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Abstract

This paper studies a model for the optimal management of a portfolio when there are transaction costs proportional to a fixed fraction of the portfolio value. The risky securities are modelled as correlated geometric Brownian motions, there is a riskless bank account, and the objective is to maximise the long-run growth rate. It is known that the optimal trading strategy is characterised by the solution of a certain PDE free boundary problem. This paper explains how to transform this free boundary problem for the case of three securities into a much simpler one that is feasible to solve with numerical methods.

Keywords: optimal stopping, change of variables, Markov process, portfolio management, finance, partial differential equations.

1. INTRODUCTION

In a closely related paper, Morton and Pliska (1993) introduce and develop a model for the optimal management of a portfolio with fixed transaction costs. Their model features a savings account with a constant interest rate r and m risky stocks whose prices Z_t^k, k = 1,..,m, are correlated geometric Brownian motions, that is,

$$dZ_t^k = Z_t^k(\mu_k dt + \sum_{j=1}^{m} \lambda_{kj} dW_t^j), \qquad k = 1,..,m,$$

where μ is an m vector of appreciation rates, λ_{ij} is the (i,j) entry in the m by m matrix Λ, and $W^1,...,W^m$ are independent Brownian motion processes. Thus the time t price Z_t^k of stock k has a log-normal distribution with mean $Z_0^k \exp(\mu_k t)$. It is assumed that the variance-covariance matrix $M \equiv \Lambda\Lambda'$ is of full rank and that all the components of the vector $(\Lambda\Lambda')^{-1}(\mu - r1)$ are strictly positive and sum to less than one.

The portfolio manager starts with initial capital V_0 and strives to maximize the long-run, asymptotic growth rate

$$\liminf_{T \to \infty} \frac{E[\ell n\, V_T]}{T},$$

where V_T is the value of (that is, the amount of money in) the portfolio at time T. No money can be added to or withdrawn from the portfolio (except for transaction costs paid to the broker), there is no short-selling of stocks, and there is no borrowing of funds at the interest rate r.

The portfolio manager is free to use a very general, nonanticipative trading strategy governing the allocation of funds between the individual stocks and the savings account, but each time a transaction occurs, that is, each time funds are shifted between two or more stocks or between the stocks and the savings account, a transaction cost is incurred equal to the fraction $1 - \alpha$ times the current value of the whole portfolio (here $0 < \alpha < 1$, and normally α is close to 1). Aside from the portfolio's value, this transaction cost does not depend on the number of, the prices of, or the positions in the stocks that are involved in the transaction at the point in time when the transaction occurs. Hence if a transaction occurs at time T when the value of the portfolio is V_{T-}, then the amount $(1-\alpha)V_{T-}$ is paid to the stock broker and the portfolio continues with the new value $V_T = \alpha V_{T-}$.

Morton and Pliska (1993) show that the trading strategy which maximizes the asymptotic growth rate is fully described by an m vector b (whose components are strictly positive and sum to less than one) and a stopping time τ (which will be described below). The idea is very simple. The portfolio manager starts out with the initial funds allocated among the stocks according to the vector b, and the balance of the funds in the savings account. In other words, $b_k V_0$ equals the initial investment in stock k, $k = 1,..,m$. The first transaction occurs according to the stopping rule τ: the amount $(1-\alpha)V_{\tau-}$ is paid to the stock broker and the remaining funds $V_\tau = \alpha V_{\tau-}$ are reallocated (that is, rebalanced) according to the vector b. Now $b_k V_\tau$ equals the investment in security k, and this cycle then repeats itself indefinitely.

The optimal values of b and τ, denoted b^* and τ^*, are related to a problem of optimally stopping a Markov process B called the "risky fraction" process. The m dimensional process B is simply the vector of fractional allocations you would get if the initial allocation is B_0 and you never do any transactions; in other words, B_T^k would be the fraction of money held in stock k if no transactions ever occur.

The solution of the optimal stopping problem is fully described by a "continuation region"

$$\mathscr{C} \subset \{b \in \mathbb{R}^m : b_1 > 0, .., b_m > 0, b_1 + .. + b_m < 1\}.$$

It turns out that $\mathscr{C}$ is an open set which contains b^*. The optimal stopping rule τ^* for the portfolio manager is simply the first exit time from $\mathscr{C}$ by the process B which started with $B_0 = b^*$. In other words, having just rebalanced in order to achieve the allocation b^*, the portfolio manager does no transactions for a while but pays close attention to the fractional allocations among the m risky stocks. This is the same as watching the evolution of the risky fraction process B, which had been reset equal to b^* at the time of the last transaction. Moreover, τ^* will be the elapsed time between transactions. When B hits the boundary of $\mathscr{C}$, the next transactions are made, the portfolio is rebalanced to b^*, and the cycle is repeated.

In order to specify b^* and $\mathscr{C}$, it is necessary to specify the value function for the optimal stopping problem:

$$f_R(b) \equiv \sup_{\tau}\{-E_b[\ln(1 - 1'B_\tau)] - (R - r)E_b[\tau]\}.$$

Here R is a parameter $R > r$ whose role will be described below, and E_b denotes expectation conditioned on $B_0 = b$. Hence $f_R(b)$ is the value (that is, the expected net payoff) of being able to optimally stop the Markov process B when its initial value is b, a reward of $-\ln(1 - 1'B_\tau)$ is collected for stopping in state B_τ, but up until the time of stopping, a "continuation fee" of $R - r$ per unit time is paid.

As shown by Morton and Pliska (1993), once you know the value function f, it is a simple matter to compute b^* and $\mathscr{C}$. The continuation region $\mathscr{C}$ is simply given by

$$\mathscr{C} = \{ b \in \mathbb{R}^m : f_R(b) > -\ln(1 - 1'b) \}.$$

The optimal rebalance vector b^* will be the solution along with the parameter R of the $m + 1$ equations:

$$\frac{1}{1 - 1'b} = \frac{\partial f_R(b)}{\partial b^k}, \qquad k = 1,..,m$$

$$0 = \ln \alpha + \ln(1 - 1'b) + f_R(b).$$

The value of the parameter R that is part of this solution will turn out to be the maximum growth rate for the portfolio.

Hence to solve this optimal portfolio problem, it suffices to compute the optimal stopping problem's value function $f_R(b)$. Morton and Pliska (1993) show this is given by the solution H of the following free boundary problem:

$$\frac{1}{2} \sum_i \sum_j H_{ij} b_i b_j [(e_i'-b')M(e_j-b)] + \sum_i H_i b_i [(e_i'-b')(\mu-r1-Mb)] = R - r$$

$$H(b) > h(b) \equiv -\ln(1 - b_1 - .. - b_m), \qquad b \in \mathcal{C}^\circ$$

$$H(b) = h(b), \qquad b \in \partial\mathcal{C}$$

$$\frac{\partial H(b)}{\partial b_i} = \frac{\partial h(b)}{\partial b_i}, \qquad i = 1,..,m; \quad b \in \partial\mathcal{C}.$$

Here the subscripts on H denote partial derivatives, the subscripts on the vector b denote its components, and e_i denotes an m-component column vector consisting of zeros except for a 1 in the ith entry.

Morton and Pliska (1993) show that this free boundary problem is easy to solve numerically when there is only $m = 1$ risky stock, because then the boundary $\partial\mathscr{C}$ is fully characterized by two scalars. But the numerical methods required to solve this problem become much more complicated when $m \geq 2$, because then the boundary consists of infinitely many points. There are some standard approaches that can be considered, such as the discrete time, Markov chain approximation method of Kushner and Dupuis (1991), but their implementation is made extremely difficult by the nonconstant coefficients in the partial differential equation.

This brings us to the purpose of this paper: to develop transformations of the above free boundary problem that lead to new free boundary problems which are easier to solve. In particular, we will focus on the case of $m = 2$ risky securities and show that the original free boundary problem can be transformed to one that is especially simple. In fact, Morton and Pliska (1993) use our transformations and Markov chain approximation methods to numerically solve this problem.

2. MAIN RESULTS

Our objective is to compute a solution of the free boundary problem when there are $m = 2$ risky securities. Equivalently, we want to compute the value function for the problem of optimally stopping the risky fraction process B when the reward-for-stopping function is $h(b)$ and there is a continuation cost-rate equal to $R - r$. It suffices, therefore, to specify three elements: (1) the partial differential equation, (2) the state space of the risky fraction process (a subset of which is the continuation region $\mathscr{C}$ which, in turn, is the domain for the solution of the partial differential equation), and (3) the reward-for-stopping function. As we go through a series of transformations, these three elements will be identified at each stage. Hence each stage of this analysis will be associated with a unique free boundary problem as well as a unique optimal stopping problem.

In particular, our original problem will be described as follows:

(1) *Find a value function* $H: \{b \in \mathbb{R}^2 : b_1 > 0, b_2 > 0, b_1 + b_2 < 1\} \rightarrow \mathbb{R}^2$ *satisfying*

$$\frac{1}{2} \sum_i \sum_j H_{ij} b_i b_j [(e_i' - b')M(e_j - b)] + \sum_i H_i b_i [(e_i' - b')(\mu - r1 - Mb)] = R - r$$

and corresponding to the reward-for-stopping function h.

In all that follows we denote $m_1 = m_{11}$, $m_2 = m_{22}$, and $m_3 = m_{12} = m_{21}$, where (m_{ij}) is the standard notation for the symmetric matrix $M = \Lambda\Lambda'$.

Proposition 1 *The function H satisfies (1) if and only if*

$$H(b_1,b_2) = \Phi(u_1,u_2),$$

where

$$u_i = \frac{b_i}{1 - b_1 - b_2}, \qquad i = 1,2,$$

and where $\Phi:\mathbb{R}_+^2 \to \mathbb{R}^2$ is the value function satisfying

$$\tfrac{1}{2}m_1 u_1^2\Phi_{11} + m_3 u_1 u_2 \Phi_{12} + \tfrac{1}{2}m_2 u_2^2 \Phi_{22} + (\mu_1 - r)u_1\Phi_1 + (\mu_2 - r)u_2\Phi_2 = R - r$$

and corresponding to the reward-for-stopping function

$$\phi(u_1,u_2) \equiv \ln(1 + u_1 + u_2).$$

Proof. A careful proof requires considerable tedious algebra; the presentation here will only sketch the main steps. First you use the chain rule for partial derivatives to derive expressions for the five partial derivatives of H in terms of Φ. For example,

$$H_1 = \Phi_1\frac{\partial u_1}{\partial b_1} + \Phi_2\frac{\partial u_2}{\partial b_1} = \Phi_1\frac{1 - b_2}{(1-b_1-b_2)^2} + \Phi_2\frac{b_2}{(1-b_1-b_2)^2}.$$

Meanwhile, the partial differential equation in (1) can be written out as

$$
\begin{aligned}
(2)\quad & \tfrac{1}{2}[m_1(1 - b_1)^2 - 2m_3(1 - b_1)b_2 + m_2b_2^2]b_1^2H_{11} \\
+ \quad & [-m_1b_1(1-b_1) + m_3(1-b_1)(1-b_2) + m_3b_1b_2 - m_2b_2(1-b_2)]b_1b_2H_{12} \\
+ \quad & \tfrac{1}{2}[m_1b_1^2 - 2m_3b_1(1 - b_2) + m_2(1 - b_2)^2]b_2^2H_{22} \\
+ \quad & [(\mu_1-r)(1-b_1) - m_1b_1(1-b_1) - m_3(1-b_1)b_2]b_1H_1 \\
- \quad & [(\mu_2-r)b_2 - m_3b_1b_2 - m_2b_2^2]b_1H_1 \\
- \quad & [(\mu_1-r)b_1 - m_1b_1^2 - m_3b_1b_2]b_2H_2 \\
+ \quad & [(\mu_2-r)(1-b_2) - m_3b_1(1-b_2) - m_2b_2(1-b_2)]b_2H_2 \quad = \quad R - r.
\end{aligned}
$$

Substituting the expressions for the partial derivatives H_1, H_2, H_{11}, H_{12}, and H_{22} into (2), wading through a lot of tedious algebra, and eventually switching to the new variables u_1 and u_2, one finally obtains the partial differential equation in the hypothesis of this proposition. The new reward-for-stopping function ϕ is immediately obtained with the same change of variables. ■

The first transformation has greatly simplified the coefficients in the partial differential equation, although they are still not constants. The next transformation will lead to a partial differential equation with constant coefficients.

Proposition 2 *The function* Φ *satisfies the problem in Proposition 1 if and only if*

$$\Phi(u_1,u_2) = \Delta(z_1,z_2)$$

where

$$z_i = \frac{\ell n(u_i)}{\sqrt{m_i}}, \qquad i = 1,2,$$

and where $\Delta:\mathbb{R}^2 \to \mathbb{R}^2$ *is the value function satisfying*

$$\tfrac{1}{2}\Delta_{11} + \frac{m_3}{\sqrt{m_1 m_2}}\Delta_{12} + \tfrac{1}{2}\Delta_{22} + \frac{1}{\sqrt{m_1}}(\mu_1 - r - \tfrac{1}{2}m_1)\Delta_1 + \frac{1}{\sqrt{m_2}}(\mu_2 - r - \tfrac{1}{2}m_2)\Delta_2 = R - r$$

and corresponding to the reward-for-stopping function

$$\delta(z_1,z_2) \equiv \ell n(1 + \exp\{\sqrt{m_1}\, z_1\} + \exp\{\sqrt{m_2}\, z_2\}).$$

The proof will be omitted, since it can be carried out in the same manner as with Proposition 1. Looking at the partial differential equation in Proposition 2, one sees that it corresponds to a two-dimensional Brownian motion with state space all of $\mathbb{R}^2$, with drift, and with correlated components. The next transformation will eliminate the cross-partial term.

Proposition 3 *The function* Δ *satisfies the problem in Proposition 2 if and only if*

$$\Delta(z_1,z_2) = \Theta(x_1,x_2)$$

where

$$x_1 = cz_1 + dz_2, \qquad x_2 = dz_1 + cz_2, \qquad \nu \equiv \frac{m_3}{\sqrt{m_1 m_2}},$$

$$c \equiv \frac{-\nu}{\sqrt{2(1-\nu^2)(1+\sqrt{1-\nu^2})}} \quad \text{and} \quad d \equiv \sqrt{\frac{1+\sqrt{1-\nu^2}}{2(1-\nu^2)}},$$

and where $\Theta:\mathbb{R}^2 \to \mathbb{R}^2$ *is the value function satisfying*

$$\tfrac{1}{2}\Theta_{11} + \tfrac{1}{2}\Theta_{22} + a\Theta_1 + b\Theta_2 = R - r$$

and corresponding to the reward-for-stopping function

$$\theta(x_1,x_2) \equiv \ln[1 + \exp\{\sqrt{m_1}(px_2-qx_1)\} + \exp\{\sqrt{m_2}(px_1-qx_2)\}],$$

where

$$p \equiv \frac{d}{d^2 - c^2}, \qquad q \equiv \frac{c}{d^2 - c^2},$$

$$a \equiv \frac{c}{\sqrt{m_1}}(\mu_1 - r - \tfrac{1}{2}m_1) + \frac{d}{\sqrt{m_2}}(\mu_2 - r - \tfrac{1}{2}m_2),$$

and

$$b \equiv \frac{d}{\sqrt{m_1}}(\mu_1 - r - \tfrac{1}{2}m_1) + \frac{c}{\sqrt{m_2}}(\mu_2 - r - \tfrac{1}{2}m_2).$$

Note that M is positive definite, so $\nu^2 < 1$; it follows that c and d are real with $d^2 > c^2$. Hence all the quantities in

Proposition 3 are well defined.

Proof. Substituting expressions for the partial derivatives as in the proof of Proposition 1, one eventually obtains

$$\tfrac{1}{2}(c^2 + 2\nu cd + d^2)\Theta_{11} + (\nu c^2 + 2cd + \nu d^2)\Theta_{12} + \tfrac{1}{2}(c^2 + 2\nu cd + d^2)\Theta_{22} + a\Theta_1 + b\Theta_2 = R - r.$$

A little more algebra reveals that $c^2 + 2\nu cd + d^2 = 1$ and that $\nu c^2 + 2cd + \nu d^2 = 0$, so we actually have the partial differential equation as stated in the hypothesis of this proposition.

With regard to the reward-for-stopping function, notice that $d^2 > c^2$ implies the change of variables is nonsingular. It is easily shown, therefore, that

$$z_1 = px_2 - qx_1 \qquad \text{and} \qquad z_2 = px_1 - qx_2.$$

Hence the reward-for-stopping function θ is as indicated. ∎

We now have the problem of stopping a two-dimensional Brownian motion on $\mathbb{R}^2$, where the reward-for-stopping function is θ and the continuation rate is $R - r$. The Brownian motion components are independent of each other, but they each have a nonzero drift. This is the problem that Morton and Pliska (1993) solved by using a Markov chain approximation. By taking the transformations in the reverse direction, they were then able to estimate the original value function H. For example, having estimated the value of Θ at the point (x_1,x_2), they knew that this equaled the value of Δ at the point $z_1 = px_2 - qx_1$, $z_2 = px_1 - qx_2$, the value of Φ at the point where $u_i = \exp\{\sqrt{m_i}\, z_i\}$, i = 1,2, and finally the value of H at the point where $b_i = u_i/(1+u_1+u_2)$, i = 1,2.

3. SUMMARY AND CONCLUSIONS

Non-constant coefficient, multi-dimensional, second-order partial differential equations arise naturally in finance. The degree of the partial differential equation reflects the modeling of uncertainty. Cross-product terms arise because of correlations between asset prices and/or state variables. The simplest model that we use for describing an asset price is geometric Brownian motion. This leads to equations with unbounded coefficients and the vanishing of leading terms (e.g., see Gleit (1978)). Therefore, many of the standard approaches for analysis cannot be used without considerable care. Further, many standard numerical analysis schemes are not robust with respect to the partial differential equations arising in finance (see Clewlow (1990)).

There is an increasing use of two and three dimensional equations, particularly in the fields of options on many assets and stochastic volatility. Non-linear equations are also arising out of foundational issues (e.g., Hodges and Carverhill (1993)).

In contrast, in this paper we focus on a non-constant coefficients, 2-dimensional partial differential equation which arises in the optimal management of a portfolio when transaction costs are fixed.

A major feature of these types of problems is that a direct attack may lead to extreme difficulties. However, we believe that whenever possible suitable transformations of the equations should be undertaken to make them more analytically and computationally tractable. The determination of the transformations and their implementation are of fundamental importance. Although we have not used Lie groups directly in this paper, we believe that Lie groups and normal form theory may be exactly that part of mathematics which will make a crucial contribution to the continuation of the revolution instigated by the introduction of martingales and stochastic integrals in the theory of continuous trading.

In this paper we have not derived the normal form for the original partial differential equation. The normal form reduces to a Schrödinger equation. We have not worked with the Schrödinger equation form because we are able, in this case, to maintain our insight in regard to the Brownian motions and the use of the Markov chain approximation. However, we believe that it is quite likely that if we had, say, worked with asset prices where the instantaneous means and standard deviation of returns were not constant, then a final reduction to the normal form would have been necessary to give meaningful insights and to provide computational advantages. In particular, we may have been led to computational advantages based on perturbation or asymptotic analysis (provided they could have been validly carried out). Clearly we are suggesting this as a further area for research. Indeed, Atkinson and Wilmott (1993) have already commenced work in this field by explicitly attacking the original Morton and Pliska (1993) model with an asymptotic analysis.

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