Contingent Claims Analysis

Simon Babbs*

and

Michael Selby**

*Midland Global Markets

**Financial Options Research Centre
University of Warwick†
†(Michel Selby was affiliated to the London School of Economics when this article was written)

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Financial Options Research Centre
Warwick Business School
University of Warwick
Coventry
CV4 7AL
Phone: 0203 523606

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Abstract

The purpose of this article, is to give a concise overview of the modern theory of contingent claims analysis (CCA). CCA is possibly the most significant development in financial economics over the last twenty years. From its origins in option pricing and the valuation of corporate liabilities, it has become a major approach to intertemporal general analysis equilibrium under uncertainty. It has made important contributions to economic theory. Moreover, its focus on the pricing and replication of contingent payoffs offers insight into the role of financial intermediaries.

Contingent claims are contracts whose outcomes depend on the evolution of one or more uncertain variables. Contingent Claims Analysis (CCA) is that branch of financial economics which focuses on the valuation of such contracts.

Commercial examples of contingent claims include: futures and options contracts based on commodities, stock indices, interest rates, exchange rates, or on individual stocks; and mortgage-backed securities. (See eg Merton, 1990.) Other example include: Arrow-Debreu securities, which play a key role in general equilibrium theory, and options. A notable insight of Black and Scholes (1973) states that all corporate liabilities can be viewed as combinations of options on the total assets of the firm. Moreover, the outcome of a portfolio strategy can also be viewed as a contingent claim. Under conditions of general equilibrium, therefore, the pricing of contingent claims is intimately related to the optimal solutions of agents' portfolio planning problems. CCA is, therefore, a central part of financial economics. It achieves its results by integrating the option pricing theory initiated by Black and Scholes (1973) with the optimal portfolio planning problem under uncertainty.

One striking feature of CCA is that many of its valuation formulae are largely - or even totally - free from explicit dependence on agents' preferences and on expected returns. The attractiveness of formulae with these properties is recognised as one stimulus prompting a proliferation of academic research, and even of new commercial instruments. Formulae altogether lacking explicit dependence on preferences are sometimes referred to as Risk Neutral Valuation Relationships (RNVRs). In many leading models, the possibility of these formulae often stems from, *inter alia*, the assumptions of frictionless securities markets which are open continuously.

Many of the most significant contributions to CCA have adopted continuous time models utilising powerful mathematical techniques from stochastic calculus and martingale theory. Nevertheless, contributors and users of CCA have frequently conducted at least part of their analysis in a multi-period discrete time setting with a finite state space; the motivations have been either to develop or expound intuitive insights at a mathematically less demanding level, or to develop numerical approximation schemes for continuous time, continuous state space models. We shall not discuss discrete time approaches further, but refer the reader to Willinger and Taqqu (1991).

OPTION PRICING THEORY

The fundamental insight of Black and Scholes (1973), leading to their celebrated option pricing formula, was, as clarified by Merton (1973), that the option could be priced relative to the underlying stock, by the construction of a dynamically managed portfolio involving the option, the stock, and default-free borrowing or lending. Assuming only frictionless markets and the absence of arbitrage opportunities, the option value could be deduced.

Merton (1977) showed that the Black-Scholes-Merton approach is applicable to a very broad class of derivative securities. The approach has been applied, both before and since, to a vast range of contingent claims with significantly different payoff structures and/or underlying securities. Its applicability has been extended beyond that covered by Merton to embrace, for example, futures contracts (see eg Cox, Ingersoll and Ross, 1981).

A significant limitation of the Black-Scholes-Merton approach is that it is essentially a partial equilibrium analysis, deriving relative pricing results based on specifying exogenously the stochastic processes for the underlying variables. It was unclear whether the models based on the approach are viable, ie capable of holding in a general equilibrium, or even whether those models exclude the possibility of arbitrage strategies. A further question was to determine conditions under which markets are complete, ie under which any contingent payoff can by obtained by following an appropriate trading strategy.

GENERAL EQUILIBRIUM AND PRICING BY ARBITRAGE

Cox, Ingersoll and Ross (1985a) integrated general equilibrium analysis and CCA, by determining a rational expectations, expected utility maximization, equilibrium in an economy incorporating both a real sector and securities markets in contingent claims. In this type of general equilibrium approach, key assumptions are made concerning technology and preferences; all financial variables are endogenous. This contrasts with option pricing theory, where technology and preferences are essentially arbitrary, but the stochastic processes followed by financial variables are exogenously specified.

Ross (1977, 1978) provided a pointer to the link between option pricing theory and pricing under general equilibrium, by showing that the absence of arbitrage opportunities implies the existence of a linear pricing operator for traded contingent payoffs. The link itself, however, was established by Harrison and Kreps (1979).

Harrison and Kreps showed that the prices, at which claims to state-contingent consumption can be obtained, are viable if and only if they are consistent with a linear pricing operator for all contingent claims. In equilibrium, agents' trading strategies are the optimal solutions to their portfolio planning problems. A claim is said to be "priced by arbitrage" if all consistent pricing operators assign it the same value.

Introducing securities markets, Harrison and Kreps demonstrated a reciprocal relationship between consistent pricing operators and "equivalent martingale measures" (EMMs). An EMM is a reassignment of the probabilities attaching to future uncertain events, such that the price processes of securities, when expressed in units of some numeraire security, become martingales. If there exists a unique EMM - such as Harrison and Kreps proved to exist for the Black-Scholes model - then the model is viable and all contingent claims must be priced by arbitrage.

Under the pricing operator corresponding to any particular EMM, and expressed in units of the numeraire security, the value of a contingent claim equals the expectation of its payoffs, with respect to the EMM probabilities. Harrison and Kreps noted the correspondence between this characterisation and the "risk neutral pricing" concept of Cox and Ross (1976). One might argue that "risk adjusted pricing" would be a more apt term, since the change of probability measure in passing from the actual probabilities to the EMM is equivalent, at least in many models, to adjusting the expected returns on securities to remove risk premia.

Harrison and Kreps established their results under the restriction that agents could follow only "simple" trading strategies, under which portfolios are revised only at an arbitrary prespecified finite set of fixed dates. It appears that, in models with infinite state space, the admissibility of at least "simple" trading strategies is a necessary condition for any extensive ability to price by arbitrage. If such strategies are excluded, the valuation results can only be recovered by restricting agents' preferences. Working in discrete time, Rubinstein (1976) and Brennan (1979) were able to obtain RNVRs provided that agents' preferences were restricted to being drawn from a particular class. They did not require any ability to trade the underlying assets - which is required in continuous time frameworks (see Constantinides, 1978).

However, a generous specification of the opportunities to trade is not a sufficient condition for extensive pricing by arbitrage. One example is provided by Merton (1976), who extended the Black-Scholes option pricing model by describing the stock price process with both jump and diffusion components. The available securities (the stock and borrowing/lending) are then unable to span the increased number of sources of uncertainty. Merton found that it was not possible to create a riskless hedge for the option - even with continuous trading. He required an assumption about the market price of jump risk in order to value the option. In terms of the EMM analysis of Harrison and Kreps, Merton's model possesses a family of EMMs parametrized by the market price of jump risk; specifying that price selects an EMM, enabling the option to be valued using the corresponding pricing operator.

A further example shows that even when coupled with "dynamic spanning" - ie the existence of a range of securities sufficient to hedge the full number of sources of uncertainty - a rich set of opportunities to trade is insufficient to ensure extensive pricing by arbitrage. The models of Vasicek (1977) and Cox, Ingersoll and Ross (1985b) (and related models by others) seek to determine the term structure of interest rates by valuing bonds as contingent claims on the evolution of the short term interest rate, the latter being the sole source of uncertainty. Unlike the stock price in the Black-Scholes model, the short term interest rate is not an asset price; its expected changes do not tell us the price of risk. Even while allowing continuous trading, Vasicek, and Cox, Ingersoll and Ross found it necessary to specify the price of interest rate risk to close the model, and that price features in the eventual term structure formulae. In contrast, in the framework recently independently developed by Heath, Jarrow and Morton (1992) and

Babbs (1990, 1991a), the dynamics of the term structure are modelled consistently with whatever initial structure is actually observed. The price processes of bonds are indirectly but explicitly specified by the constructed dynamics; expected bond returns, therefore, tell us the price(s) of risk(s), and all term structure contingent claims are priced by arbitrage.

COMPLETENESS

Whereas Harrison and Kreps (1979) focused on viability and the pricing of contingent claims, Harrison and Pliska (1981, 1983) set aside the issue of viability, expanding dramatically the range of trading strategies admitted by Harrison and Kreps, to determine conditions under which markets are complete. (Babbs, 1990, 1991b, has shown how the results of Harrison and Kreps can be extended to cover strategies similar to those admitted by Harrison and Pliska.) Completeness turns out to depend - via various "martingale representation theorems" - on the fine structure of the way in which economic uncertainty is progressively resolved, and on the availability of a rich set of trading opportunities in a range of securities sufficient to provide dynamic spanning.

The securities markets models discussed in previous sections neglected welfare issues. Indeed, the possibility of addressing the relationship between competitive equilibrium and Pareto optimality, in the spirit of the Arrow-Debreu model, did not appear promising. In the classical discrete time and space framework used by that model, the results hinged on the ability to construct a pure security for each individual state. In contrast, most CCA models adopt a continuous time framework based, say, on diffusion processes. The state space is then uncountable, so that it appears that an uncountable infinity of Arrow-Debreu securities would be required for complete markets. Building on the work of Harrison and Pliska, however, Duffie and Huang (1985), and Duffie (1986) were able to show that a complete Arrow-Debreu equilibrium, in the dynamic sense propounded by Radner (1972), could be implemented via the dynamic spanning achievable by trading a finite set of long-lived securities, even though the consumption space was infinite-dimensional.

From their analysis, Duffie and Huang were able to confirm earlier insights and conjectures of Arrow (1953/1964), Kreps (1982), and Merton (1982), that dynamic spanning is the key to completeness of markets. Indeed, it finally became clear that it is essentially the dynamic spanning of uncertainty by the stock and riskless borrowing/lending that drives the Black-Scholes option pricing formula; it had previously been thought by some writers that subsidiary features of those models, such as continuous price processes, were directly the key.

TRANSACTIONS COSTS

The insights of Harrison and Kreps (1979) have enabled the earlier partial equilibrium option pricing results of Black and Scholes (1973), Merton (1973), and others, to be seen as flowing from a special kind of general equilibrium analysis, in which the key assumptions concern security price processes, rather than being made at the more fundamental level of technology and agents' preferences. Nevertheless, a growing body of CCA, concerned with the impact of transactions costs, retains a partial equilibrium approach.

Many of the trading strategies which replicate contingent claims in standard models, involve infinite trading volumes. When, therefore, the assumption of frictionless markets is relaxed to admit transactions costs, infinite costs are incurred. This implies that buying and selling prices of contingent claims no longer need to coincide to preclude arbitrage opportunities, and the linear pricing operator / EMM general equilibrium approach, deriving from Harrison and Kreps (1979), breaks down.

In response, some authors (eg Leland, 1985) have sought to produce preference-independent bounds for option values by considering *ad hoc* adaptations of replication strategies drawn from models with frictionless markets. Others (eg Constantinides, 1986, and Davis and Norman, 1990) have addressed various optimal portfolio/consumption problems faced by an individual investor seeking to maximize expected utility.

Appreciation of the impact of transactions costs upon the ability to replicate contingent claims provides an insight into the role of financial intermediaries. Unlike other agents, market-makers, for example, are perceived as trading with negligible transactions costs. They are, therefore, able to provide a diversity of instruments with contingent payoffs, while hedging their risks by means of strategies akin to those suggested by models with frictionless markets.

CONCLUDING REMARKS

CCA is possibly the most significant development in financial economics over the last twenty years. From its origins in option pricing and the valuation of corporate liabilities, it has become a major approach to intertemporal general analysis equilibrium under uncertainty. It has made important contributions to economic theory. Moreover, its focus on the pricing and replication of contingent payoffs offers insight into the role of financial intermediaries.

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