A Theory of the Term Structure with an Official Short Rate

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Abstract

Many of the larger movements in market interest rates are associated with changes in official interest rate policy. This applies especially in the UK, where the Bank of England effectively controls very short term rates, and moves its key dealing rate in multiples of a discrete unit. We present a theory of the term structure, with the short rate following a pure jump process with stochastic intensities depending on the short rate and a diffusion state variable. An illustrative model mimics empirical regularities of UK rates. We suggest how to extend our approach to other countries, most readily the US.

1. INTRODUCTION

The behaviour of monetary authorities, and the influence of their interest rate policy upon the term structure of interest rates, have long been of interest to economists (see eg Dale [1993] and references therein; and a paper by Balduzzi, Bertola and Foresi [1993] which we discuss in Section 2). Nevertheless, as far as we are aware, this paper represents the first work in the modern finance literature on the term structure, in which the rate-setting role of the monetary authorities plays any explicit part.

The need for such work is suggested by, *inter alia*, the attention which markets devote to official interest rate policy. Moreover, many of the larger movements in market interest rates are associated with changes in official interest rate policy.

The task of providing any kind of plausible model to describe the actions of monetary authorities is a challenging one, both theoretically and empirically. Moreover, differences in institutional arrangements for the implementation of monetary policy make the precise nature of that task vary markedly between countries. For a description of various countries' arrangements, see Schnadt [1994].

This paper starts to tackle the above task as it presents itself in the UK. We discuss in Section 9 how our approach might be extended to other countries, most readily the US. In part, the choice of the UK reflects a bias natural to British authors. However, there are also objective reasons for commencing the modelling of official interest rate-setting behaviour with the UK case.

The Bank of England seeks to exert control over very short term sterling interest rates, primarily through outright and repo transactions in eligible bills[1]. The Bank regards this control as effective - see Bank of England [1988]. Its key interest rate is its "stop rate for Band 1 bills". While "Band 1" covers maturities of up to 14 days, a sizeable proportion of the dealings in which the rate is used are shorter term repo transactions. This, and the general conduct of the Bank's operations, lead us to believe that a plausible idealisation of this behaviour, for term structure modelling purposes[2], is to identify the Band 1 rate with the instantaneous spot interest rate. For a number of years, the Band 1 rate has followed a pure jump process. From the mid-1980s to early 1994, all the observed jump sizes were integer multiples of half a point. In February 1994, however, and perhaps as a reflection of the historically low level of rates now prevailing, the rate was changed by 0.25%. We may therefore view

[1] That is to say, in corporate debt with under 14 days to maturity, which has been guaranteed by a bank of good standing. For more details, consult Bank of England [1988].

[2] In principle, it would be desirable to recognise that the application of the Band 1 dealing rate across a spectrum of short maturities affords interesting trading opportunities, constrained by the availability of suitable eligible bills and by the volumes the Bank of England is prepared to trade (see Schnadt and Whittaker [1993]). Unfortunately, we cannot see how such effects could be integrated with the powerful frictionless competitive markets paradigm of finance, that we wish to use for term structure modelling. Our belief is that these effects have little persistent impact on the term structure, affording justification for our idealisation. We discuss in Section 9 ways of relaxing our idealisation, compatible with the perfect frictionless markets paradigm.

changes in the official rate as occurring in multiples of a quarter of a percentage point, with odd multiples having a positive probability only at low levels of interest rates. It can, therefore, be argued that, in the UK case, the nature of the official rate-setting behaviour is reasonably clear-cut, simplifying the modelling problem. This contrasts with the situation in, say, the US, where the Fed exerts only indirect influence on the key Fed funds rate (see Schnadt [1994] pp111-6); or Germany, where the implementation of monetary policy involves significant roles for "discount" and "Lombard" rates, and also for fixed and variable rate repos in which interest rate changes are of no particular size.

An additional reason for commencing with the UK case is that, if the sterling short rate is indeed appropriately modelled by a pure jump process, then the popular diffusion term structure models in finance are singularly inapt to sterling interest rates.[3] Our framework may therefore meet an especial need.

At least from outside, as it were, of the monetary authorities, neither the size nor the timing of changes in the official short term interest rate can be predicted with any certainty.[4] Therefore, in modelling the determination of the term structure by the financial markets, it is natural to treat the short rate jump process as having stochastic intensities, ie to regard the occurrence of jumps of each of the permitted sizes as Poisson events with an arrival rate which depends upon the state of the economy. To provide a parsimonious form for this dependence, we model all the jump intensities as functions of the officially set short rate, r, and of a single additional state variable, x, (and perhaps of time also).

Our general theoretical framework (Sections 2-6) assumes that x follows a diffusion, but does not impose any interpretation upon x. Empirical work, however, will tend to require some kind of pre-specification of the functional form of the drift and diffusion coefficients of the dynamics of x. We would therefore suggest that x be made to represent, at least in some loose sense, the financial markets' perception of an "prospective" or "shadow"[5] level of interest rates, given the condition of the real and financial economy and the policy stance of the authorities. The dynamics of x can then plausibly be supposed to take the kind of form accorded to the short rate itself in the existing diffusion-based literature, perhaps modified to give weight to the authorities' policies as reflected in the current level of r.

In assessing our approach, it is important to bear in mind that we are modelling the arrival of short rate changes from the viewpoint of the financial markets. The information available to the monetary authorities is greater than that possessed by the markets, and their probability assessments will consequently be different. By confining

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[3] This would be most acutely the case, of course, at modest maturities, where the influence of the precise behaviour of the instantaneous spot rate is most keenly felt.

[4] For example, there is no obvious tendency for the UK authorities to change rates on particular days of the week.

[5] The epithet "shadow rate" was used by Steele and Webber [1993] to describe such a variable.

its assumptions to the markets' information and beliefs, therefore, our approach is consistent with all manner of decision-making processes within the monetary authorities.[6] We believe, however, that our approach provides a flexible yet tractable means of describing the authorities' behaviour as it may appear to those not privy to the authorities' inner decision-making processes - ie as it may appear to investors in financial markets - and of analysing the implications for the term structure.

The core of the present paper is theoretical. Motivated by the above discussion, we develop a model of the term structure in which the short rate (ie the instantaneous spot interest rate) is official, in the sense of being effectively controlled by the monetary authorities, and in which it follows a pure jump process with a range of fixed jump sizes. The jump intensities of this process vary stochastically, responding to a diffusion state variable representing the condition of the real and financial economy. This development is inevitably technical and somewhat abstract. Nevertheless, it provides the necessary groundwork for an extensive programme of research embracing empirical estimation, and the pricing of interest rate contingent claims, and touching upon the possible policy implications of how changes in official rate-setting policy might affect the variability of interest rates. We intend to report on this programme elsewhere.

In the latter parts of the present paper, however, we seek both to illustrate our theoretical framework, and to indicate the extent to which it may afford a basis for better understanding the term structure of sterling interest rates. We attempt this by rehearsing some empirical regularities of sterling interest rates, and comparing them with those arising from an illustrative model constructed under our framework.

The rest of this paper is organised as follows. Section 2 discusses the theoretical approach we will adopt, and differentiates it from the existing term structure literature in finance, with especial attention to jump-diffusion models. Sections 3-5 develop our general framework, covering respectively the formal statement of our assumptions, the demonstration that these assumptions combine to define models consistent with equilibrium, and the pricing - at an abstract level - of interest rate contingent claims. Section 6 derives from the abstract pricing formula a partial differential difference equation (PDDE) satisfied by the prices of pure discount bonds and, indeed, all other claims except continuously resettled claims such as futures and other margined contracts. We illustrate our model in Sections 7-8. The former of those sections rehearses various empirical regularities in sterling interest rates; the latter section compares these with the properties of an illustrative model, obtained by numerical solution of the PDDE coupled with Monte Carlo simulation of the underlying state

[6] Thus, for example, we do not preclude the possibility that, at each time, the authorities have a "desired" interest rate, arrived at by a combination of economic and political considerations. Neither do we preclude the idea that financial or reputational costs are involved in bringing forward or resisting interest rate changes that seem warranted on the basis of information available to the markets; in particular, rapid reversals of interest rate changes may be politically damaging.

variables. Section 9 discusses extensions to our approach, with application to other countries, most readily the US. Section 10 offers a summary and concluding remarks. Proofs are relegated to the Appendix.

2. THEORETICAL APPROACH

We focus on two state variables: the officially set short rate, r, which follows a pure jump process, and a diffusion process, x. The latter process represents some indicator of the state of the economy, which, together with the current level of r, governs the financial markets' assessment of the likelihood of the authorities changing the level of r.

As noted in the introduction, our treatment of the authorities' rate setting behaviour confines itself to a model of the financial markets' probability assessments concerning that behaviour. We do this by specifying the jump intensities of r as functions of (at most) the concurrent values of r and x and of time. That is to say that, at each instant, the probability of an immediate change[7] in the short rate, r, depends on the current short rate, the prevailing state of the economy, and, perhaps, time.

While r and x are the only two state variables which we model explicitly, we do not restrict the nature of the remainder of the economy. Therefore, the overall information flows in the economy can be arbitrarily large. Moreover, since neither of our state variables is the price of a security, the pricing operator in the economy will depend on the market prices of risks attaching to the uncertain evolution of our state variables, and upon those attaching to other sources of uncertainty in the economy that we have not modelled explicitly.

Cox, Ingersoll and Ross [1985b] point out that market prices of risk cannot be specified arbitrarily, without introducing further arbitrage possibilities. Babbs and Selby [1993] (BS) have recently introduced a "risk pricing measure" (RPM) concept, by use of which one can examine whether particular specifications of the market price(s) of risk(s) are viable, ie can be supported in general equilibrium. We will use this approach to show that, under mild regularity conditions, our model is viable. Moreover, we exploit further results in BS to show that the term structure, and the prices of term structure derivatives, are independent of the market prices of risks not explicitly modelled.

Our model can be sharply differentiated from any other in the finance literature on the term structure. The vast majority of existing modelling of the term structure has employed only diffusion processes, which exclude any jump component. The exceptions are Ahn and Thompson [1988] and Shirakawa [1991].

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[7] ie a jump of one of the fixed allowable jump sizes

Ahn and Thompson show that the diffusion-based general equilibrium asset pricing model of Cox, Ingersoll and Ross [1985a] can be generalised to allow the underlying state variables of the economy to have jump components, possibly with stochastic intensities. In principle, this approach could yield a rich variety of term structure models. Ahn and Thompson exhibit one example, which adds a jump component to the "square root model" of Cox, Ingersoll and Ross [1985b] (CIR).

The spirit of Ahn and Thompson's model, inherited from the work of Cox, Ingersoll and Ross, is of seeking to derive interest rate processes that are freely determined by the potentialities of the economy and the objectives of agents. In our framework, the short rate of interest is set by the *fiat* of the authorities, and the other state variable, x, follows a process which, like that of the state variables of an explicit general equilibrium approach, is exogenously given. Under these latter conditions, the scope of an explicit equilibrium analysis would be limited to determining the market prices of risk corresponding to alternative specifications of agents' preferences. This would not be without interest. Against that, however, must be set the fact that the concrete results in CIR and in Ahn and Thompson, rest on assuming that the state variable(s) driving the term structure are the sole variables driving the entire economy, and upon extremely restricted preferences. By contrast, our approach, inherited from Babbs and Selby, enables us to confine our explicit modelling to r and x and the associated market prices of risk, leaving the remainder of the economy unrestricted.

Shirakawa [1991] extends the arbitrage-based analysis of Heath, Jarrow and Morton [1992] to include a jump component with constant jump intensity - ie a pure Poisson process. The nature of the implied process for the short rate, r, is a diffusion, punctuated by jumps whose arrival is independent of the state of the economy. By contrast, our model is founded on a pure jump process for r, whose intensities depend - potentially quite intricately - upon the state of the economy.

Finally, we must discuss recent work Balduzzi, Bertola and Foresi [1993] (BBF) which, like the present paper, models the term structure largely in terms of an official interest rate. BBF construct a discrete-time model for the overnight US Fed funds rate as mean-reverting to an officially-set target rate. The target rate follows a pure jump process with jump sizes almost invariably multiples of a quarter of a percentage point. BBF seek to infer from data on term interest rates, the markets' expectations of forthcoming jump sizes, under the joint hypotheses that the sum of the jump intensities is a constant, and that investors are risk-neutral. By contrast, we model the markets' perceptions of prospective jumps by use of an additional diffusion state variable representing, in some way, the state of the economy, and derive the term structure under general market prices of risk.

3. ASSUMPTIONS OF MODEL

As in Harrison and Kreps [1979] (HK), and Babbs and Selby [1993], we formulate our model as a pure exchange economy over a fixed finite interval $[0,T]$, with certain consumption at time *0*, and square integrable state dependent consumption at

T. We presuppose a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, P\}$ satisfying the "usual conditions"[8], with $\mathcal{F}_T = \mathcal{F}$. In this formalism $\mathcal{F}_t$ represents the information available to financial markets[9] at time t; available information grows over time, as uncertainty is progressively resolved.

We shall seek to price various contingent claims - pure discount bonds and other term structure claims - by arbitrage, supplemented by specifying the market prices of the risks that we model explicitly. Babbs and Selby [1993] have argued that such results should be substantiated in a setting which allows for arbitrarily larger information and uncertainty than that modelled explicitly through the various stochastic processes taken as primitive. Following their lead, we allow the filtration (briefly $\{\mathcal{F}_t\}$) to be arbitrarily large.

Our first specific assumption equips our economy with a security we shall use as numeraire. While choice of numeraire should be irrelevant for relative prices, it can affect the informational content of normalized prices (ie prices expressed in units of the numeraire, rather than in units of account).[10] Babbs and Selby [1993] (their Remark 3.1) therefore place significance on using a numeraire whose price depends solely on the primitive processes of the model.

Assumption (A) *A continuously compounded money market asset is available, ie an asset whose price process is:*

$$S_0 \equiv \exp\left\{\int_0^{\cdot} r(u)du\right\} \tag{1}$$

where r *is the instantaneous spot interest rate.*

The primary characteristic of this paper is, of course, that the instantaneous spot interest rate, r, is modelled as a pure jump process. We allow for a number of different fixed sizes of jump. At each point in time, t, the probability of a jump of size c_j occurring in the next δt is given (following the well-known heuristic) by a state-dependent Poisson law with intensity λ_j as $\lambda_j \, \delta t + o(\delta t)$.

Formally, we make:

Assumption (B) *The instantaneous rate,* r, *is assumed to follow a pure jump process, with a finite number of fixed jump sizes, ie we can write:*

$$r(t) = r(0) + \sum_{j=1}^{J} c_j N_j(t) \tag{2}$$

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[8] see eg Bremaud [1981] p75

[9] As emphasised in the introduction, we are modelling the information set and probability assessments of the financial markets; the monetary authorities may well possess greater information.

[10] see Duffie and Huang [1986] p301

where: $c_1, \ldots, c_J$ *are distinct non-zero constants representing jump sizes; and each* N_j *is a point process counting the number of jumps of size* c_j *since time zero, and admits a predictable intensity*[11] λ_j . *For* $j \neq k$, N_j *and* N_k *do not charge a common jump time*[12].

By making the intensity of jumps vanish suitably, we make r bounded below, by zero if so desired, and above. This boundedness provides various technical simplifications, and could be relaxed if so desired. Thus we make:

<u>**Assumption (C)**</u> *There exists a lower bound* $r_{\min}$ *and an upper bound* $r_{\max}$ *for r, in that each* λ_j *vanishes when* $r + c_j \notin [r_{\min}, r_{\max}]$.

To simplify subsequent technical manipulations, we introduce a global bound on the jump intensities. In principle, this is a substantive restriction, since it rules out, in all circumstances, a jump in interest rates being certain to occur immediately. However, since we can set the global bound arbitrarily high, we see the restriction as being of no practical significance.

<u>**Assumption (D)**</u> *The jump intensities* $\lambda_1, \ldots, \lambda_J$ *are essentially bounded*[13].

We suppose that the probability of official changes in interest rates is governed by some indicator x of the state of the economy. We assume that this indicator follows a diffusion, influenced in part by official interest rates. The future joint evolution of r and x is Markovian (ie we allow it to depend on their current levels, rather than on their histories to date, and perhaps on time). More precisely, we have:

<u>**Assumption (E)**</u> *There exists a diffusion state variable, x, such that the joint process of* r *and* x *is (possibly time-inhomogeneous) Markov; ie we can write the dynamics of* x *as:*

$$dx(t) \quad = \quad a(r(t-), x(t), t)\, dt \quad + \quad b(r(t-), x(t), t)\, dZ \tag{3}$$

where Z *is a standard Brownian motion,*[14] *and each of the jump intensities,* λ_j , *can be written as:*

$$\lambda_j(t) \quad = \quad \lambda_j(r(t-), x(t), t)$$

We shall use the money market asset, established in Assumption (A), as numeraire. None of the other stochastic processes, set out in Assumptions (B)-(E), represents a security price. Therefore, as discussed in Babbs and Selby [1992], the pricing of contingent claims will require the specification of market prices of risks.

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[11] see Bremaud [1981] Definition D7 p27

[12] see eg Elliott [1982] Definition 6.45 p60

[13] ie there exists a constant K such that:

$$P\left\{ \sup_{0 \le t \le T} \max_{j=1,\ldots,J} \lambda_j(t) \;\le\; K \right\} \;=\; 1$$

[14] By stating that x follows a diffusion, we implicitly require also that a and b have sufficient regularity for (3) to be well-defined.

Cox, Ingersoll and Ross [1985b] (p398) present a cautionary tale in which inappropriate choices of the market price of risk guarantee, rather than exclude, arbitrage opportunities. Clearly, therefore, models depending upon specifying market price(s) of risk(s) must be carefully constructed if viability, ie consistency with general equilibrium, is to be ensured. As Babbs and Selby [1993] (BS) point out, the seminal analysis of viability in Harrison and Kreps [1979] (HK) is ill-suited to models such as ours, since HK presupposed that all the primitive processes in a model represented security prices. BS extend the analysis of viability to models in which the processes taken as primitive may consist in part, or even wholly, of processes other than security prices. They introduce a "risk pricing measure" (RPM) concept, which generalizes the "equivalent martingale measure" concept of HK. Using this concept, BS characterize viable models incorporating assumed market price(s) of risk(s) in terms of the existence of a well-behaved RPM. Expressed in normalized terms (ie in units of the numeraire security), possible equilibrium pricing operators correspond to taking expected payoffs under the well-behaved RPMs.

BS also show that, subject to a mild regularity condition, each RPM is characterized by the market price of risk processes pertaining to each of the orthogonal sources of risk that underly innovations in the primitive processes, and by a local martingale, N say, orthogonal to those risks. They observe that N can be thought of as specifying, in aggregate, the prices of risk attaching to all the sources of risk not being modelled explicitly.

In our model, the primitive processes are the diffusion x and the jump processes $N_1, \ldots, N_J$. The underlying sources of risk, as understood by BS, provide an orthogonal 2-generator - essentially a basis - of local martingales for the innovations in these primitive processes. The set $\{ M_0 \equiv Z, M_1, \ldots, M_J \}$ is readily shown[15] to provide such a basis.

Specifically, Babbs and Selby show (their Theorems 7.8 and 7.12) that, to be consistent with general equilibrium, the pricing operator in the economy must take the form $\psi : L^2(\mathcal{F}, P) \rightarrow \Re$, given by:

$$\psi(payoff) = S_0(0)\, \mathrm{E}^* \left[\frac{payoff}{S_0(T)} \right] \tag{4}$$

[15] since Z is continuous, while the local martingales $M_1, \ldots, M_J$ are locally bounded, and do not charge common jump times.

where E^* is the expectation operator for a probability measure P^*, equivalent to P, and defined by:

$$\frac{dP^*}{dP} = \eta(T) \tag{5}$$

satisfying

$$\frac{1}{S_0(T)} \frac{dP^*}{dP} \in L^2(P) \tag{6}$$

where, under P, η is a martingale and locally square integrable, and is given by:

$$\eta = \mathcal{E}\left\{ N - \sum_{j=0}^{J} \int_0^{\cdot} \theta_j \, dM_j \right\} \tag{7}$$

where: $\mathcal{E}$ denotes the exponential semimartingale; each θ_j is predictable, and N is locally a square integrable martingale, starting at zero, and orthogonal to each M_j.

Remark 3.1 Given (6), the expectation on the RHS of (4) exists and is finite for all $payoff \in L^2(P)$.

In applications work, we shall be prepared to make strong assumptions about $\theta_0, \ldots, \theta_J$, which can be interpreted as the market prices of the explicitly modelled sources of risk. N can be thought of as pricing, in aggregate, the other sources of risk in the economy. Clearly, it would be undesirable to have to make strong assumptions about the market prices of risks that we are not modelling explicitly. We hope, therefore, that we can leave N essentially arbitrary without affecting the values of contingent claims depending on our primitive processes. BS' Theorems 7.11 and 7.12 provide precisely the kind of result we would like to exploit. They establish that the choice of N is irrelevant to the pricing of claims whose normalized payoffs depend solely on the history of the primitive processes, if and only if N is also irrelevant to the risk-adjusted probability laws of the primitive processes.

At this juncture, and essentially only for technical convenience, we select as candidate market prices $\theta_0, \ldots, \theta_J$ of the explicitly modelled sources of risk (ie $\{M_0, \ldots, M_J\}$) predictable processes that are globally bounded. More substantively, we restrict $\theta_0, \ldots, \theta_J$ to depend only on the concurrent values of the counting processes $N_1, \ldots, N_J$ and the Brownian motion, Z. This will ensure that sources of risk that have not been explicitly modelled do not enter the term structure via the prices of risk, and ensure at certain key points in the mathematics of the model. We leave N essentially arbitrary, except that we do require either that no source of risk has a positive probability of a jump at the same time as a jump in interest rates, or that such a source is unpriced. To proceed formally, we commence with:

Definition 3.2 *Let* $\{ \mathcal{G}_t : t \in [0,T] \}$ *(briefly* $\{ \mathcal{G}_t \}$ *) denote the completion of the filtration generated by* $N_1, \ldots, N_J$ *and* Z*. Define* $\mathcal{G} \equiv \mathcal{G}_T$.

We can now state:

Assumption (F) *Let* $\theta_0, \ldots, \theta_J$ *be essentially bounded, predictable and* $\{\mathcal{G}_t\}$ *-adapted processes. For* $j > 0$ *, each* θ_j *is strictly less than unity.*

Under P*, let* N *be locally a square integrable martingale, orthogonal to* $M_j : j = 0, \ldots, J$ *and which does not charge a common jump time with any of them.*

Remark 3.3 In a model based exclusively on continuous sources of risk, the requirement that any of the prices of risk be bounded above by +1 would be a substantive restriction. In a model like ours, however, involving counting processes, the requirement is a technicality required as a matter of logic. The reason is that, under the risk-adjusted probabilities, the jump intensities λ_j are transformed to $(1-\theta_j)\,\lambda_j$. Thus values of θ_j in excess of +1 would give nonsensical negative jump probabilities, while values equal to +1 would (in the situations concerned) eliminate the possibility of a jump, violating the well-known principle that the actual and risk-adjusted probabilities must be "equivalent", ie admit the same possible eventualities.

Finally, we seek to close our model by assuming that the pricing operator in our economy is given by (4), when $\theta_0, \ldots, \theta_J$ and N processes satisfying Assumption (F) are used in (7). We say "seek to close" advisedly, since, of course, we will need to establish the existence of non-trivial well-defined probability measure(s) satisfying our assumption, and that such measures constitute viable RPMs. By "non-trivial" we mean that we would like to show that our assumptions are not collectively so restrictive as to limit unduly the supposedly fairly arbitrary price of risk processes, $\theta_0, \ldots, \theta_J$ and N.

Assumption (G) *The pricing operator for contingent claims in our economy is given by (4)-(7).*

4. VIABILITY OF ASSUMPTIONS

As indicated prior to the statement of Assumption (G), our first tasks are to establish the existence of non-trivial probability measures, P^* , with the properties listed in Assumption (G), and that these measures constitute viable RPMs for our model. We accomplish these tasks, respectively, by a sequence of lemmas leading to Proposition 4.7, and by Theorem 4.8.

Remark 4.1 The hardest part of this programme lies in establishing that the exponential semimartingale η , as defined by (7), is a P-martingale. Various well-known and fairly straightforward sufficient conditions exist to ensure that applying the exponential semimartingale operator to a continuous local martingale yields a martingale. In our case, however, the argument of the operator involves the discontinuous processes N_j and the essentially arbitrary process N. To handle this situation, we first (Lemma 4.2) decompose η into two orthogonal factors, one depending solely on the explicitly modelled sources of risk, the other involving N. In Lemma 4.3 and Lemma 4.4, we show, respectively, that the first factor is a strictly positive martingale under P, and has finite moments of all orders. In Lemma 4.6, we impose a modest regularity condition on the second factor, subject to which the product η has the desired properties.

As just adumbrated, we first show that η can be decomposed into two factors, which are orthogonal in a local martingale sense:

Lemma 4.2 *Suppose that Assumptions (B)-(F) hold. Then*

$$\eta \quad = \quad \eta_0 \quad \mathcal{E}\{N\} \tag{8}$$

where

$$\eta_0 \quad \equiv \quad \mathcal{E}\left\{ -\sum_{j=0}^{J} \int_0^{\cdot} \theta_j \, dM_j \right\} \tag{9}$$

Moreover,[16] *the quadratic covariation process,*

$$[\eta_0, \quad \mathcal{E}\{N\}] \quad = \quad 0 \tag{10}$$

Our next two lemmas are devoted to establishing that, under P, η_0 is a strictly positive martingale with finite moments of all orders.

Lemma 4.3 *Suppose that Assumptions (B)-(F) hold. Then,* η_0 *, defined by (9), is a strictly positive martingale under* P.

Lemma 4.4 *Suppose that Assumptions (B)-(F) hold. Then:*

$$\eta_0(t) \quad \in \quad L^q(P), \qquad \forall t \in [0,T], q \in \Re$$

If we now assume that the second factor on the RHS of (8) satisfies (significantly) weaker conditions, than the above lemmas have shown to apply to the first factor, then their product, η , has the properties we seek:

Lemma 4.5 *Suppose that Assumptions (B)-(F) hold. Suppose, in addition, that* N *satisfies the following technical regularity conditions:*

(a) *any jumps in* N *are strictly bounded below by -1;*
(b) $\mathcal{E}\{N\}$ *is a* *P-martingale;*
(c) $\exists \delta > 0$ *such that* $\mathcal{E}\{ \ N \ \}(T) \ \in \ L^{2+\delta}(P)$

Then η *, defined by (7), is a strictly positive and square integrable martingale under* *P.*

Remark 4.6 Condition (a) of Lemma 4.5 is necessitated by the same consideration as that which impelled us to require that the price of jump risk processes $\theta_1, \ldots, \theta_J$

[16] We show (8) and (10) as distinct results, as we shall use them both in the sequel. They are, of course, two sides of the same coin, in that the exponential semimartingale of the sum of two local martingales is the product of the individual exponential semimartingales if and only if those local martingales are orthogonal (see eg Elliott [1982] Corollary 13.8 p160).

be strictly bounded above by +1, namely the need to ensure that η, purporting to be the conditional expectations of the Radon-Nikodym derivative of an equivalent measure, remains strictly positive.

From Lemma 4.5 we can deduce that Assumption (G) is mathematically well-formulated, in the sense that (7) does indeed yield, via (5), a probability measure, P^*, with the same null sets as *P*, and such that the square-integrability condition, (6), holds:

Proposition 4.7 *Suppose that Assumptions (A)-(F), and the regularity conditions on N, used in Lemma 4.5, all hold. Then, η, defined by (7), is a square integrable martingale under P; moreover, (5) defines a probability measure, P^*, equivalent to P, and satisfying (6).*

Proposition 4.7 confirms that our Assumptions are mathematically well-formulated, and suggests, moreover, that it should be possible to construct non-trivial instances of our model.[17] It remains to prove that Assumption (G) is economically well-formulated in the sense of giving a viable model. Fortunately, a straightforward appeal to the results of Babbs and Selby [1993] now establishes that our Assumptions do indeed yield a viable model, in which contingent claims are priced by (4):

Theorem 4.8 *Assumptions (A)-(G), and the technical conditions on N introduced in Lemma 4.5, define a viable model.*

Throughout the sequel, we assume that Assumptions (A)-(G), and the technical conditions on *N* introduced in Lemma 4.5, hold.

5. UNIQUE PRICING OF TERM STRUCTURE DERIVATIVES

While our model is viable, the pricing operator, ψ, given by (4), depends partly on the essentially arbitrary term, *N*, corresponding to the market prices of risks we have not explicitly modelled. Fortunately, as hinted in the previous Section, we shall be able to prove that the choice of *N* is irrelevant to the determination of the term structure, and indeed to the pricing of any other contingent claims whose normalized payoffs are $\mathcal{G}$-measurable, ie which depend solely on the history of our explicitly modelled sources of risk, $Z, N_1, \ldots, N_J$. The key to this result is Theorem 7.11 of Babbs and Selby [1993], which focuses our attention on the probability law of our primitive processes under the risk-adjusted probabilities corresponding to different choices of *N*. This motivates us to obtain the following preliminary result:

Proposition 5.1 *Under the risk-adjusted probabilities, P^*, corresponding to any choice of N, we can write:*

$$Z = Z^* - \int_0^{\cdot} \theta_0 \, du \qquad (11a)$$

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[17] For example, *N* will satisfy the conditions in Lemma 4.5 if it can be expressed as a sum of integrals of essentially bounded processes with respect to standard Brownian or Poisson processes.

where Z^* *is a* P^* *-standard Brownian motion; moreover each* N_j *admits the predictable* P^* *-intensity*

$$\lambda_j^* = (1-\theta_j)\ \lambda_j \tag{11b}$$

By Assumptions (E) and (F), both the drift term in (11a) and the intensities in (11b) are $\{\mathcal{G}_t\}$ -adapted, ie their values at each date are known from the history of $N_1,\ldots,N_J$ and Z, without reference to that of N. Hence, one has grounds to believe that the probability law of $N_1,\ldots,N_J$ and Z under P^* is independent of the corresponding choice of N. This is indeed the case:

Proposition 5.2 *The risk-adjusted probability law of* $N_1,\ldots,N_J$ *and* Z *is independent of the choice of* N.

As well flagged already, Proposition 5.2 enables to make use of Theorem 7.11 in Babbs and Selby [1993] to price uniquely pure discount bonds and other contingent claims depending on the history of the sources of risk we have modelled explicitly. Put another way, our next result shows that we can price the claims we are interested in by specifying: the coefficients of the diffusion x; the interest rate jump sizes and intensities; and the market prices of diffusion and interest rate jump risks; without needing to specify the nature or market pricing of other risks in our economy.

Theorem 5.3 *The restriction of the pricing operator* ψ *to payoffs whose normalized amounts are* $\mathcal{G}$ *-measurable, is independent of the choice of* N.

6. PARTIAL DIFFERENTIAL DIFFERENCE EQUATION

One possible method of valuing contingent claims in our model would be to apply Monte Carlo techniques to (4). In many cases, however, it will be more helpful to obtain a partial differential difference equation (PDDE) that the values satisfy. Such an approach represents, of course, the natural analogue in a jump-diffusion model, of the application of partial differential equation (PDE) techniques in diffusion models.

We present our derivation of the PDDE in terms of pure discount bond prices. The equation applies, however, with different boundary conditions, to all contingent claims other than resettled contracts such as futures.

Definition 6.1 $B(M,t \mid r^*,x^*)$ *denotes the price at time* t *of unit face value of a pure discount bond maturing at a subsequent time* M, *if* $r(t)=r^*$ *and* $x(t)=x^*$.

For brevity's sake, we use $B(M,t)$ *to denote the corresponding price along the path of* r *and* x *actually realised, ie* $B(M,t)=B(M,t \mid r(t),x(t))$.

Analysis of the dynamics of the normalized bond price

$$B^*(M,t) \equiv \frac{B(M,t)}{S_0(t)} \tag{12}$$

under any risk-adjusted probability measure P^* , we are able to derive:[18]

Theorem 6.2 *Pure discount bond prices satisfy the partial differential difference equation:*

$$\frac{\partial B(M,u-)}{\partial u} + a^* \frac{\partial B(M,u-)}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 B(M,u-)}{\partial x^2}$$

$$+ \sum_{j=1}^{J} \{ B(M,u- \mid r(u-)+c_j, x(u)) - B(M,u-) \} \lambda_j^*(u-) = B(M,u-) r(u-) \tag{13}$$

where

$$a^* = a - \theta_0 b \tag{14}$$

and

$$\Delta_j B(M,u-) = B(M,u- \mid r(u-)+c_j, x(u)) - B(M,u-)$$

Substituting (13) into the dynamics of the bond price enables us to re-write the price dynamics of pure discount bonds in an intuitive way:

Corollary 6.3 *Pure discount bond price dynamics can be written as:*

$$\frac{dB(M,u)}{B(M,u-)} = \left\{ r(u-) + \theta_0 \frac{b}{B(M,u-)} \frac{\partial B(M,u-)}{\partial x} + \sum_{j=1}^{J} \theta_j \lambda_j \frac{\Delta_j B(M,u-)}{B(M,u-)} \right\} du$$

$$+ \frac{b}{B(M,u-)} \frac{\partial B(M,u-)}{\partial x} dZ + \sum_{j=1}^{J} \frac{\Delta_j B(M,u-)}{B(M,u-)} (dN_j - \lambda_j du) \tag{15}$$

Under any risk-adjusted probability measure (for our choice of numeraire), these selfsame dynamics can be re-written as:

$$\frac{dB(M,u)}{B(M,u-)} = r(u-)du + \frac{b}{B(M,u-)} \frac{\partial B(M,u-)}{\partial x} dZ^* + \sum_{j=1}^{J} \frac{\Delta_j B(M,u-)}{B(M,u-)} (dN_j - \lambda_j^* du) \tag{16}$$

Remark 6.4 (15) carries the familiar economic interpretation that expected instantaneous bond returns equal the current short rate, plus risk premia proportional to the exposures to the various relevant sources of risk. (16) carries the even more familiar finance interpretation that, under risk-adjusted probabilities, the expected instantaneous return on bonds must equal the instantaneous spot rate.

7. EMPIRICAL REGULARITIES OF STERLING INTEREST RATES

In seeking to apply the methodology of Chan, Karolyi, Longstaff and Sanders [1992], to estimate a single factor diffusion model for daily data on sterling LIBOR, Babbs [1992] found that the results were strongly affected by numerous outliers, ie

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[18] In (13) *et seq.* the notation b denotes $b(r(u-),x(u),u)$ etc. Throughout the equations which follow, we can interchange u and $u-$ in many places, since jump times are pathwise of measure zero; in choosing which to use, we have given economic meaningfulness priority over notational compactness.

day-to-day rate changes of magnitudes many times the unconditional sample standard deviation. As neighbouring changes did not seem especially large, he concluded that these changes were unlikely to be accounted for wholly by stochastic volatility, but that UK interest rates include a jump component. Veruete [1992] and Veruete and Webber [1992] also found apparent jumps in UK rates, and observed that they tend to be associated with changes in commercial bank base rates[19] and hence[20] with changes in the authorities' key official rate. Even visual inspection of data (see Figure 1) confirms their findings. Indeed, since mid-1985, all day-to-day changes in 3-month LIBOR of one percentage point or more, and over half of the changes larger than half a point, have been associated with official interest rate changes.

Figure 1 also illustrates other empirical regularities, which have been more formally established (see Steele and Webber [1993]): market rates tend partially to "anticipate" official rate changes (see also Dale [1993]), in that it is rare for LIBOR rates to be below (above) official rates immediately prior to an official rise (fall); it is rare for 3-month LIBOR to stray more than 1% from official rates, usually being within 0.5% (see Figure 2).

Over some 20 years, changes in base rates[21] have averaged about 7 per year, and exhibit a degree of positive autocorrelation (Veruete and Webber [1992]). Between such changes, LIBOR rates appear to be very strongly mean-reverting[22] towards levels close to official rates. Since official rates have tended to move in multiples of 0.5%, the reversionary tendencies of LIBOR rates cause them to have a multi-modal distribution (see Figure 3).

There is some evidence to suggest that the volatility of LIBOR rates increases, the further LIBOR diverges from official rates.

Despite partially anticipating official rate changes, LIBOR rates respond strongly when official rates do actually change, with shorter maturity rates moving more vigourously than longer ones (see Dale [1993]).

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[19] Base rates are administered interest rates for commercial loans, in some respects analogous to prime rates in the US.

[20] For over 7 years, base rates have consistently been one eighth of a percentage point over the Bank of England's Band 1 stop rate.

[21] We take the jump properties of base rates to be representative of Band 1 dealing rates. The latter were introduced in the 1980s, replacing other mechanisms of not wholly dissimilar effect.

[22] Veruete [1992] fits Vasicek-type models to the deviations of 3-month LIBOR from base rates, over data segments between base rate changes. The estimated speeds of reversion are typically very high. To the extent that market rates and the authorities' propensity to change official rates reflect some common fundamental, the fact that a data segment lies between base rate changes may represent considerable conditioning. Nevertheless, as Figure 2 illustrates, there is compelling *prima facie* evidence of a strong central tendency.

8. ILLUSTRATIVE MODEL

We suggested in the introduction that it might be helpful to make the diffusion state variable x take the form of the markets' perception of a "prospective" or "shadow" level of interest rates, given the condition of the economy and the policy stance of the authorities. The dynamics of x can then plausibly be conjectured to take a form related to that of the dynamics of the short rate itself in the existing diffusion-based finance literature.

For illustrative purposes, let us adopt as our starting point the dynamics used by Courtadon [1982]. If the markets' gave no weight to the authorities' policy stance in determining the "prospective" or "shadow" rate process, x might have dynamics as postulated by Courtadon:

$$dx \;=\; \xi\,(\bar{\mu} - x)\,dt \;+\; \sigma\, x\, dZ \qquad (17)$$

where ξ is a positive constant rate of mean-reversion towards a level the long-run constant rate $\bar{\mu}$, and σ is the constant level of volatility.

It is extremely probable, however, that the markets' view of prospective interest rates attributes considerable persistence to the authorities' current interest rate stance (*cf* the work by Goodfriend and others cited on p9 of Dale [1993]). Emphasising again our illustrative intent, let us suppose that this effect takes the form that the level to which x reverts is a weighted average of the long-run rate $\bar{\mu}$ and the current officially-set short rate r. Thus, we modify (17) to:

$$dx \;=\; \xi\,(\mu - x)\,dt \;+\; \sigma\, x\, dZ \qquad (18)$$

where

$$\mu \;=\; (1-\alpha)\,\bar{\mu} \;+\; \alpha\, r\,; \qquad \alpha \in (0,1) \qquad (19)$$

In accordance with UK experience from the mid-1980s to early 1994, we make the possible jump sizes, c_j, multiples of half a percentage point. Introducing an additional jump sizes of a quarter of a point, with a positive probability only at low interest rate levels would accommodate the quarter point change witnessed in February 1994, but to do so here would over-complicate what is intended as a purely illustrative model. Changes of a half or whole percentage point have been historically by far the most common, but larger moves are by no means unprecedented. We therefore allow moves of up to 4%. Existing literature offers no guidance as to the form of the jump intensities λ_j. However, heuristic considerations lead readily to a plausible form. Firstly, having formulated x as the markets' perception of some kind of "prospective" interest rate, it is reasonable to treat the markets' perception of the likelihood of a each change as a function of the gap between x and the current short rate, r. The observation that market rates and r tend to remain close together, suggests that the most likely jump sizes at any time will be those close to $x - r$. Positive autocorrelation of official rate changes, and the general absence of rapid reversals of official rate changes suggests that the peak intensity for a particular jump size may be located at a point where the shadow rate x is "suggesting" that a larger jump may in fact be "appropriate". On the other hand, the fact (see Section 7) that market rates are occasionally (albeit rarely) on the opposite side of r from an immediately following jump, suggests that there is a positive chance of (say) a half point rise in official rates when x is slightly below r. Figure 4 illustrates a form for the jump intensities consistent with these heuristics. In the Figure, the bold line represents the intensity of a half point rise in r; intensities for larger rises are given by translations to the right; intensities for falls in r are mirror images of the intensities for rises of the same magnitudes[23].

Remark 8.1 Note that, if we consider the limit in which the allowed jump sizes are reduced towards zero, and the jump intensities increased, so that r jumps nearer to x in response to ever more minute divergences, the process for r becomes identical with that for x. Thus our illustrative model can be thought of as including the single factor diffusion model of Courtadon [1982] as a limiting case.

To obtain a fully parameterized illustrative model, we took $\xi = 0.1$ and $\sigma = 30\%$ in (18), a long-run rate $\bar{\mu} = 6\%$, and placed a high weight $\alpha = 0.9677$ upon the current level of r in (19)[24], together with jump intensities as illustrated in Figure 4, and full risk-neutrality ($\theta_0 = \theta_1 = \ldots = \theta_J = 0$). The permissible range of r was set from 3% to 17%, inclusive.

For this illustrative specification, we solved the bond price PDDE, (13), numerically.[25] We also undertook a Monte Carlo study of the model's time series properties by simulating the evolution of the state variables r and x, from initial values both equal to the long-run rate of 6%.

Figure 5 shows the term structures corresponding to a current short rate of 6.5%, and current x of 5% (lower curve) and 8% (upper curve) - the most extreme values that our Monte Carlo simulations suggest would be likely to be associated with r at 6%. The Figure suggests to us that our model offers an unprecedentedly wide dispersion of yield curves - even at long maturities - consistent with a given short rate and long-run rate; we conjecture that this will prove to be a strength of models constructed under our framework.

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[23] A more elaborate scheme, involving the possibly of an asymmetry between the chances of rates falling and of rates rising, might be used to explore the widespread belief that - at least during certain periods in the past - the UK authorities have tended to seize keenly upon opportunities to cut rates in repeated small steps, and to raise rates reluctantly but in large doses.

[24] This apparently non-intuitive level of α arises from superimposing a reversion of x towards r at rate *3.0*, upon the simple dynamics in (17).

[25] We took logarithms of the diffusion state variable x, in order to obtain a constant diffusion coefficient in the transformed version of (18). A modified Crank-Nicholson scheme was then used. The derivatives of bond values with respect to the transformed diffusion state variable were set to zero at extreme values of that variable.

A typical simulation of 3-month LIBOR and r is shown in Figure 6. The Figure shows LIBOR partially anticipating some jumps in r, while others seem to come as a surprise, as in the actual data in Figure 1. The simulated behaviour of LIBOR minus r (Figure 7) also shows strong similarities to the actual behaviour in Figure 2.

Table 1 shows the average ratio of day-to-day changes in various maturities of LIBOR, to changes in r, on days when the latter changes, both for the simulated series, and for actual data (as reported by Dale [1993]). The illustrative model appears to mimic the actual behaviour quite well.

Finally, we note that, in our illustrative model yields a realistic volatility of 3-month LIBOR of about 20%, despite the 30% volatility of the state variable x. One can interpret this as the authorities' rate setting behaviour smoothing out a sizeable proportion of the impact of news concerning the condition of the economy.

9. EXTENSIONS OF OUR APPROACH

We conceded, in the introduction, that it was an idealisation to identify the Bank of England's Band 1 dealing rate with the instantaneous spot rate, r. One way of relaxing this idealisation, in keeping with the apparent intent of the UK authorities' implementation of monetary policy, while retaining the perfect frictionless markets paradigm, would be to view the Band 1 rate as a "target" rate, and model the "deviations" of r from the target rate as a mean-reverting random walk.

The "target" and mean-reverting "deviations" extension to our approach, just described, might well serve as a model appropriate for some other countries. We have the US particularly in mind here, since Balduzzi, Bertola and Foresi [1993] report considerable success in modelling the deviations of the overnight Fed funds rate from the midpoint of the FRBNY's target range precisely as a simple Ornstein-Uhlenbeck process.

There are countries, however, for which extending our approach would require greater effort. In Germany for example, it can be argued that a key official interest rate, often perhaps the most important, is the Bundesbank's rate for repo transactions in government securities. The repo transactions are typically of two or four weeks in maturity, and changes in the repo rate are potentially of almost any size, depending in part on whether the Bundesbank sets a variable or fixed rate repo (see Schnadt [1994] for further details). We conjecture that for Germany, or for other countries with analogous institutional arrangements, the relevant extension of our approach might need to make use of general marked point processes, rather than jump processes with fixed jump sizes as in the present paper.

10. SUMMARY AND CONCLUDING REMARKS

In introducing this paper, we suggested that term structure modelling within finance needs to take account of the rate-setting role of monetary authorities, especially since changes in official interest rates can be associated with jumps in market rates. We decided to tackle this task as it presents itself in the UK, where, we argued, it is a plausible idealisation to treat the instantaneous spot interest rate as a pure jump process set by the authorities, and having a range of fixed jump sizes.

Market interest rates reflect investors' probability assessments of future short rates, based on information available to them. (Both the information set and the probability assessments of the authorities may well differ.) In the interests of parsimony, we chose to model the markets' perceptions of the chances of changes in the officially-set short rate, r, in terms of stochastic jump intensities depending on the current level of r, and of a single additional state variable, x (and possibly of time). We assumed that x follows a diffusion, representing some indicator of the condition of the real and financial economy.

Using these ingredients, we constructed a general theoretical framework. Using results in Babbs and Selby [1993], we showed that our framework is consistent with equilibrium, and that the term structure, and the values of interest rate derivatives are determined by the market prices of the diffusion and jump risks inherent in x and r, independently of the market prices of risk attaching to any sources of risk other than those we have modelled explicitly. A partial differential difference valuation equation (PDDE) was derived, as an aid to numerical computation.

We illustrated our general theoretical framework with an example, in which x played the role, in at least some loose sense, of the markets' perception of a "prospective" or "shadow" level of interest rates, given the condition of the economy and the policy stance of the authorities. This enabled us to propose dynamics for x based on taking dynamics posited for the short rate itself in the diffusion-based finance literature, and modifying the local reversion level to give weight to the current level of r. Solving the PDDE numerically for the term structure suggested that models constructed under our framework may allow a remarkably wide range of term structures for given levels of the short rate and long-run rate. While being wholly illustrative almost to the point of being simplistic, Monte Carlo simulation of the model yielded time series properties bearing an encouraging resemblance to a number of empirical regularities of actual UK data.

We are currently pursuing further research, covering empirical estimation, and the valuation of interest rate derivatives. We intend to report on this elsewhere. Extensions to our approach, particularly to cover other countries - most readily the US, along the lines discussed in Section 9, seem a fruitful area for further research.

Finally, we note that, in our illustrative model, the volatility of 3-month LIBOR was significantly less than that of x which indicated the markets' view of prospective rates. This suggests that - at least in our illustrative model - the authorities' rate-setting behaviour may smooth out a considerable degree of the turbulence that might occur if they were responded more readily to news on the condition of the economy. In general principle, such an effect might represent an interest rate analogue of the calming effects which some models suggest to result from central bank management of floating exchange rates.

APPENDIX

Proof of Lemma 4.2 From a standard property of the exponential semimartingale,[26]

$$\mathcal{E}\left\{-\sum_{j=0}^{J}\int_0^{\cdot}\theta_j\,dM_j\right\}\quad \mathcal{E}\{N\} \;=\; \mathcal{E}\left\{N-\sum_{j=0}^{J}\int_0^{\cdot}\theta_j\,dM_j-\sum_{j=0}^{J}\left[N,\int_0^{\cdot}\theta_j\,dM_j\right]\right\} \tag{A1}$$

By definition of the optional quadratic covariation process, and exploiting the predictability of each θ_j, we have, for each *j*,

$$\left[N,\int_0^{\cdot}\theta_j\,dM_j\right](t) \;=\; <N^c,\left(\int_0^{\cdot}\theta_j\,dM_j\right)^c>(t) \;+\; \sum_{0<u\le t}\Delta N(u)\,\theta_j(u)\,\Delta M_j(u) \tag{A2}$$

Consider first the case *j=0*. $M_0 = Z$ is continuous; thus the second term on the RHS of (A2) disappears immediately. The first term also vanishes, once we have performed the decomposition:

$$<N^c,\left(\int_0^{\cdot}\theta_0\,dM_0\right)^c> \;=\; <N^c,\int_0^{\cdot}\theta_0\,dZ> \;=\; <N,\int_0^{\cdot}\theta_0\,dZ> \;-\; <N^d,\int_0^{\cdot}\theta_0\,dZ>$$

by the orthogonality between N and Z (from Assumption (F)), and that between purely discontinuous and continuous local martingales.

For *j>0* also, the second term on the RHS of (A2) disappears immediately, since, by Assumption (F), N does not charge a common jump time with M_j. The first term, again, also vanishes, since the second argument within the angle brackets vanishes, being the continuous part of a purely discontinuous local martingale.

We have now shown that the final summation within the RHS of (A1) is identically zero. The result follows, since, by (7), the RHS of (A1) is precisely η. *

Proof of Lemma 4.3 Various well-known and fairly straightforward sufficient conditions exist to ensure that applying the exponential semimartingale operator to a continuous local martingale yields a martingale. In our case, however, the argument of the operator involves the discontinuous processes N_j. To handle this situation, we wish to rely on sufficient conditions established by Lepingle and Memin [1978] (Theoreme III.1, pp185-6). To that end, we define a local martingale M by:

$$M \;\equiv\; -\sum_{j=0}^{J}\int_0^{\cdot}\theta_j\,dM_j \tag{A3}$$

and consider the process $\tilde{M}$ defined by:

$$\tilde{M}(t) \;\equiv\; \frac{1}{2}<M^c,M^c>(t) \;+\; \sum_{0\le s\le t}[\{1+\Delta M(s)\}\ln\{1+\Delta M(s)\}-\Delta M(s)] \tag{A4}$$

By Assumption (F), $\Delta M > -1$. Since, $\eta_0 = \mathcal{E}\{M\}$, it follows from the cited result of Lepingle and Memin that it will suffice to prove that $\tilde{M}$ admits a predictable compensator, Λ_M, such that $E[\exp\{\Lambda_M(T)\}] < \infty$.

Ex hypothesi, $M_0 \equiv Z$ is continuous, while the compensated counting processes $M_1,\ldots,M_J$ do not charge any common jump times. Moreover, the jumps of $M_1,\ldots,M_J$ are all equal to +1, and the processes θ_j are all predictable. It follows immediately that

$$\sum_{0\le s\le t}[\{1+\Delta M(s)\}\ln\{1+\Delta M(s)\}-\Delta M(s)] \;=\; -\;(2\ln 2-1)\sum_{j=1}^{J}\sum_{0\le s\le t}\theta_j(s)\Delta N_j(s) \tag{A5}$$

[26] see eg Elliott [1982] Corollary 13.8 p160

By the same hypotheses, coupled with the fact that $M_1, \ldots, M_J$ are purely discontinuous,

$$< M^c, M^c > \;=\; < -\int_0^{\cdot} \theta_0 dZ, \; -\int_0^{\cdot} \theta_0 dZ > \;=\; \int_0^{\cdot} \theta_0^2 \, du \qquad (A6)$$

which, on substituting into (A4), together with (A5), yields

$$\tilde{M}(t) \;=\; \frac{1}{2}\int_0^t \theta_0^2 \, du \;-\; (2\ln 2 - 1) \sum_{j=1}^{J} \sum_{0 \le s \le t} \theta_j(s) \Delta N_j(s)$$

which (since the Lebesgue integral of each intensity λ_j compensates the respective N_j, and each θ_j is predictable) admits the predictable compensator

$$\Lambda_M(t) \;\equiv\; \frac{1}{2}\int_0^t \theta_0^2 \, du \;-\; (2\ln 2 - 1) \sum_{j=1}^{J} \int_0^t \theta_j \, \lambda_j \, du \qquad (A6)$$

By Assumptions (D) and (F), all the terms on the RHS of (A6) are essentially bounded. The result follows. *

Proof of Lemma 4.4 For *q=0*, the result is trivial. Henceforth, consider non-zero *q*.

For *j = 1, ..., J*, M_j is the compensated local martingale of the counting process N_j. Hence,[27]

$$\mathcal{E}\left\{ -\sum_{j=0}^{J} \int_0^t \theta_j \, dM_j \right\} \;=\; \exp\left\{ -\int_0^t \theta_0 \, dZ \;-\; \frac{1}{2}\int_0^t \theta_0^2 \, du \;-\; \sum_{j=1}^{J} \int_0^t \theta_j \, dM_j \right\}$$

$$\times \prod_{j=1}^{J} \prod_{0<u\le t} [\{1 - \theta_j(u)\Delta N_j(u)\} \exp\{\theta_j(u)\Delta N_j(u)\}] \qquad (A7)$$

Now, for *j>0*, exploiting the predictability of θ_j gives:

$$\int_0^t \theta_j \, dM_j \;=\; \sum_{0<u\le t} \theta_j(u) \, \Delta N_j(u) \;-\; \int_0^t \theta_j(u) \, \lambda_j(u) \, du$$

Substituting in (A7), and collecting terms,

$$\mathcal{E}\left\{ -\sum_{j=0}^{J} \int_0^t \theta_j \, dM_j \right\} \;=\; \mathcal{E}\left\{ -\int_0^t \theta_0 \, dZ \right\} \prod_{j=1}^{J} \left[\exp\left\{ \int_0^t \theta_j \lambda_j du \right\} \prod_{0<u\le t} \{1 - \theta_j(u)\Delta N_j(u)\} \right] \qquad (A8)$$

Define:

$$\tilde{\theta}_0 \;\equiv\; q \, \theta_0$$

and, for *j>0*,

$$\tilde{\theta}_j \;\equiv\; 1 \;-\; (1 - \theta_j)^q$$

which, by the strict upper bound of unity imposed on θ_j in Assumption (F),

$$< \; 1$$

Substituting in (A8), and recalling that each N_j has unit jumps,

$$\mathcal{E}\left\{ -\sum_{j=0}^{J} \int_0^t \theta_j \, dM_j \right\}^q \;=\; \exp\left\{ \frac{1}{2} q \, (q-1) \int_0^t \theta_0^2 \, du \right\} \; \mathcal{E}\left\{ -\int_0^t \tilde{\theta}_0 \, dZ \right\}$$

$$\times \prod_{j=1}^{J} \left[\exp\left\{ \int_0^t q \, \theta_j \, \lambda_j \, du \right\} \prod_{0<u\le t} \{1 - \tilde{\theta}_j(u)\Delta N_j(u)\} \right]$$

$$= \; \exp\left\{ \frac{1}{2} q(q-1) \int_0^t \theta_0^2 \, du + \int_0^t (q\theta_j - \tilde{\theta}_j)\lambda_j du \right\} \; \mathcal{E}\left\{ -\sum_{j=0}^{J} \int_0^t \tilde{\theta}_j \, dM_j \right\} \qquad (A9)$$

Now, the ordinary exponential on the RHS of (A9) is essentially bounded, *ex hypothesi*; the exponential semimartingale on the RHS is integrable, by Lemma 4.3 with each θ_j replaced by the (also essentially bounded) $\tilde{\theta}_j$. The result follows. *

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[27] See eg Elliott [1982] p156 for the general expression for the exponential semimartingale. We have exploited our various orthogonality and no common jump assumptions, the predictability of each θ_j, etc, to make preliminary obvious simplifications.

Proof of Lemma 4.5 Strict positivity of η follows from it being the exponential semimartingale of a semimartingale whose jumps all strictly exceed -1.[28]

By assumptions (b) and (c) of the lemma, $\mathcal{E}\{N\}$ is certainly a square integrable martingale.[29] By Lemmas 4.3 and 4.4, η_0 is also a square-integrable martingale, which, by Lemma 4.2, is orthogonal to $\mathcal{E}\{N\}$. Hence,[30] η is a martingale.

It remains to establish that η is square-integrable. Now, by Holder's inequality,

$$E[\ \eta_0^2(T)\,\mathcal{E}\{N\}^2(T)\ ] \ \le\ E[\ \eta_0^{2(2+\delta)/\delta}(T)\ ]^{\delta/(2+\delta)}\ E[\ \mathcal{E}\{N\}^{2+\delta}(T)\ ]^{2/(2+\delta)} \qquad (A10)$$

Both the terms on the RHS of (A10) are finite, by Lemma 4.3 and assumption (c) of the present lemma, respectively. The result follows[31]. *

Proof of Proposition 4.7 By Lemma 4.5, η is a strictly positive martingale under *P*. Hence[32], (5) defines a probability measure equivalent to *P*. Also by Lemma 4.5, $\eta(T)$ is square integrable, while, by Assumption (C), $S_0(T)$ is bounded below away from zero. Hence, (6) is satisfied. *

Proof of Theorem 4.8 By Proposition 4.7, and Babbs and Selby [1993] Theorem 7.12, P^*, as defined by (5) is a (local) RPM. However, as neither of our primitive processes r and x are securities prices, any local RPM is an RPM.[33] Moreover, our (6) is precisely BS' Condition (S). Hence, by BS' Theorem 7.8 and Corollary 7.9, our model is viable, with pricing operator precisely as given by (4). *

Proof of Proposition 5.1 By a form of Girsanov's theorem,[34]

$$M_0^* \equiv M_0 - \ < M_0, N - \sum_{j=0}^{J} \int_0^{\cdot} \theta_j\, dM_j >$$

is a local martingale under P^*, for $j = 0, \ldots, J$.

Exploiting the various orthogonality relations among $M_0, \ldots, M_J, N$ yields

$$\begin{aligned} < M_0, N - \sum_{j=0}^{J} \int_0^{\cdot} \theta_j\, dM_j > &= \ < M_0, -\int_0^{\cdot} \theta_0\, dM_0 > \\ &= -\int_0^{\cdot} \theta_0\, d < M_0, M_0 > \\ &= -\int_0^{\cdot} \theta_0\, du \end{aligned}$$

Moreover, since M_0 is continuous, the predictable quadratic variation of M_0^* under P^* equals that of M_0 under *P*, ie time.[35] Hence, by Levy's characterisation theorem,[36] M_0^* is a standard Brownian motion under P^*.

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[28] see eg Jacod [1979] Proposition 6.5(b) p192

[29] by eg Elliott [1982] Lemma 9.6(2) p86

[30] see eg Elliott [1982] Lemma 9.12 p88

[31] by eg Elliott [1982] Lemma 9.6 p86

[32] see eg Elliott [1982] p165

[33] Babbs and Selby make this point immediately after their Definitions 7.3 and 7.4

[34] see eg Elliott [1982] Theorem 13.19 p165

[35] see eg Elliott [1982] Theorem 13.24 p168

[36] see eg Bremaud [1981] Appendix A3 Theorem T2 p313

It remains to consider to establish that the P^* -intensity of each N_j is λ_j^* . By definition[37], we must establish that if C is a non-negative predictable process, then

$$\mathrm{E}^*\left[\ \int_0^T C\, dN_j\ \right] \;=\; \mathrm{E}^*\left[\ \int_0^T C\, \lambda_j^*\, du\ \right] \qquad (A11)$$

Since η is the Radon-Nikodym derivative of P^* with respect to P, and substituting (11b), (A11) becomes

$$\mathrm{E}\left[\ \eta(T) \int_0^T C\, dN_j\ \right] \;=\; \mathrm{E}\left[\ \eta(T) \int_0^T C\,(1-\theta_j)\,\lambda_j\, du\ \right] \qquad (A12)$$

The integral inside the expectation operator on each side of (A12) is an increasing adapted process, since, by Assumption (F), $\theta_j > -1$; moreover, η is a non-negative martingale. Hence, by a result of Dellacherie and Meyer,[38] (A12) in turn becomes

$$\mathrm{E}\left[\ \int_0^T \eta\, C\, dN_j\ \right] \;=\; \mathrm{E}\left[\ \int_0^T \eta\, C\,(1-\theta_j)\,\lambda_j\, du\ \right] \qquad (A13)$$

Now, in the integrand inside the RHS of (A13), we can replace η by η_- , since the two differ only at jump times, which, on each path, form a null set. Following this substitution, the integrand is predictable as well as non-negative. Hence, since N_j admits the P-intensity λ_j , (A13) becomes

$$\mathrm{E}\left[\ \int_0^T \eta\, C\, dN_j\ \right] \;=\; \mathrm{E}\left[\ \int_0^T \eta_-\, C\,(1-\theta_j)\, dN_j\ \right]$$

which we rearrange as

$$\mathrm{E}\left[\ \int_0^T \{\eta - (1-\theta_j)\,\eta_-\}\, C\, dN_j\ \right] \;=\; 0 \qquad (A14)$$

Thus, it remains to prove (A14).

Since N_j is a counting process,

$$\int_0^T \{\eta - (1-\theta_j)\,\eta_-\}\, C\, dN_j \;=\; \sum_{0<u\le T} \{\eta(u) - (1-\theta_j(u))\,\eta_-(u)\}\, C(u)\, \Delta N_j(u) \qquad (A15)$$

Hence it will suffice to show that

$$\{\eta - (1-\theta_j)\,\eta_-\}\,\Delta N_j \;\equiv\; 0$$

which rearranges as

$$\Delta\eta\, \Delta N_j \;\equiv\; -\theta_j\, \eta_-\, \Delta N_j \qquad (A16)$$

We shall now show that (A16) holds.

As the exponential semimartingale of

$$M \;\equiv\; N \;-\; \sum_{i=0}^{J} \int_0^{\cdot} \theta_i\, dM_i$$

η satisfies

$$\eta \;=\; 1 \;+\; \int_0^{\cdot} \eta_-\, dM$$

whence

$$\Delta\eta \;=\; \eta_-\,\Delta M \;=\; \eta_- \left\{ \Delta N - \sum_{i=1}^{J} \theta_i\, \Delta N_i \right\} \qquad (A17)$$

Now, the only N_i to share any jump times with N_j is N_j itself, while, by Assumption (F), N does not charge any common jump time with N_j . Therefore, substituting (A17) into the RHS of (A16) yields the promised identity. *

Proof of Proposition 5.2 Available from the authors on request. *

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[37] see Bremaud [1981] chap II Definition D7 p27

[38] see eg Bremaud [1981] Appendix 2 Theorem T19 p302

Proof of Theorem 5.3 Immediate from Proposition 5.2, by Theorem 7.11 in Babbs and Selby [1993]. *

Proof of Theorem 6.2 As we noted while motivating Assumption (F), the prices of pure discount bonds depend on the state variables (and time and maturity) alone. Hence, by the generalized version of Ito's lemma, applied to $B(M,t)$, for fixed M, as a function of r, x and t, we obtain:

$$B(M,t) = B(M,0) + \int_0^t \frac{\partial B(M,u-)}{\partial u} du + \int_0^t \frac{\partial B(M,u-)}{\partial x} dx + \frac{1}{2}\int_0^t b^2 \frac{\partial^2 B(M,u-)}{\partial x^2} du + \sum_{0<u\le t} \Delta B(M,u) \tag{A18}$$

Now,

$$\Delta B(M,u) = \sum_{j=1}^{J} \{ B(M,u-|\ r(u-)+c_j, x(u)) - B(M,u-) \}\ \Delta N_j(u) \tag{A19}$$

Hence,

$$\sum_{0<u\le t} \Delta B(M,u) = \sum_{j=1}^{J} \int_0^t \{ B(M,u-|\ r(u-)+c_j, x(u)) - B(M,u-) \}\, dN_j \tag{A20}$$

$$= \sum_{j=1}^{J} \left[\int_0^t \{ B(M,u-|\ r(u-)+c_j, x(u)) - B(M,u-) \}\, d\left(N_j - \int_0^{\cdot} \lambda_j^*(s) ds \right) + \int_0^t \{ B(M,u-|\ r(u-)+c_j, x(u)) - B(M,u-) \}\, \lambda_j^*(u-)\, du \right] \tag{A21}$$

In each summand on the RHS of (A21), the first integral has a *caglad* integrand and a local martingale integrator, and hence is itself a local martingale; the second integral is absolutely continuous.

Using these findings, we can re-write (A18) as:

$$B(M,t) = B(M,0) + \int_0^t A(M,u)\, du + L(M,t) \tag{A22}$$

where

$$A(M,u) = \frac{\partial B(M,u-)}{\partial u} + a^* \frac{\partial B(M,u-)}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 B(M,u-)}{\partial x^2} + \sum_{j=1}^{J} \{ B(M,u-|\ r(u-)+c_j, x(u)) - B(M,u-) \}\, \lambda_j^*(u-) \tag{A23a}$$

and $L(M,.)$ is a local martingale given by

$$L(M,t) = \sum_{j=1}^{J} \int_0^t \{ B(M,u-|\ r(u-)+c_j, x(u)) - B(M,u-) \}\, d\left(N_j - \int_0^{\cdot} \lambda_j(s) ds \right) + \int_0^t b \frac{\partial B(M,u-)}{\partial x} dZ^* \tag{A23b}$$

where a^* is given by (14).

Applying Ito's lemma to the normalized bond price as given by (A18), and exploiting the absolute continuity of S_0 over time, we obtain

$$B^*(M,t) = B^*(M,0) + \int_0^t \frac{dB(M,u)}{S_0(u)} - \int_0^t B^*(M,u-)\, r(u-)\, du \tag{A24}$$

whence, substituting (A22),

$$B^*(M,t) = B^*(M,0) + \int_0^t \left\{ \frac{A(M,u)}{S_0(u)} - B^*(M,u-)\, r(u-) \right\} du + \int_0^t \frac{dL(M,u)}{S_0(u)} \tag{A25}$$

Now, $B^*(M,.)$ must be a P^* -martingale. The second integral on the RHS of (A25) is a local martingale under P^* .[39] Hence, the first integral on the RHS must also be a local P^* -martingale. Being (pathwise) an ordinary integral, it is continuous, and thus predictable and of finite variation. But predictable local martingales of finite variation are constant[40]. Thus, the integrand must vanish identically. Multiplying the integrand through by $S_0(u)$ yields:

$$A(M,u) \;-\; B(M,u-)\,r(u-) \;=\; 0 \qquad (A26)$$

Expanding (A26) by substituting (A23a) immediately gives us the stated Theorem. *

Proof of Corollary 6.3 Substitute (A26) into (A22), and note that L is a local martingale under P^* . *

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[39] Since the integrand is *cadlag*, and the integrator a local martingale.

[40] see eg Elliott [1982] Lemma 11.39 p121

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Figure 1

3-month LIBOR and base rates : actual data

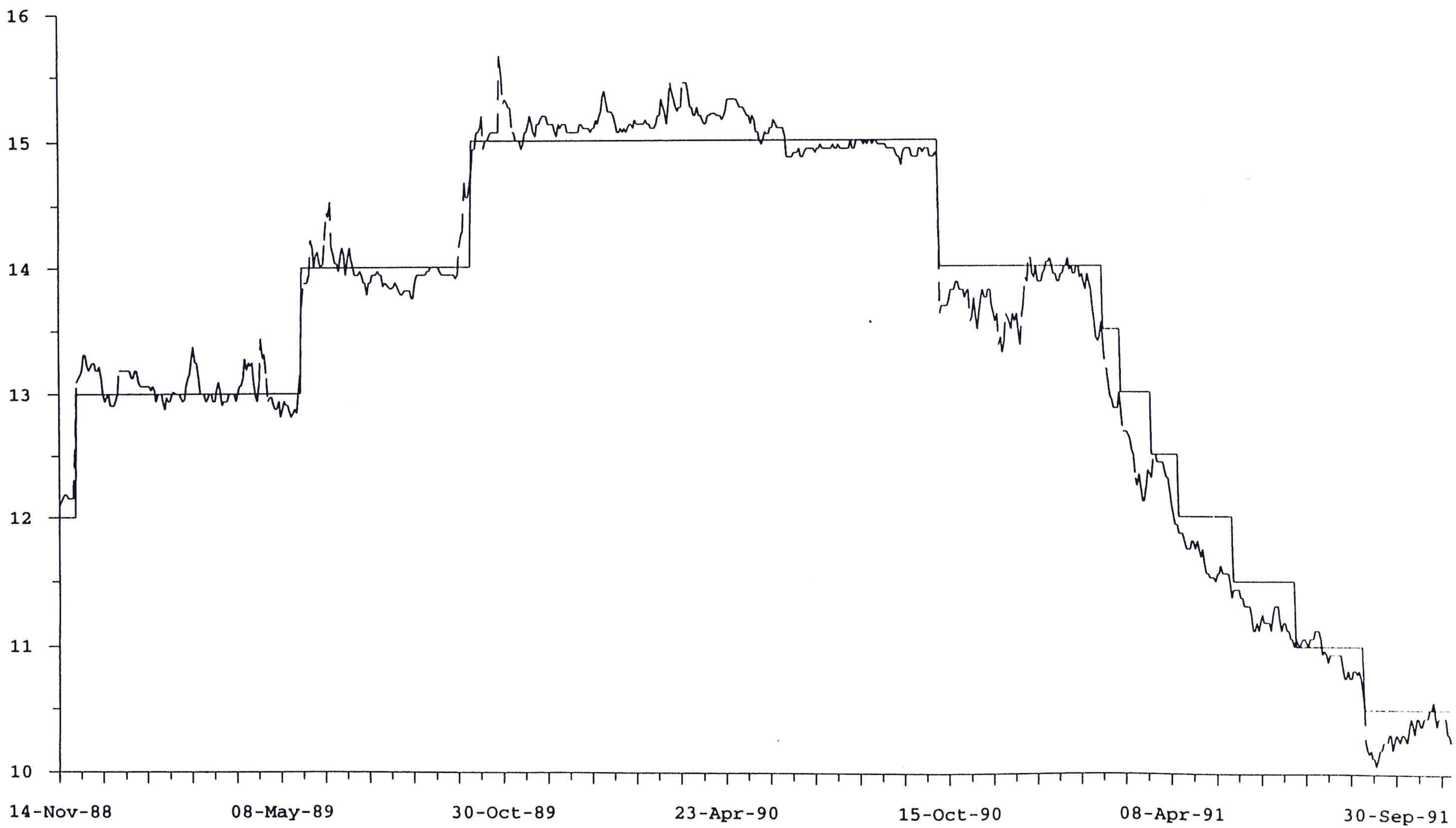


Figure 2

3-month LIBOR minus base rates: actual data

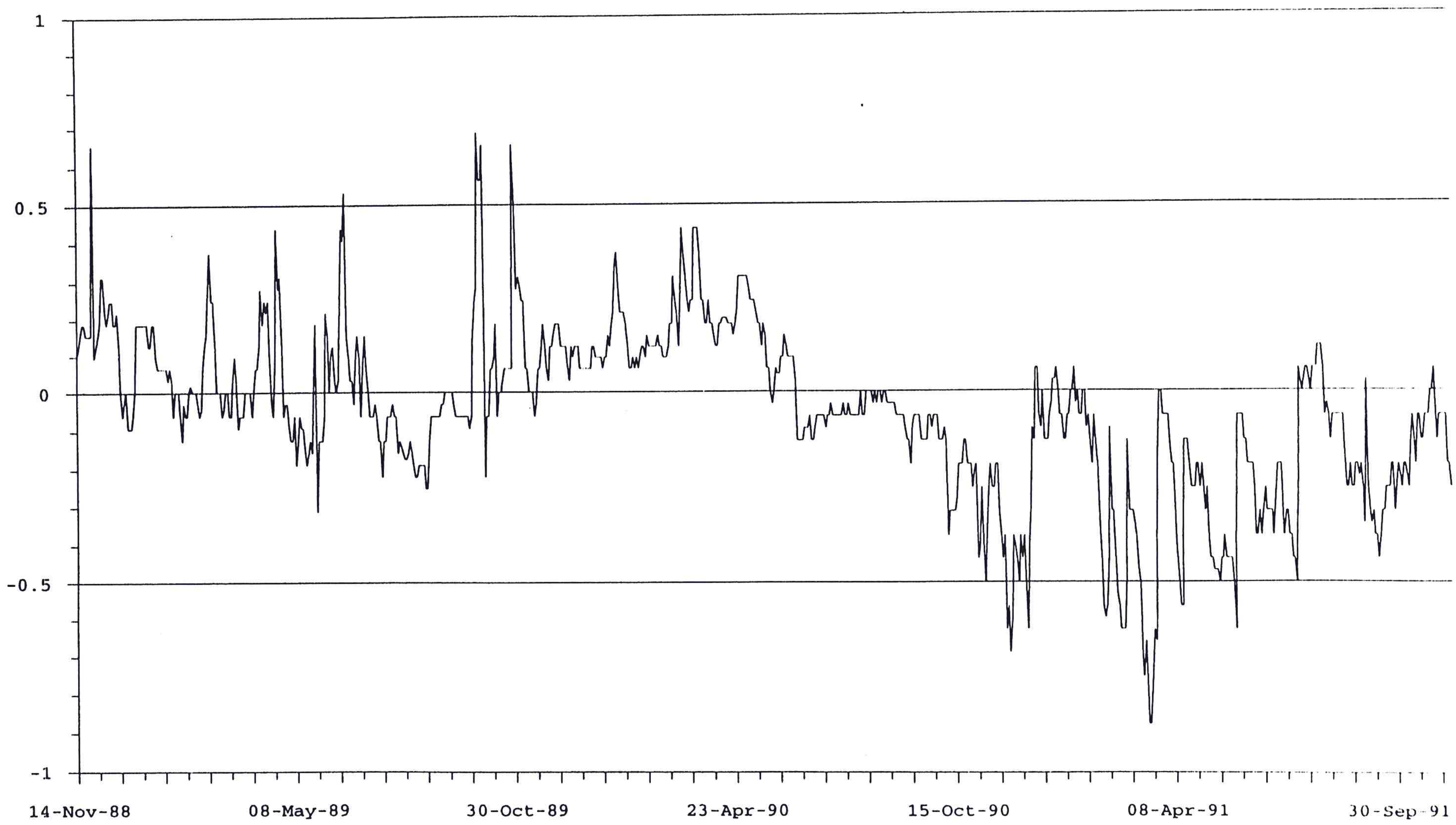
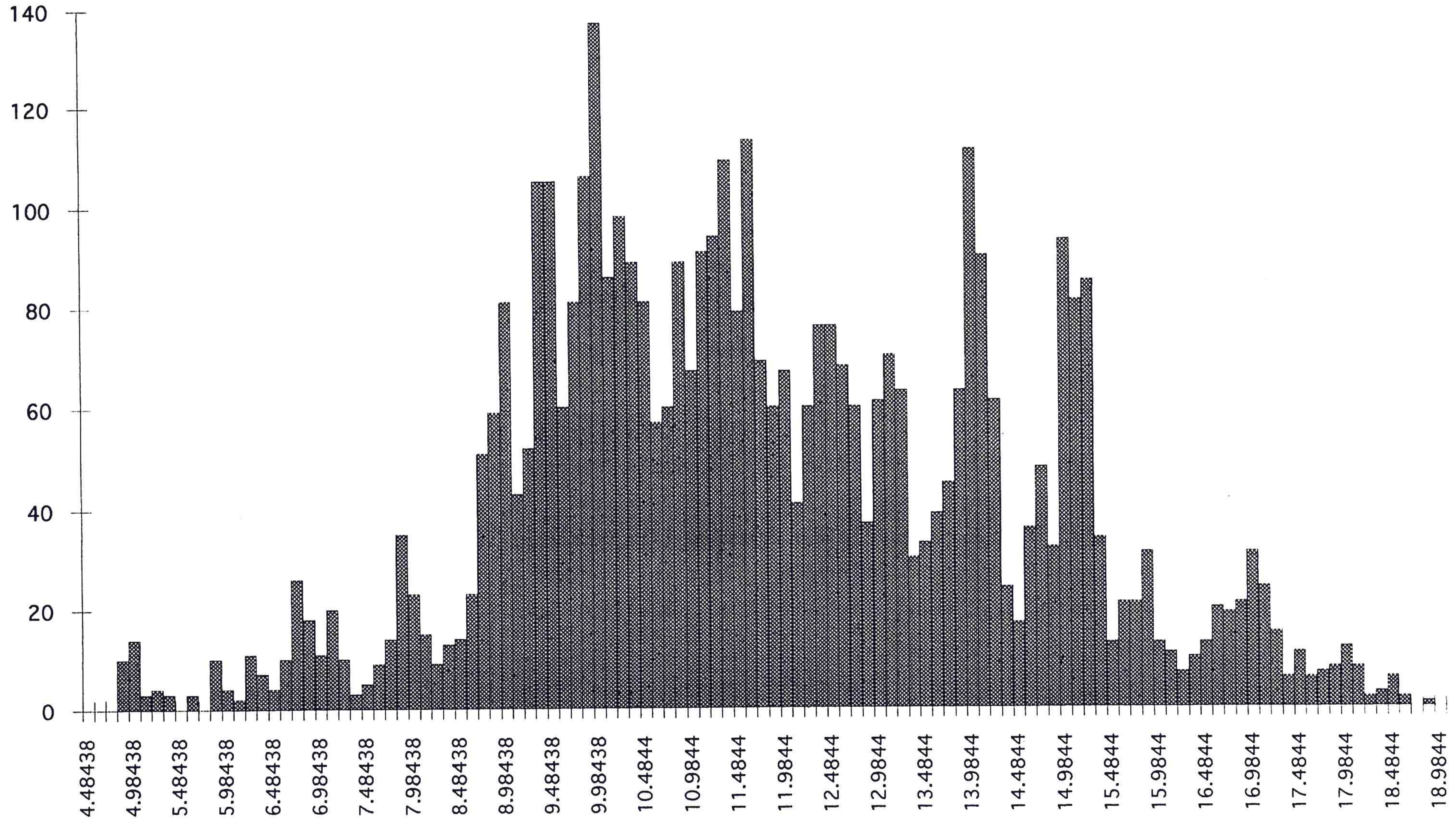


Figure 3

Frequency Histogram of sterling three month libor, 1975 - 1991



lambda for jump sizes of -0.5%, +0.5%, +1% and +1.5%

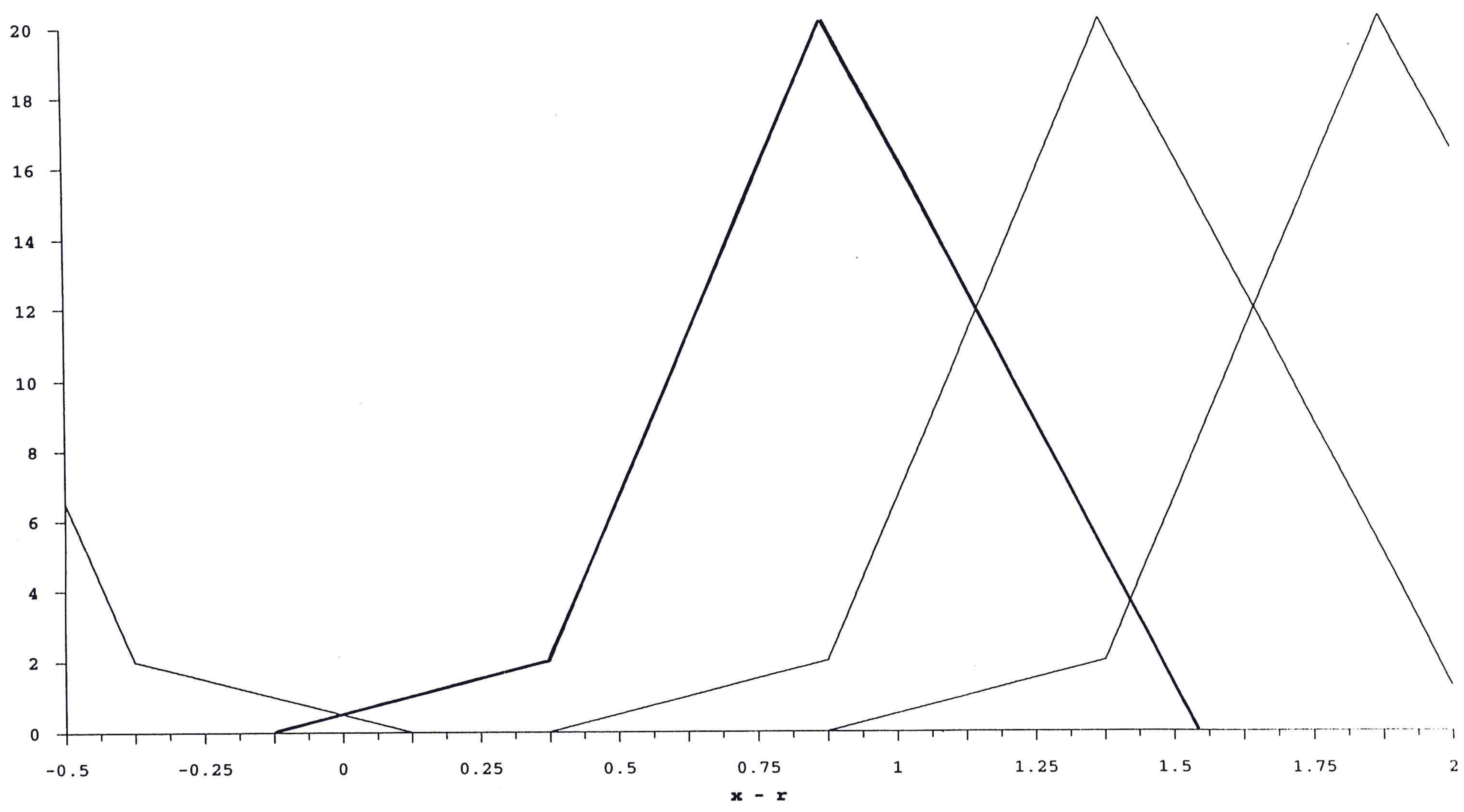


Figure 4

Figure 5

Theoretical term structures with r = 6.5%;
x(0) = 5% and 8% respectively

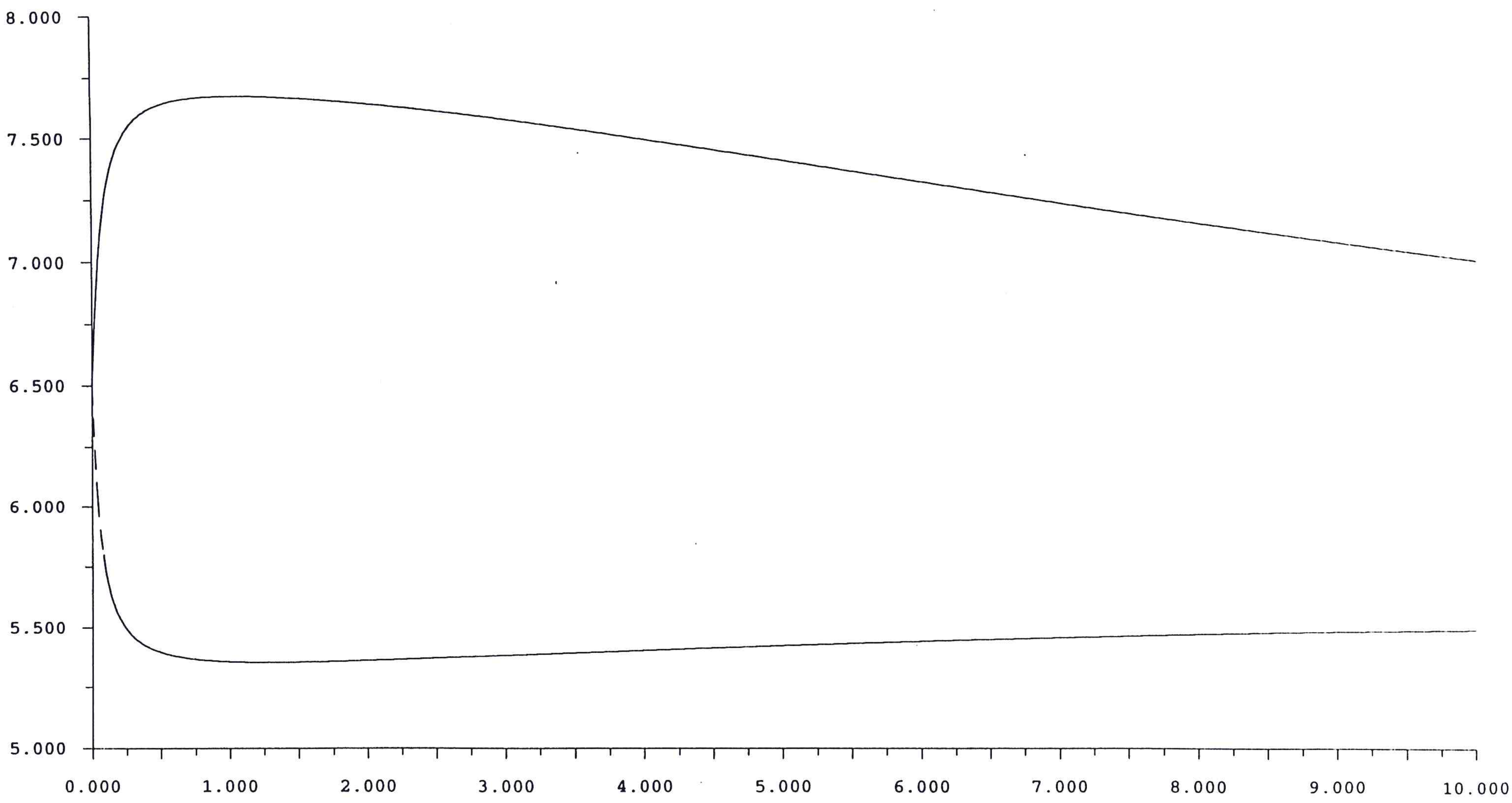
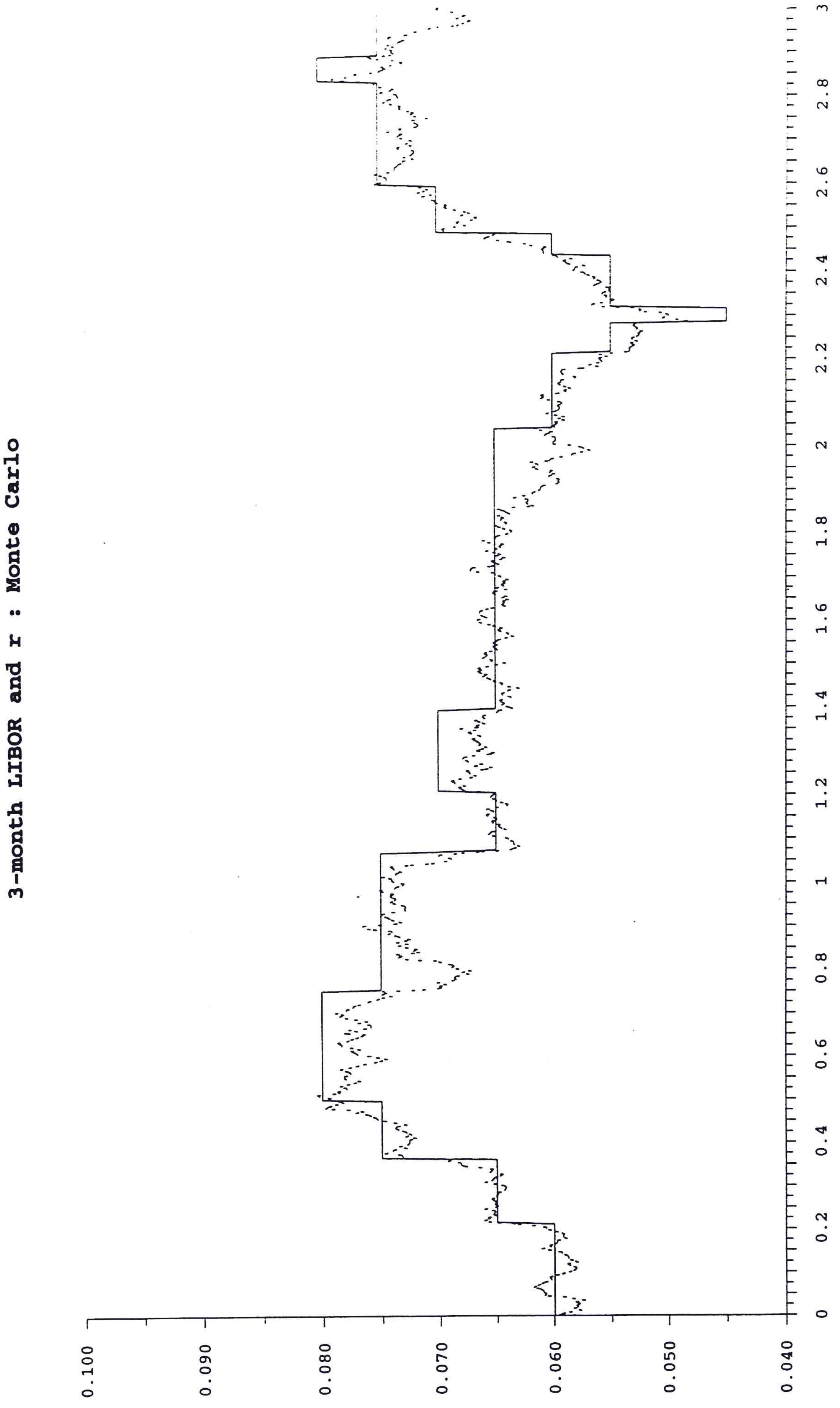


Figure 6



3-month LIBOR and r : Monte Carlo
0.100
0.090
0.080
0.070
0.060
0.050
0.040
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3

Figure 7

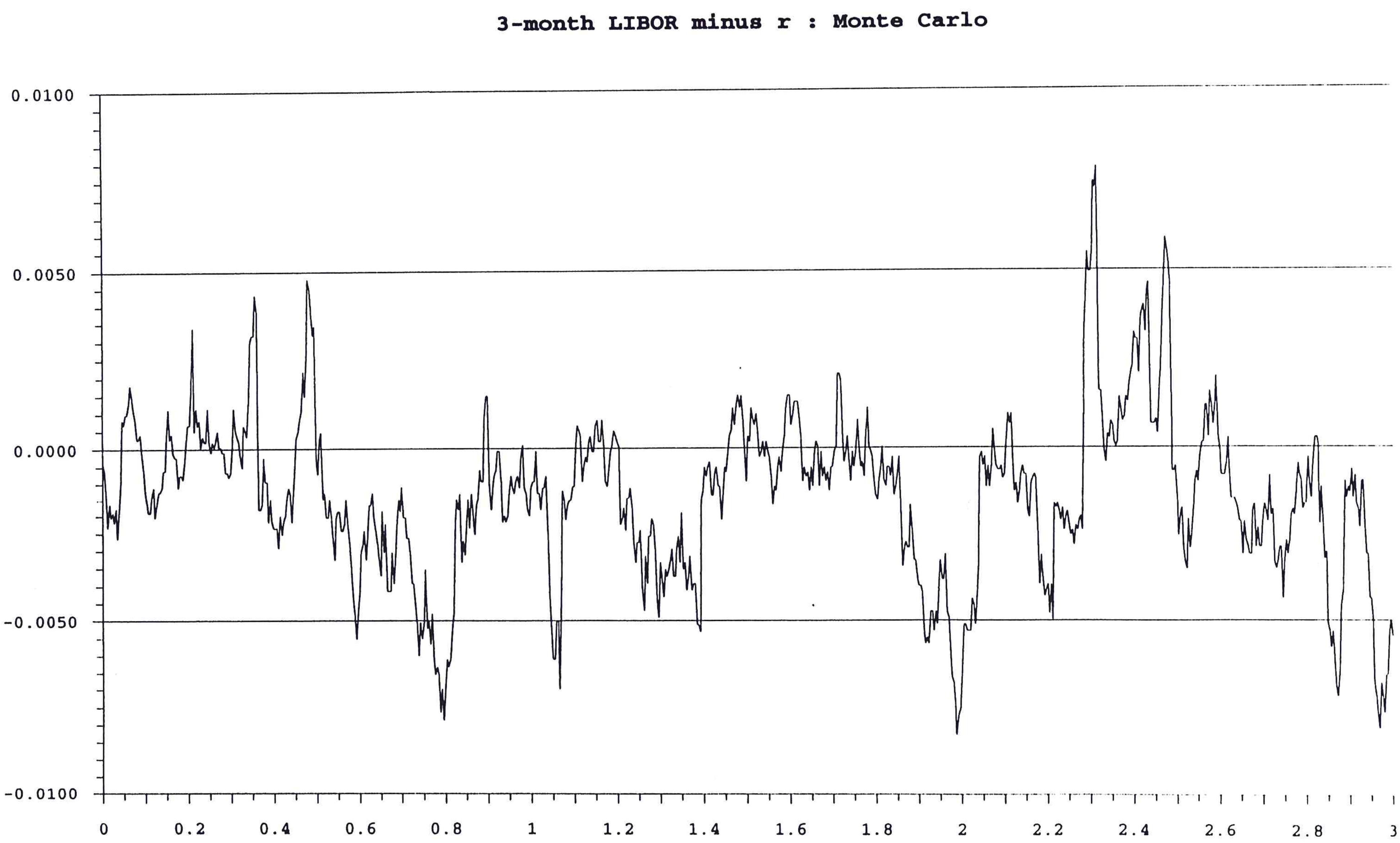

3-month LIBOR minus r : Monte Carlo
0.0100
0.0050
0.0000
-0.0050
-0.0100
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3

Table 1

RESPONSES OF LIBOR TO CHANGES IN *r*

($\Delta LIBOR / \Delta r$)

	Maturity			
	1 month	3 months	6 months	12 months
Monte Carlo[1]	65.6%	40.7%	30.3%	24.1%
Actual[2]	52.2%	37.6%	32.0%	31.1%

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[1] Based on 10 3-year runs;
average no. of jumps per year: 5.2

[2] 1987 - July 1991 as reported by Dale [1993]