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Abstract

This article provides further analysis of the behaviour of the risk premium on the market portfolio of risky assets. Earlier work by Hodges and Carverhill (1993), and by others, has characterised the evolution of the market risk premium in economies where the variance of the return on the market has constant variance and market index options can be priced using the Black–Scholes model. In such economies the risk premium satisfies a non-linear partial differential equation called Burgers' equation. This chapter provides some significant new insights into this analysis. First we describe the nature of the existing results and provide a much simpler and more intuitive derivation. Next, we consider the time homogeneous case. Our original objective was to find a time homogeneous economy which allows the risk premium to vary inversely with the level of the market so that some kind of mean reversion could take place. Sadly, this is impossible. We obtain the interesting, but negative, result that the risk premium must be constant or increasing in the market level for time homogeneous equilibria which rule out arbitrage. Finally, this result is shown to tie in to earlier work on asymptotic portfolio selection. The article also illustrates that caution is required in this kind of modelling to avoid writing down models which admit arbitrage. The analysis also shows the limitations of the representative investor paradigm.

1 Introduction

This article describes properties of the behaviour of the risk premium on the market portfolio of risky assets. It analyses how the market risk premium must evolve in a single representative agent economy where the variance of the equity market return and the risk free interest rate are both constant. These are precisely the assumptions which enable index options to be valued using the Black and Scholes (1973) option valuation model. For the purposes of option pricing – under these assumptions – the behaviour of the risk premium is of no consequence. We transform to a risk neutral measure and obtain prices as expectations under that measure. However, for portfolio management the risk premium is extremely important, and we would like to know how it behaves and, if it varies through time, we will want to know how it varies.

There is some evidence that the required risk premium does vary through time, and in a way which is akin to mean reversion. Thus, studies by Poterba and Summers (1988), Fama and French (1988) and Lo and MacKinlay (1988) have all suggested that equity market returns are to some extent predictable. There is also evidence that the dividend yield may be related to the expected risk premium.

It is quite plausible that if investors have decreasing relative risk aversion then these kinds of relationships will be supported within a market equilibrium. The earlier analysis of Hodges and Carverhill (1993) showed how the risk premium would have to evolve under a non-linear partial differential equation known as Burgers' equation if the economy were characterised by a single representative investor maximising (non-state dependent) expected utility for wealth at terminal date H and the interest rate and volatility of the risky asset were constant. The original motivation for the current paper was to try to find a simple model of an economy of the type just described. A pleasing aesthetic for such a model is the property of time homogeneity: it seems undesirable and unnatural to be able to deduce the date from the form of the structure of the risk premium (as a function of the market price). We therefore investigate economies which have this steady state property. However, it turns out that the only equilibria possible under the time homogeneity assumption either give a risk premium which increases with the market level or else a trivially constant one.

The structure of the chapter is as follows. After this introductory section, we describe the Burgers' equation characterisation of the evolution of the price of risk and some of its key properties. Section three then presents a new and much simpler derivation of the result than has previously appeared. In section four we add the requirement of time homogeneity. The conclusion of this section is that the only three possible steady state alternatives are:

1. that the price of risk becomes completely flat and (boringly) constant as a function of market level;

2. that it increases with market level; or

3. that its relationship contains a singularity of sufficient severity to permit arbitrage.

In a short fifth section this analysis is discussed in the context of work on asymptotic portfolio selection carried out in the mid seventies. It is shown that the two perspectives are consistent with one another. The final section summarises the results contained in the chapter and draws together their various implications.

2 Burgers' Equation and its Properties

We begin by summarising the results of Hodges and Carverhill (1993).

The analysis is of an economy with a single asset whose price S_t follows the process

$$\frac{dS}{S} = [r + \sigma\alpha(.)]dt + \sigma dB^P \tag{1}$$

where r is a constant risk free interest rate, σ is the volatility of the asset which we assume constant, B^P is Brownian Motion under the objective probability measure P, and $\alpha(.)$ is the (adapted) process for the price of risk.

For simplicity we shall also assume that no dividends are paid within the time horizon H. The question posed is how must the price of risk $\alpha(\omega, t)$ behave if we require this economy to correspond to an equilibrium characterised by individuals maximising the expected utility of their wealth at dates greater than or equal to H. This question has also been studied by Bick (1990), and by He and Leland (1993) whose results are very closely related to, and slightly more general than, those described here. Our assumption of the constancy of the variance rate enables us to obtain stronger results.

The kind of equilibrium which we are concerned with here is that characterised by a single representative investor. We assume agents maximise expected utility over utility functions which are not state dependent (and that their initial endowments are not state dependent either). Their demands aggregate to the demands of a single representative agent who holds the market portfolio at all times (see, for example, Huang and Litzenberger (1988)). It is worth noting how this assumption differs from the currently more usual and rather weaker assumption of no-arbitrage. Following Harrison and Kreps (1979), we note that any arbitrage-free price system can be sustained as a competitive equilibrium characterised by a single representative agent. In general, this agent may have a state dependent utility function or possibly just initial endowments which are state dependent. However, in our equilibrium assumption we are ruling out the possibility of this kind of state dependence.

Further, for any economy which does not permit arbitrage, there exists a risk neutral probability measure under which the rate of drift of all assets is the risk free rate, r. We shall define P as the objective probability measure, and Q as the risk neutral one. Thus the process for S under the measure Q

is:

$$\frac{dS}{S} = rdt + \sigma dB^Q \tag{2}$$

ratio of the risk neutral density to the objective one (i.e., the Radon-Nikodym derivative dQ/dP) gives the state-price density which also defines the marginal utilities of this agent. Later we will use Girsanov's Theorem to understand the nature of this change of measure.

It is convenient first of all to introduce a change of variables. Instead of working with S_t we shall work with the transformed variable

$$x_t = \ln S_t - \left(r - \frac{1}{2}\sigma^2\right)t \tag{3}$$

We find that the process for x is

$$dx = \alpha(.)\sigma dt + \sigma dB^P$$

under P, and

$$dx = \sigma dB^Q \tag{4}$$

under Q. The key result (shown by Hodges and Carverhill (1993)) is that given the assumptions we have made, $\alpha = \alpha(x,t)$ is a deterministic function of x and time (so it is path independent) which satisfies a non-linear partial differential equation known as Burgers' equation. The equation is

$$\alpha_\tau = \frac{1}{2}\sigma^2\alpha_{xx} + \sigma\alpha\alpha_x \tag{5}$$

where $\tau = H - t$ and the subscripts denote partial derivatives.

Note that this equation is very similar to the diffusion equation, but the final term makes it non-linear. As with the diffusion equation, given suitable boundary conditions (e.g. α is a specified function of x at the horizon date H and there are regularity conditions for extreme values of x) we can solve backwards in time from the horizon date to all earlier dates. Equation (5) was proposed by Burgers in 1948 as a model for the one dimensional flow of a viscous fluid. It also occurs in a number of other applications including modelling traffic flows. Closed form solutions are possible depending on the boundary conditions imposed. Properties of this equation and solution methods may be found in Bland (1988), Kevorkian (1990), and Whitham (1974).

In 1950 and 1951, Hopf and Cole showed independently how an analytic solution to Burgers' equation could be derived using a clever transformation which reduces the problem to a conventional diffusion equation. For this transformation we define v by setting

$$\alpha = \sigma\frac{v_x}{v} = \sigma\frac{\partial}{\partial x}[\ln(v(x,t))] \tag{6}$$

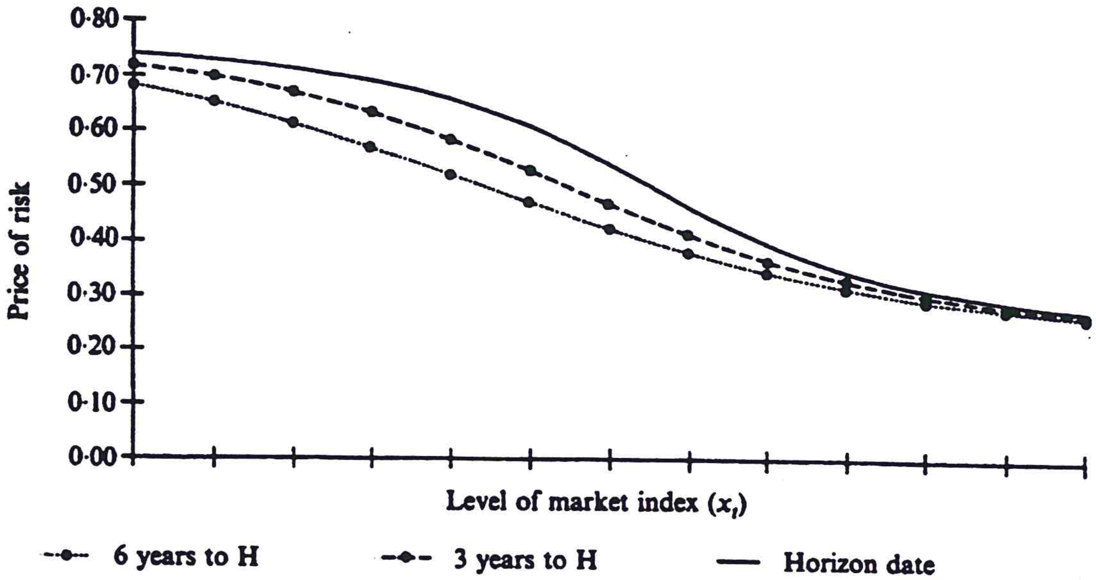

Figure 1. Solution of Burgers' equation

Cole and Hopf showed that if $v(x,\tau)$ satisfies the diffusion equation

$$v_\tau = \frac{1}{2}\sigma^2 v_{xx} \tag{7}$$

then the resulting $\alpha(x,\tau)$ from equation (6) solves Burgers' equation (5). Since, from (6), a terminal date boundary condition on α defines $v(x,0)$ up to a multiplicative constant, we can then solve (7) for $v(x,\tau)$ and finally use (6) to obtain $\alpha(x,\tau)$.

As an example, the case where $\alpha(x,0) = a - bx$ leads to the solution $\alpha(x,\tau) = (a - bx)/(1 + b\tau\sigma)$, which, for positive b, leads to an increasingly flat relationship as increases. Figure 1, which is reproduced from Hodges and Carverhill, shows the results of numerical calculations starting from more complicated S-shaped initial conditions. The initial condition for α is shown with a solid line, the dashed line shows α three years earlier, and the dotted one three years before that. Note that the direction in which α moves depends on the slope of α as a function of x. In this example the slope of the risk premium curve is generally smaller the further we are from the horizon date.

3 A Simple Derivation

In Hodges and Carverhill (1993), two derivations are given of the Burgers' equation result. Both approaches directly make use of Dybvig's (1988a,b) insights that wealth must be monotonic decreasing in the state-price density. The first approach involves taking limits of a binomial approximation, while the second is based on algebraic manipulation of the Girsanov change of measure. Neither approach is very simple, and the intuition provided is limited. The derivations in He and Leland (1993) suffer from similar drawbacks. We therefore now provide an alternative derivation which we hope provides better clarity and insight into what is going on. We will make use of both the Girsanov change of measure and of the Hopf and Cole transformation.

We start by assuming the general model (1). We note by Girsanov's Theorem that the change of measure corresponding to

$$\frac{dP}{dQ} = \exp\left\{\int_0^t \alpha_s dB_s^Q - \frac{1}{2}\int_0^t \alpha_s^2 ds\right\} = M(B_t^Q, t) \tag{8}$$

gives a Q-martingale M which satisfies the diffusion equation

$$\frac{\partial M}{\partial t} + \frac{1}{2}\frac{\partial^2 M}{\partial B^2} = 0$$

or, changing to a function of x and using equation (4)

$$\frac{\partial M}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 M}{\partial x^2} \tag{9}$$

The economics tells us that the marginal utility (with respect to log wealth) is proportional to the state-price density function dQ/dP. It is therefore inversely proportional to $M(x,t)$ (i.e. $U_x = k/M$ for some positive constant k). Portfolio theory tells us that the market equilibrium condition for holding the market portfolio is that the price of risk α must equate to σ times the Arrow–Pratt coefficient of relative risk aversion[1]

$$\alpha = -\sigma\frac{U_{xx}}{U_x} = -\sigma\frac{d}{dx}[\ln U_x]$$

Substituting for U_x we finally obtain

$$\alpha = -\sigma\frac{d}{dx}[\ln(\frac{1}{M})] = \sigma\frac{M_x}{M} \tag{10}$$

This completes the proof, as we now have shown that $\alpha(x,\tau)$ takes the form of the Cole and Hopf transformation[2]. Equation (10) corresponds to our earlier equation (6), and the diffusion equation (9) corresponds to the earlier (7).

Notice that the v variable, of the Cole and Hopf ratio which gives α, corresponds to the reciprocal of marginal utility (with respect to log wealth). This follows a diffusion, but what we are interested in is the evolution of the measure of (relative) risk aversion given by $-U_{xx}/U_x$.

[1]Note that the risk aversion $-SU_{SS}/U_S = -U_{xx}/U_x$ where x is as defined in equation (3).

[2]He and Leland (page 610) also identify $1/U_x$ as following a diffusion and describe the Cole and Hopf transformation for Burgers' equation.

4 The Time Homogeneous Case

In this section we describe our analysis of the time homogeneous case. We wish to find how the risk premium (may depend on the level of the market in such a way that the form of this function does not depend on time. This restriction enables us to reduce the partial differential equation (Burgers' equation) to an ordinary differential equation. We solve this differential equation and consider which solutions are consistent with market equilibrium. Not all solutions to Burgers' equation will be feasible, as this is a necessary condition for equilibrium and not a sufficient one. We conclude that there are only three possible alternatives in the steady state: either the price of risk becomes completely flat and (boringly) constant as a function of market level, or that it increases with market level or its relationship contains a singularity of sufficient severity to permit arbitrage.

We begin with a price of risk function $\alpha(x, \tau)$ which satisfies Burgers' equation (5). We interpret our assumption of homogeneity in time as meaning that α takes the form

$$\alpha(x, \tau) = y(u) \qquad \text{where} u = x + \theta\tau \tag{11}$$

for some function y and constant θ. Forming the partial derivatives and substituting into Burgers' equation we obtain the ordinary differential equation

$$\theta y' = \frac{1}{2}\sigma^2 y'' + \sigma y y' \tag{12}$$

Integrating this equation once gives

$$\frac{1}{2}\sigma^2 y' = \theta y - \frac{1}{2}\sigma y^2 + \text{ constant} \tag{13}$$

We may therefore obtain the solutions y by integrating:

$$\sigma \int \frac{dy}{a + (2\theta/\sigma)y - y^2} = \int du \tag{14}$$

where α is a constant. This integrates to give

$$\frac{\sigma}{2c} \ln \left[\left| \frac{y + c - b}{y - c - b} \right| \right] = u + \quad \text{constant} \tag{15a}$$

$$\text{where} \quad b = \theta/\sigma \text{ and } c = \sqrt{a + b^2} \tag{15b}$$

Finally, provided c is real and not complex, we obtain a general solution of the form

$$y = \begin{cases} \dfrac{(c+b)e^{s(u)} + b - c}{e^{s(u)} + 1} & \text{for } y \in (b - c, b + c) \\ \dfrac{(c+b)e^{s(u)} + c - b}{e^{s(u)} - 1} & \text{for } y \notin (b - c, b + c) \end{cases} \tag{16a}$$

$$\text{where} \quad s(u) = k + \frac{2c}{\sigma} u \tag{16b}$$

and k is a constant. The first equation provides a stable travelling wave solution with y between $b-c$ and $b+c$. We require $b>0$ for a positive risk premium. For positive c the solution increases from b_c at $-\infty$ to $b+c$ at $+\infty$. If we take c to be negative it still increases, from $b+c$ to $b-c$. Turning to the second equation, we may easily rule out as having no economic interpretation the other possible mathematical solutions which arise where c is complex. These involve trigonometric functions (instead of hyperbolic ones), and imply that the price of risk as a function of the market level has repeated singularities and oscillates an infinite number of times. No utility function will support these kinds of behaviour.

We now distinguish special cases of equation (16). We first note that the risk premium y has an unacceptable singularity at $u=-\frac{1}{2}k\sigma/c$, except in the case where $c=0$. For $c=0$ but $k\neq 0$ we obtain

$$y=b=\frac{\theta}{\sigma} \tag{17}$$

This, of course, is the trivial case of a constant risk premium. It corresponds to the well known assumption of a representative investor with constant relative risk aversion, or, in other words, either power or logarithmic utility (see, for example, Bick (1987)). However, there is one further case which we need to examine, corresponding to $k=0$ and taking the limit as c tends to 0. This gives a solution

$$y=\frac{\theta}{\sigma}+\frac{\sigma}{u}$$

which implies

$$\alpha(x,\tau)=\frac{\theta}{\sigma}+\frac{\sigma}{x+\theta\tau} \tag{18}$$

Unfortunately, although this solution does indeed satisfy Burgers' equation (as can easily be verified), it also has a singularity, namely, at $x=0$ in the case where $\theta=0$. The effect of the singularity is that as x falls close to this point, the drift increases sufficiently to prevent the point of the singularity from ever being reached. Sadly this model cannot represent an economic equilibrium because it admits arbitrage. This is a pity, because the variation of the risk premium, which increases as the market falls, would otherwise have made an interesting model. The hyperbolic form of the drift (under the objective probability measure P) means that the process for x is one which is well known to probability theorists and is called a 3–dimensional Bessel process. Many of its properties are given in Revuz and Yor (1991). The variable x corresponds to the distance of a point P from the origin, where P follows a random walk in 3 dimensions with no drift. Many analytical properties are known for this process, including that this process never reaches the origin. Hence the support for future values under the objective measure P is bounded below by the point of singularity, while that of the risk neutral measure Q is given by the whole of the real line. The two probability measures cannot be equivalent, so there does not exist a risk neutral probability measure. It is therefore clear that this otherwise appealing model admits arbitrage and so cannot correspond to an equilibrium.

We are left with the surprising conclusion that the only possible time homogeneous representative economy supporting a constant variance rate corresponds to an increasing or a constant risk premium. The only way a negative slope in the risk premium function can persist through time is through the existence of a singularity of sufficient severity to allow arbitrage.

5 Relationship to Asymptotic Portfolio Theory

It is interesting to reinterpret the results of the last section in the context of earlier work on asymptotic portfolio theory. Hakansson (1974) showed that in a terminal utility model, under really quite weak assumptions, there are strong convergence results which govern the evolution of the induced utility functions for earlier periods. An abbreviated account of this work is also given in Hakansson (1987). Hakansson finds that, under very general assumptions which are distinct from ours, there is always some coefficient of relative risk aversion such that the induced utility functions converge to the corresponding power utility function as one backs away from the horizon date. Thus reinvesting individuals with distant horizons should follow a power utility investment policy as long as their horizon remains distant. (But, in practice the convergence is slow, so distant may mean very distant!) In the travelling wave solution for our economy, as one backs further and further away from the horizon, one becomes less and less likely to be in the transition region and increasingly likely to be close to either y's lower bound or its upper bound.

Conclusions

This chapter has provided further analysis of the behaviour of the risk premium on the market portfolio of risky assets. Understanding the behaviour of the market risk premium is a problem of fundamental importance for fields such as portfolio management. Earlier work by Hodges and Carverhill (1993) and by others has characterised the evolution of the market risk premium in economies where the variance of the return on the market has constant variance and market index options can be priced using the 1973 Black–Scholes model. Representative agent equilibrium implies that, although the risk premium can vary, it must evolve according to a non linear partial differential equation called Burgers' equation.

We have provided some significant new insights into this analysis. First, we have summarised the existing results and provided a much simpler and more intuitive derivation. Next, we have analysed the time homogeneous case. It would have been nice if we had found a time homogeneous economy which allows the risk premium to vary so that some kind of mean reversion could take place. Sadly, this is impossible. We obtained the interesting but negative result that the risk premium must be increasing in the market level or constant for time homogeneous equilibria which rule out arbitrage. However, this result is one which is consistent with earlier work by Hakansson on asymptotic portfolio selection.

The assumptions we have made appear to be strong but some relaxation would be possible, particularly along the lines of He and Leland (1993) who allow both the risk free rate and volatility to vary. However, stochastic variation of these variables does make the mathematics horribly complex. Burgers' equation holds whether or not there are intermediate dividends, and there should be no real difficulty in extending the formal analysis to include more general situations. In a model with many dividend dates, the risk premium for the claim which pays on a given date will satisfy Burgers' equation subject to a boundary condition imposed at that date. The market portfolio now becomes a portfolio of claims which pay at different dates, and its risk premium is a weighted average of the risk premia for the different dates. For any given payment, the risk premium must behave according to the results presented above. However, for the portfolio, there could be a non-trivial steady state risk premium function formed from the weighted averages. It seems unlikely that such a model could be solved analytically, but numerical solution would not be difficult.

The chapter showed how easy it is to obtain a models which admit arbitrage, such as our Bessel process one. It is surprisingly easy to produce and perform partial analysis of such models without appreciating their inherent contradictions. We wish to reiterate a strong caveat in this regard. The authors are aware of published papers where arbitrage exists in a model: for example a recent paper on valuing options on spreads where the underlying asset price has a process with a reflecting barrier under the (assumed) risk neutral probabilities. We have also seen an option pricing model (derived from a no-arbitrage argument and with no market frictions) where the option prices fail to satisfy put-call parity!

A further implication of our work concerns the limitations of the representative investor paradigm. Under a single representative investor with decreasing relative risk aversion, the price of risk would be likely to become increasingly sensitive to the market level as the market evolves through time. It appears to us that in order to obtain more realistic models of market equilibrium behaviour, and ones which would enable us to study issues such as how to invest within a lifetime cycle, we may need to employ the much richer but more difficult techniques available of overlapping generations models.

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