest that the model is mis-specified. However, the directly observed time series of volatility exhibits a close correlation with the extracted value. Brown and Dybvig's estimation procedure, of taking 4 points along the curve, suggests that it is the shape of the term structure that is related to volatility.

The paper Litterman-Scheinkman-Weiss is largely empirical. They develop a model of the yield curve in which future short rates are expressed as a function of three variables - the current short rate, the long rate, and the volatility of the long rate, and suggest that by considering the yield spread on a butterfly, a measure of volatility of the yield curve can be obtained. The authors perform a regression of volatility (implied from the price of options trading on Treasury bond futures contracts) on the yields of one-month, 3-year, and 10-year zero-coupon bonds, finding that this linear function of yields explains 70% of the variation in the implied volatility. They conclude that the curvature in the yield curve is related to volatility.

The work of Brown and Schaefer [1992] is purely theoretical. They primarily work with the very general class of one factor Equilibrium type models, those of 'affine yield' that we referred to in Section III. The authors show that for this class of models the forward rates must have the form

$$f(r,t,s) = r + (\mu - \lambda\sigma)B(t,s) - \frac{1}{2}\sigma(r,t)^2 B(t,s)^2 \qquad (29)$$

This relationship is derived by substituting the partial derivatives of the pure discount bond price into the fundamental equation governing the value of any interest rate contingent claim when the short rate follows the process given by (4). The drift term has been risk adjusted by the risk premium $\lambda_t$. Brown-Schaefer thus develop a relatively simple relation between the forward interest rate curve, the level of interest rate volatility and the level of bond price volatility, or duration.

As pointed out by Brown and Schaefer the affine yield structure is not restricted to single factor models. In section III we studied two, multi-factor models of the term structure, namely Fong-Vasicek and Longstaff-Schwartz. In both of these models the diffusions can be characterised by square-root processes of the form:

$$\begin{aligned} dr &= \alpha_r(t)dt + \sqrt{v}\,dz_1 \\ dv &= \alpha_v(t)dt + \sigma_v(t)dz_2 \end{aligned} \qquad (30)$$

$\alpha_r(t)$ and $\alpha_v(t)$ are the (risk-adjusted) drift terms of the short rate and the variance of the short rate respectively. $\sigma_v(t)$ is the instantaneous variance of the short rate variance $v$.

Under appropriate conditions (see Cox, Ingersoll and Ross [1985]) the fundamental differential equation governing the value of a pure discount bond (or indeed any interest rate derivative security driven by the above diffusions) is given by:

$$P_\tau + \alpha_r P_r + \alpha_v P_v + \tfrac{1}{2}\sigma_r^2 P_{rr} + \rho\sigma_r\sqrt{v}\,P_{rv} + \tfrac{1}{2}vP_{vv} - rP = 0 \qquad (31)$$

For both of the two-factor models the resulting pure discount bond prices, and the yield on the bonds, $R(r,t,s)$ takes the form

$$P(r,t,s) = \exp\{G(\tau) + D(\tau)r + F(t)v\} \qquad (32)$$

$$R(r,t,s) = -\frac{G(\tau)}{\tau} - \frac{D(\tau)}{\tau}r - \frac{F(\tau)}{\tau}v \qquad (33)$$

i.e. the yields have affine structure. (NB. $\tau = s - t$.)

Taking the partial derivatives required for (31) and noting that $\frac{P_\tau}{P} = f(r,t,s)$ we obtain the following expression for the forward rate,

$$f(r,t,s) = r + \alpha_r D(\tau) + \alpha_v F(\tau) - \frac{1}{2} v D^2(\tau) - \rho\sqrt{v}\sigma_v D(\tau)F(\tau) - \frac{1}{2}\sigma_v^2 F^2(\tau) \quad (34)$$

The forward rate is therefore expressed as a function of the instantaneous rate, the parameters of the risk-adjusted process of the state variables, and $D(\tau)$ and $F(\tau)$, the exposure to interest rates (i.e. duration) and volatility respectively.

In a related paper Carverhill and Strickland [1992a] discuss the relationship of volatility and the yield curve in relation to the term structure models represented by the papers of Ho-Lee and Heath-Jarrow-Morton, which we summarised earlier as whole yield curve models. In the whole yield curve context the relationship between the curvature and the volatility is more subtle than in the equilibrium case: the shape of the initial term structure cannot be revealing about volatility, since they are both exogenous in the whole yield curve model. If we note this class of models can be regarded as an extension of the Equilibrium Model which can deal with the perturbations of the term structure from the Equilibrium (see Carverhill [1992a]); then any initial term structure which differs from that prescribed by the equilibrium model will adjust gradually, so that it tends towards having the same relationship to the volatility as in the Equilibrium Model (see Carverhill and Strickland [1992b]).

In order to be able to empirically test the relationship described above we need to be able to extract a measure of volatility from the yield curve. The relationship of curvature to volatility implies that by studying a trading strategy that has high convexity, we may be able to judge the degree of volatility anticipated by the market. Drawing on this intuition one of the tests we perform is to set up a trading strategy that 'captures' the curvature of the yield curve.

Our second test is suggested by the results which have come from a number of studies set up to identify the common factors that explain the variation in the returns on the term structure of interest rates. (See for example Steely[1990], Dybvig[1988], Carverhill and Strickland [1992a]). All of these studies find the same three factors underlie and 'drive' the term structure. The first factor corresponds roughly to a parallel shift in the yield curve, with the second and third factors corresponding to changes in the overall slope of the yield curve and changes in the curvature of the curve respectively. These findings lead us to base our second measure of implied curvature from the term structure on this third factor.

## VII The Empirical Evidence

In section VI we explored the relationship between interest rate volatility and the curvature of the term structure and outlined a number of tests designed to test this relationship. In this section we perform those tests on money market term structures implied by the prices of traded instruments and estimates of short rate volatility.

We build up the daily money market term structure from swaps rates and money market rates for two different currencies - Sterling and US Dollar. We use interbank money market rates for 6 months and 12 months and swap rates for 1, 2, 3, 4, 5, 7, and 10 years. The US Dollar swaps data covers the period 20 April 1989 to 8 August 1990. The Sterling swaps data covers the longer period 05 November 1987 to 05 March 1990[8]. All our swap data refers to semi-annual compounding of the interest rates. For the sterling data, at the short end, we use 6 month LIBOR (the London Interbank Offer Rate). For the US Dollar data we use an adjusted 6 month Treasury Bill rate. The money market, and T-Bill data were obtained from *Datastream*.

In order to estimate the money market term structures we follow the procedure outlined in Carverhill and Strickland [1992a]. We formulate the term structure in terms of spot rates of interest $R(t,s)$, and define the forward rate at time $t_i$ to run from time $t_j$ to time $t_k$ to be given by the formula:

$$f_i(t_j, t_k) = \frac{\left(1+R(t_i,t_k)\right)^{(k-i)}}{\left(1+R(t_i,t_j)\right)^{(j-i)}} \tag{35}$$

In order to compare the volatility we obtain from the curvature of the yield curve we use two different methods for obtaining a measure of the volatility of interest rates.

The first, which we shall use for the US Dollar data, is an implied measure that we obtain from the prices of money market caps. The cap price data forms a term structure of volatility for money market interest rates. It consists of volatilities that, when inserted into Black's [1976] formula for pricing the individual caplets in an interest rate cap agreement, give the correct market price. In this way the volatility is implied from traded prices and is an 'average' figure over the life of the cap[9].

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8 Swap data was provided by Credit Suisse Financial Products in London

9 Cap data was also provided by Credit Suisse Financial Products in London.

The second method, which we shall use for the Sterling data, involves implying the price volatility implicit in options on short sterling futures contracts, as traded on the London International Financial Futures Exchange (LIFFE)[10]. We use the widespread procedure for valuing short sterling futures options of assuming that the implied futures rate follows a geometric Brownian motion with constant volatility, and then applying Black's (1976) model[11].

As indicated at the end of section VI, in order to isolate and measure curvature from the shape of the yield curve, we perform two sets of analysis; we set up a trading strategy that has high convexity and perform a principle components analysis on the time series of the forward rate curve.

Our trading strategy with high convexity is a 'barbell' strategy consisting of a portfolio of (money market) pure discount bonds which is long two bonds of different maturities at either end of the yield curve and short an intermediate maturity bond. We perform a regression of the interest rate volatility implied from the prices of caps (in the Dollar case) and short sterling options (in the Sterling case) against the yields from these particular bonds.

**US Dollar**

$$IV(t) = 29.166 - \underset{(1.413)}{2.568} y_{0.5}(t) + \underset{(1.384)}{8.518} y_{2.0}(t) - \underset{(0.323)}{7.041} y_{10.0}(t)$$

$$\overline{R}^2 = 77\% \qquad se = 1.02$$

10 Data on short sterling futures options was provided by LIFFE.
11 See Bates and Lcewlow (1992) for the full procedure.

**Sterling**

$$IV(t) = 35.000 - \underset{(0.278)}{0.446} y_{0.5}(t) + \underset{(1.419)}{2.957} y_{2.0}(t) - \underset{(0.688)}{0.615} y_{10.0}(t)$$

$$\overline{R}^2 = 41\% \qquad se = 3.01$$

The $y_i$'s are yields (in %) on *i* th year maturity bonds, and the figures in parenthesis are standard errors. The sign configuration for both sets of data is evidence that the downward curvature of the spot rate curve is positively related to the volatility. The high $\overline{R}^2$ for the Dollar data suggests a close correspondence between the level óf interest rate volatility and the yield spreads. Our first feelings on the low explanatory power for the Sterling data are connected with the nature of the fixed expiry dates of the short sterling options. Figures 5 and 6 illustrate the correspondence between the two volatility values.

We perform principle components analysis in order to identify factors that represent relationships among our variables. Table 1 presents the three largest principal components for the evolution of the money market term structure based on the US Dollar data. The table shows that the principal component accounts for slightly less than 95 percent of the term structure evolution, with the second component accounting for nearly 93 percent of the remaining variation, but only 5% of total variation of the evolution. There is thus one highly dominant factor affecting the co-movements in the spot rates. Table 2 presents the three largest principal components for the evolution of the money market term structure based on the Sterling data. The magnitude of the percentage of yield curve evolution accounted for by each of the factors is similar to the US Dollar case.

Principal components analysis gives us, via the resulting eigenvectors, the sensitivities of the term structure to changes in the components (the factors). In Figures 7a, 7b, 7c we show the impact of factors 1, 2, and 3 respectively on the set of average spot rates for the Sterling money market term structure. The figures show that the 3 principal factors affecting the co-movements in the spot rates corresponds roughly to parallel shifts in the spot rate curve (all the rates have roughly the same loading on the first factor), changes in the overall slope of the money market yield curve (the second factor causes opposite changes in short term and long term rates), and changes in curvature of the spot interest rate curve[12].

We now express the forward rate curve in terms of its principal components.

$$\text{forward curve} = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \text{noise} \qquad (36)$$

In this way we obtain a time series of values for the coefficients $\alpha_i$ so that the noise term is minimised in terms of least squares. The coefficients $\alpha_1, \alpha_2, \alpha_3$, can be taken, as suggested, to represent the level, the degree of tilt and the curvature of the term structure respectively, and the differences in these coefficients from one time to the next represent in an optimally efficient way, how these factors change.

12 Similar results are obtained for the US Dollar data set.

We regress the (implied) volatility against the independent variables $\alpha_1, \alpha_2, \alpha_3$. The full regression equations are given by;

**US Dollar**

$$IV(t) = 13.187 + 0.267\alpha_1(t) - 0.049\alpha_2(t) + 1.088\alpha_3(t)$$
$$(0.077) \qquad (0.074) \qquad (0.050)$$

**Sterling**

$$IV(t) = 15.416 + 0.198\alpha_1(t) - 0.029\alpha_2(t) + 0.0088\alpha_3(t)$$
$$(0.010) \qquad (0.014) \qquad (0.0080)$$

By performing a stepwise regression, for the dollar data, we find that the regression against $\alpha_3$ alone explains nearly 75% of the variation of the implied volatility. Including the independent variables $\alpha_1$ and then $\alpha_2$ only increases the $\overline{R}^2$ to 76.3%, and 76.4% respectively. For the sterling data the figures are 39%, 41%, and 42% respectively. Figures 8 and 9 compare the history of implied volatility obtained from traded cap and short sterling option prices and the implied volatilities estimated from the time series of the independent variables.

## VII/ Summary and Conclusions

We have discussed the important role played by interest rate volatility in the pricing of interest rate derivative securities. Single factor equilibrium models of the term structure, although providing analytical solutions to a number of derivative pricing problems, were found not to be consistent either with the empirically observed term structure or with the observation of interest rate volatility itself being volatile.

We analysed two recent models of the term structure which start from the traditional term structure theory but which explicitly recognise interest rate uncertainty as a second stochastic factor - so allowing a richer variety of possible yield curves than single factor models.. The first model - Longstaff and Schwartz- is able to price discount bonds and options on these bonds with closed form solutions, although the cumulative distribution functions of the latter can be cumbersome to evaluate. The second - Fong and Vasicek -starts from a specification of state variables that are more easily interpretable than their equivalents in the Longstaff-Schwartz model, but with the restrictions that this specification can admit negative interest rates.

Even with the addition of an extra stochastic factor equilibrium models of the yield curve provide a limited family of term structures which generally fail to fit the observed data. We describe a number of 'whole yield curve' models which are consistent with an arbitrary initial term structure. Some of these models allow the input of market driven volatility functions. We show how, for two popular models proposed by Hull and White, that these volatility functions may be constrained to evolve through time in an unstable way.

Finally, we discuss the tension between the whole yield curve approach to modeling the term structure and recent studies of the relationship between the shape of the yield curve and interest rate volatility. An empirical study of this relationship using swaps data for two currencies, US Dollar and Sterling, confirms that the shape of the yield curve and interest rate volatility are indeed related.

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**Table 1. Principal Components Analysis of the Money Market Term Structure for US Dollar data**

| **Eigenvalue** | 18.9281 | 0.9954 | 0.0478 |
|---|---|---|---|
| Percentage | 94.6 | 5.0 | 0.2 |
| Cumulative % | 94.6 | 99.6 | 99.9 |

| Spot Rate Maturity | **Eigenvectors** | | |
|---|---|---|---|
| | FACTOR 1 | FACTOR 2 | FACTOR 3 |
| 0.5 | .87751 | .45633 | .14444 |
| 1.0 | .92437 | .37648 | .06109 |
| 1.5 | .95034 | .30901 | -.00543 |
| 2.0 | .96386 | .25310 | -.05824 |
| 2.5 | .97629 | .20540 | -.05761 |
| 3.0 | .98661 | .14765 | -.05697 |
| 3.5 | .99260 | .10869 | -.04609 |
| 4.0 | .99629 | .06460 | -.03406 |
| 4.5 | .99846 | .02093 | -.03220 |
| 5.0 | .99769 | -.02822 | -.03006 |
| 5.5 | .99730 | -.05549 | -.02317 |
| 6.0 | .99566 | -.08522 | -.01578 |
| 6.5 | .99247 | -.11748 | -.00784 |
| 7.0 | .98737 | -.15234 | .00070 |
| 7.5 | .98403 | -.17527 | .00872 |
| 8.0 | .97960 | -.19947 | .01710 |
| 8.5 | .97394 | -.22491 | .02585 |
| 9.0 | .96692 | -.25155 | .03495 |
| 9.5 | .95839 | -.27932 | .04441 |
| 10.0 | .94819 | -.30816 | .05422 |

**Table 2. Principal Components Analysis of the Money Market Term Structure for Sterling data**

| **Eigenvalue** | 19.20124 | 0.54715 | 0.19950 |
|---|---|---|---|
| Percentage | 96.0 | 2.7 | 1.0 |
| Cumulative % | 96.0 | 98.7 | 99.7 |

| Spot Rate Maturity | **Eigenvectors** | | |
|---|---|---|---|
| | FACTOR 1 | FACTOR 2 | FACTOR 3 |
| 0.5 | .89176 | .38654 | .23136 |
| 1.0 | .94235 | .31229 | .11917 |
| 1.5 | .97275 | .22730 | -.00092 |
| 2.0 | .97950 | .13732 | -.12030 |
| 2.5 | .98475 | .11910 | -.11361 |
| 3.0 | .98639 | .09909 | -.10771 |
| 3.5 | .99161 | .07029 | -.09605 |
| 4.0 | .99302 | .04293 | -.08534 |
| 4.5 | .99454 | .01690 | -.08714 |
| 5.0 | .99393 | -.00833 | -.08877 |
| 5.5 | .99564 | -.04405 | -.06575 |
| 6.0 | .99505 | -.07998 | -.04268 |
| 6.5 | .99216 | -.11587 | -.01963 |
| 7.0 | .98703 | -.15144 | .00329 |
| 7.5 | .98722 | -.15310 | .02397 |
| 8.0 | .98656 | -.15528 | .04612 |
| 8.5 | .98490 | -.15795 | .06980 |
| 9.0 | .98205 | -.16104 | .09509 |
| 9.5 | .97782 | -.16453 | .12209 |
| 10.0 | .97196 | -.16837 | .15083 |

FIG. 4.

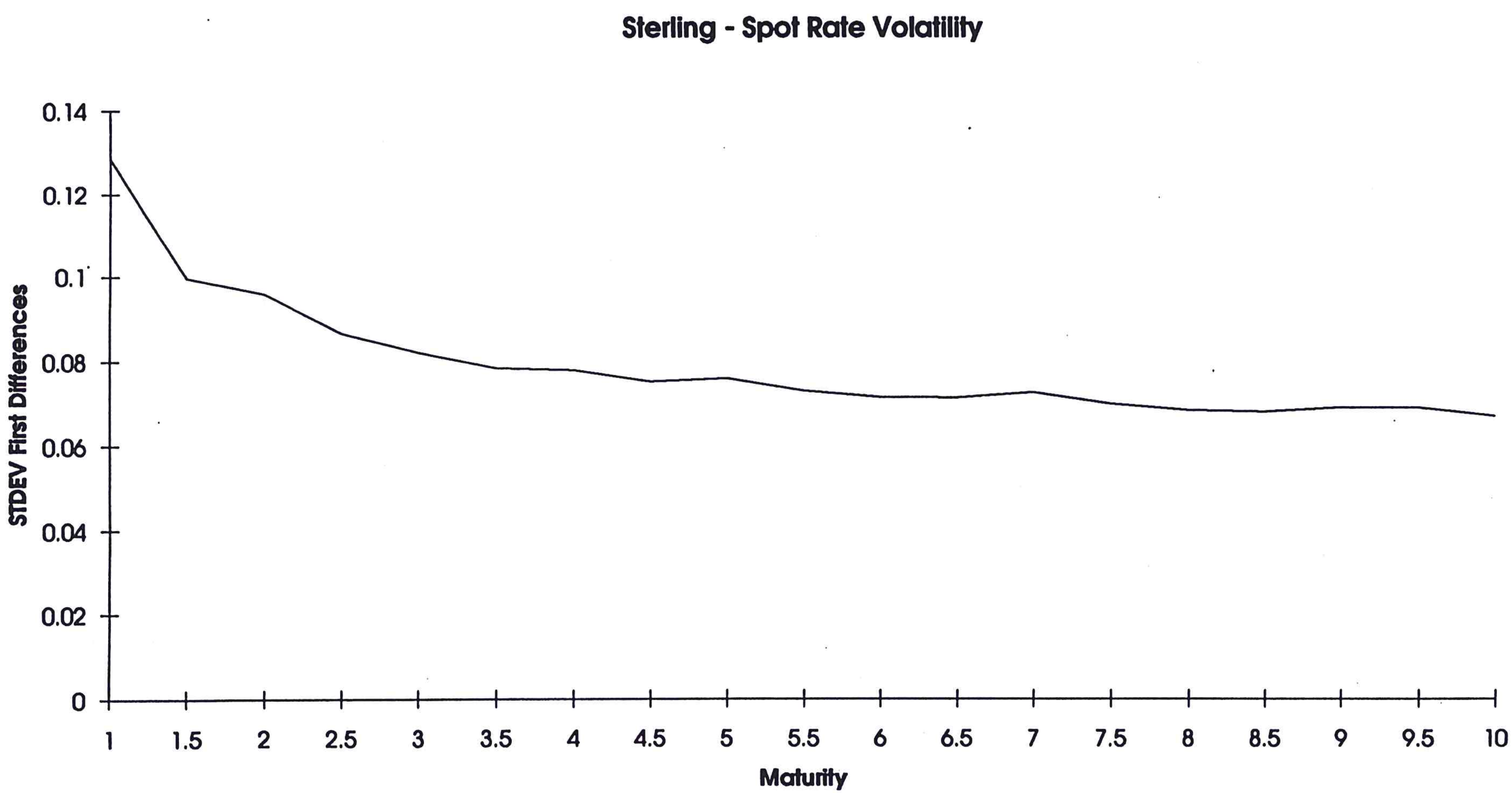
Sterling - Spot Rate Volatility
STDEV First Differences
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10
Maturity

FIG. 6.

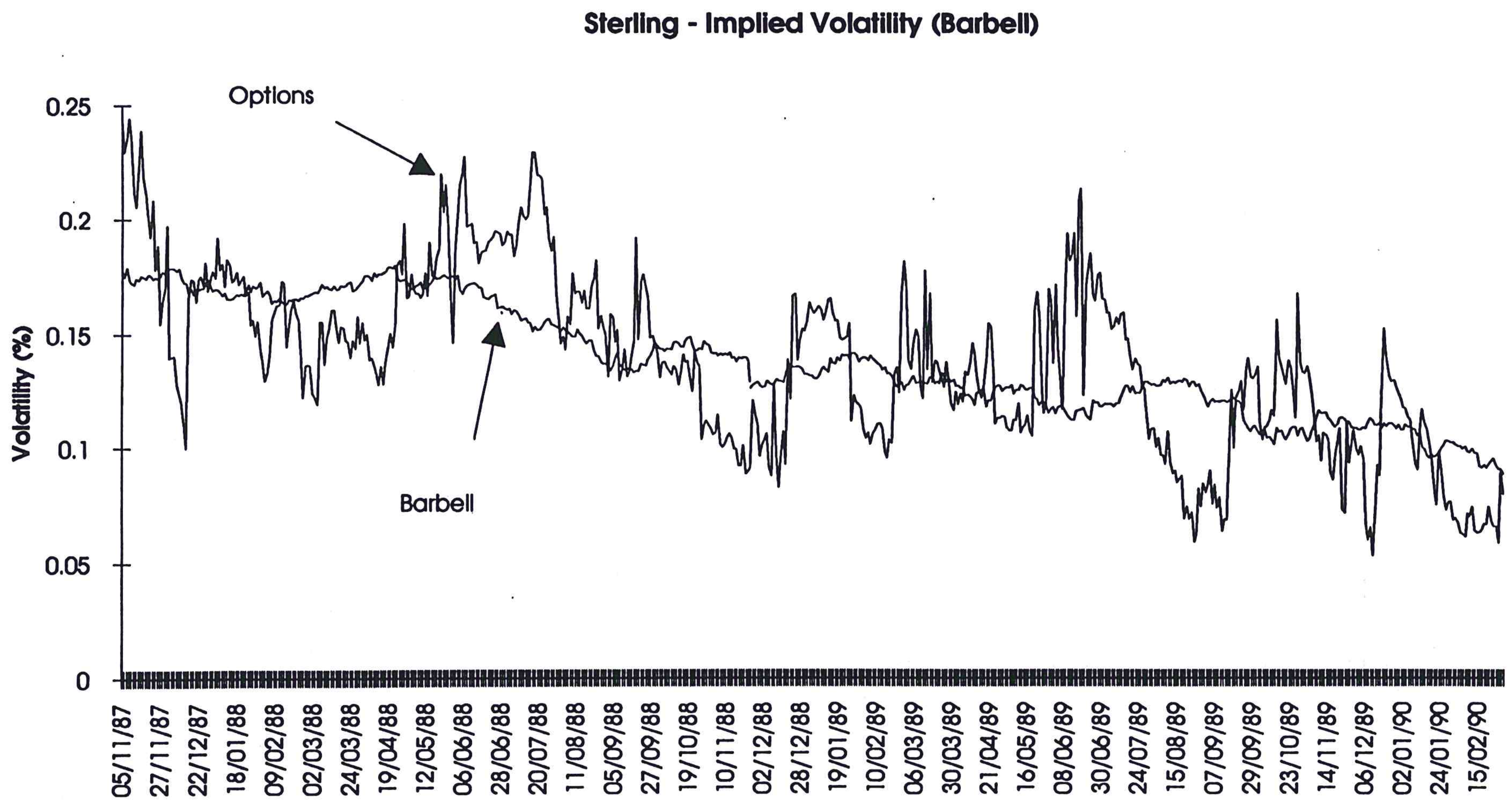
Sterling - Implied Volatility (Barbell)
Volatility (%)
0.25
0.2
0.15
0.1
0.05
0
Options
Barbell
05/11/87
27/11/87
22/12/87
18/01/88
09/02/88
02/03/88
24/03/88
19/04/88
12/05/88
06/06/88
28/06/88
20/07/88
11/08/88
05/09/88
27/09/88
19/10/88
10/11/88
02/12/88
28/12/88
19/01/89
10/02/89
06/03/89
30/03/89
21/04/89
16/05/89
08/06/89
30/06/89
24/07/89
15/08/89
07/09/89
29/09/89
23/10/89
14/11/89
06/12/89
02/01/90
24/01/90
15/02/90

Fig 7a

## Sterling (PC1)

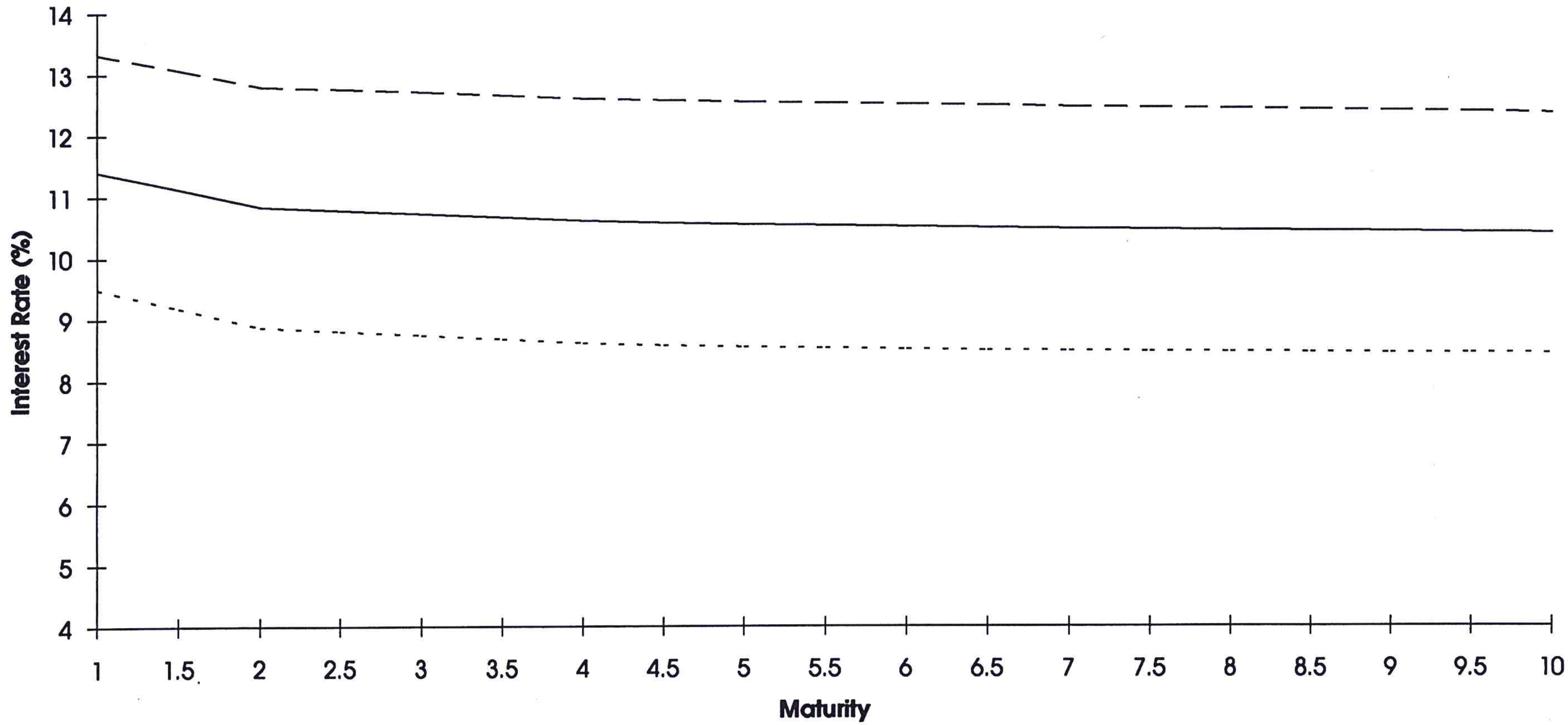

Fig 7b

## Sterling (PC2)

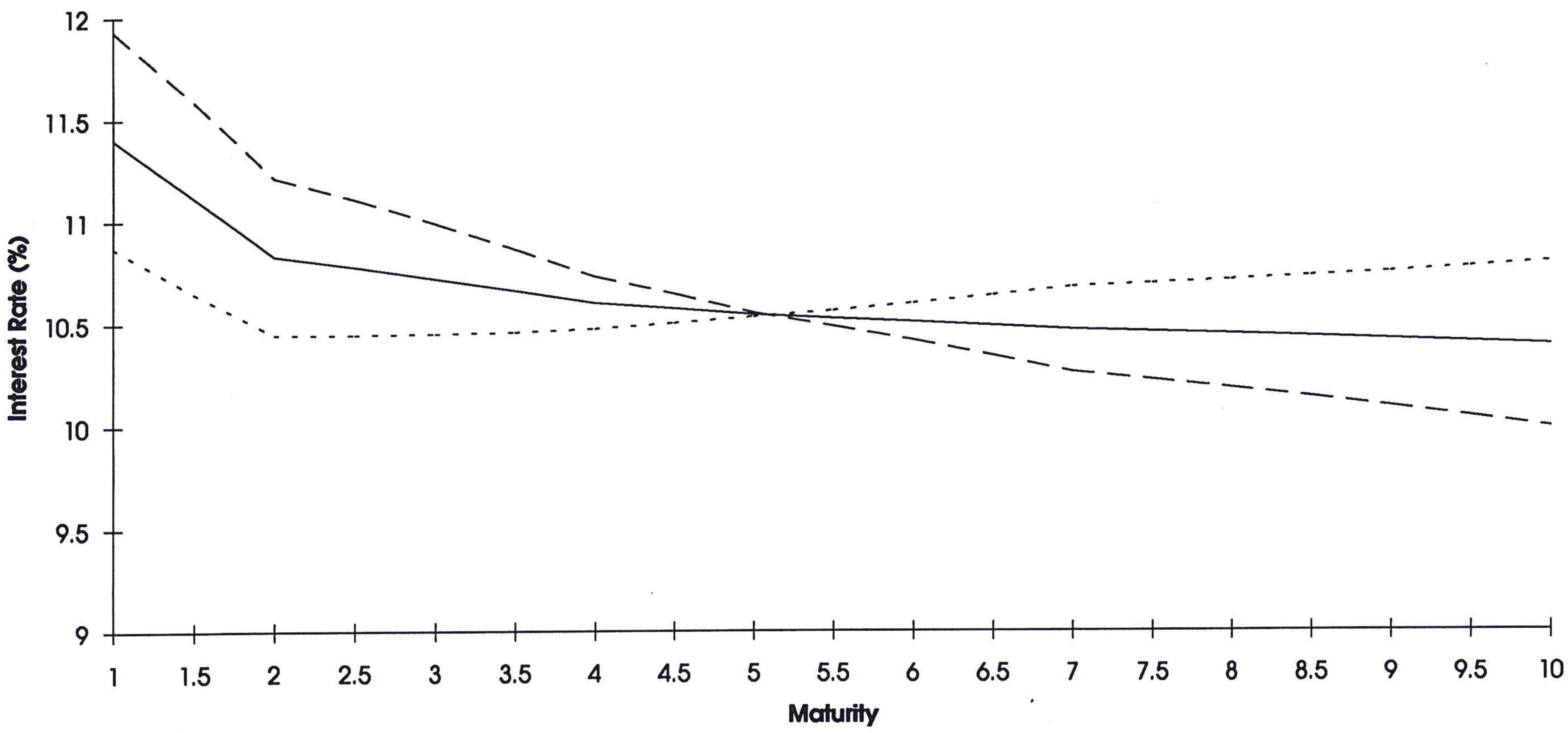

Fig 7c

## Sterling (PC3)

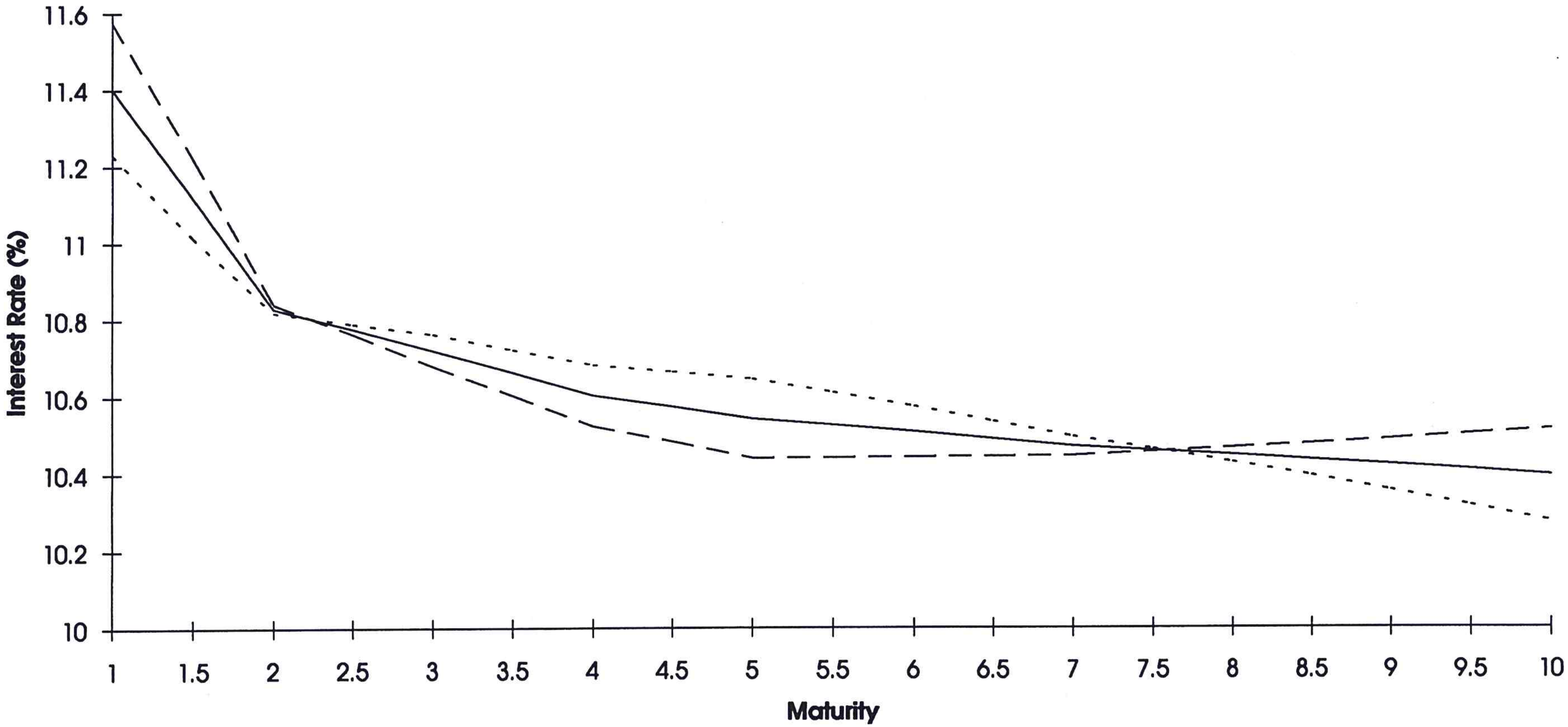

FIG 8



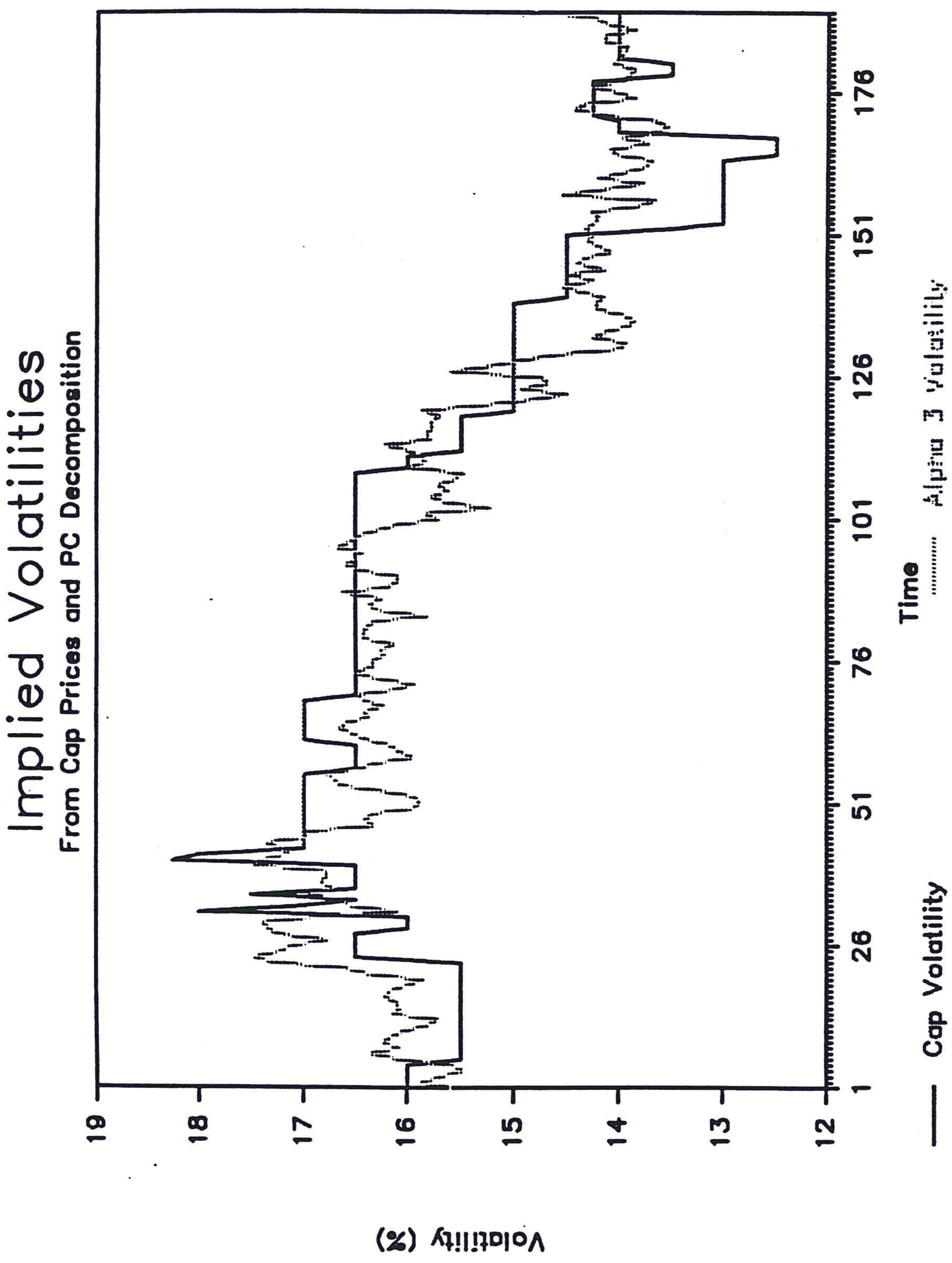


Figure 7.3

Source: Carverhill & Strickland (1992)

FIG 9

## Sterling - Implied Volatility