Monte Carlo Valuation of Interest Rate Derivatives Under Stochastic Volatility

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Abstract

This paper describes flexible and efficient Monte Carlo techniques for the valuation of interest rate derivatives in a world with stochastic interest rates and interest rate volatility. The particular model we study is due to Fong and Vasicek (1992) but the techniques are generally applicable. We extend the model of Fong and Vasicek [1992] allowing the valuation of a wide range of interest rate derivatives within their framework. We also extend the control variate technique described by Clewlow and Carverhill [1994] to efficiently value options on a wide range of derivatives, including bond options, caps, floors, collars, and swaptions. We analyse the benefits of different control variates and the effect of different numbers of time steps and simulations.

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1 Introduction

Recently, both Longstaff and Schwartz [1992] (LS) and Fong and Vasicek [1992] (FV) have developed two-factor stochastic volatility models of the term structure where the two factors are the short rate and the variance of the short rate. The motivation for these models has been the recognition that the assumption of perfect correlation between rates implicit in one-factor models is too restrictive and that the volatility of interest rates changes randomly over time and is correlated with the level of interest rates. The LS model is developed in a general equilibrium framework with the processes for the interest rate and the volatility of the interest rate being endogenously determined. They are able to derive closed-form solutions to the prices of discount bonds and also to price discount bond options 'analytically' within their framework[1].

The model of FV is perhaps more intuitive. They start by assuming plausible stochastic processes for the short rate and short rate volatility. However, in their

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1 The solution is analytical as in the sense of Black and Scholes [1973], and has the same form. LS themselves admit that the solution to the resulting bivariate cumulative distribution functions can be slow enough to warrant solving the original fundamental partial differential equation. See footnote 15 page 1271 in the LS paper.

article, FV only describe pricing discount bonds and the solution they present requires complex (as opposed to real) algebra, posing potential problems for practical implementation. In a recent paper, Selby and Strickland [1995] detail a series solution for the bond price that can easily and computationally efficiently be implemented in a programming language or spreadsheet, avoiding the need to deal with complex numbers.

The contribution of this paper to this literature is twofold. Firstly, we extend the work of Fong and Vasicek and show how to price a wide variety of interest rate derivatives within their framework. We begin by showing how to price discount bond options, and then extend our methodology to price coupon bond options and swaptions, as well as interest rate caps, floors and collars. Captions and floortions can also be handled easily within our framework.

The methodology we propose involves efficient simulation techniques. Monte Carlo simulation has long been an important numerical tool for complex option valuation problems. The technique has been used extensively in the literature to obtain prices for instruments where analytical solutions are not possible, or are only possible in an idealised world[2]. Our second contribution in this paper, therefore, is to provide practitioners and academics with an efficient tool to value interest rate derivatives. Although we concentrate here on the FV model, our methodology is generally applicable.

The plan of this paper is as follows. In section 2 we outline both the FV model and the series solution of Selby and Strickland [1995]. In section 3 we begin by presenting a simple simulation technique for pricing discount bond options and extend

2 See for example Johnson and Shanno [1987], Hull and White [1987] and, more recently Clewlow and Carverhill [1994].

it to develop efficient techniques by using delta and vega based control variates. We report the relative benefits of different control variates and the number of time steps and simulations. Section 4 extends our analysis to pricing a wide range of other interest rate derivatives. Finally, section 5 contains our summary and conclusions.

2 The Fong and Vasicek Stochastic Volatility Interest Rate Model

FV's two factor model of the term structure explicitly recognises interest rate volatility as a stochastic factor, and derives the pure discount bond price function under the equilibrium condition of no arbitrage. The diffusion processes for the two sources of uncertainty, the short term interest rate r, and the instantaneous variance of the short rate v, are given as:

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz_1 \qquad (1)$$

$$dv = \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2 \qquad (2)$$

$\bar{r}$ and $\bar{v}$ are the long-term mean of the short rate and the long-term mean of the variance of the short rate respectively. The short rate has instantaneous variance v with the random component of the volatility process having a variance proportional to the current level of the variance. Both processes tend to revert to a long term mean value, with the strength of the reversion being proportional to the variables current deviation from its mean. The two Wiener processes, their increments represented by dz_1 and dz_2, are assumed to be correlated with a coefficient of ρ.

Let the price at time t of a pure discount bond which matures with value 1 at time s be given under this analysis by $P(t,s,r,v)$, where r and v are the level of the short rate and it's variance respectively at time t. Under the equilibrium condition of no arbitrage the partial differential equation governing the price of a pure discount bond has a solution of the form

$$P(t,s,r,v) = \exp(-rD(\tau) + vF(\tau) + G(\tau)) \qquad (3)$$

where $\tau = s - t$ is the maturity of the bond. The functions of time D, F, and G are solutions to ordinary differential equations to which the partial differential equation governing the price of the pure discount bond reduces. The duration measure D is the same as in Vasicek [1977], i.e.,

$$D(\tau) = \frac{1}{\alpha}\left(1 - e^{-\alpha\tau}\right) \qquad (4)$$

In order to use the formula, the difficult part is the calculation of the time-dependent functions F and G. As proposed by the authors, the solutions to these functions require computation of the confluent hypergeometric function, involving complex (as opposed to real) algebra. One possibility for implementation then is to make use of a computer algebra package such as *Mathematica* or *Maple*. However, it is not easy to incorporate such packages into the usual investment house trading systems.

Selby and Strickland [1995] propose a series solution which avoids the need to directly evaluate the confluent hypergeometric function, and which can easily be implemented in a programming language or spreadsheet. As a guide to the computational efficiency of their solution, the authors are able to calculate a term structure out to 20 years with bi-weekly intervals (i.e. 520 bond prices) in less than a second, on a 486 notebook running at 100 MHz.

FV don't discuss the issue of pricing interest rate derivatives in their framework although they derive the fundamental partial differential equation that must be satisfied by any interest rate contingent claim, ϕ, that depends on r, v, and time;

$$\phi_t + (\alpha\bar{r} - \alpha r + \lambda v)\phi_r + (\gamma\bar{v} - (v + \xi\eta)v)\phi_v + \frac{1}{2}v\phi_{rr} + \rho\xi v\phi_{vv} - r\phi = 0 \qquad (5)$$

The constants λ and η determine the market prices of risk of r and v respectively.

For all cases where the maturity boundary condition is different from 1 (which holds for pure discount bond prices) we have to solve this partial differential equation numerically. One way of doing this would be to implement a two-dimensional finite difference technique. However, this technique can be slow and difficult to implement. We propose a numerical solution technique that is both easy to implement and is computationally efficient. The technique that we propose is an application of Monte Carlo simulation using hedge based control variates described by Clewlow and Carverhill [1994].

3 Monte Carlo Valuation of Discount Bond Options

We begin this section by describing a general framework for solving discount bond options under the FV model, and then describe how to increase the computational efficiency of the technique. We concentrate in this section on the price at time t of an option, maturing at time T, on a pure discount bond that matures with value 1, with certainty, at time s $(t \leq T \leq s)$. The value of this option we represent by $c(t,T,s)$ for a call and $p(t,T,s)$ for a put

Consider a security that pays off an amount $\varphi(T)$ at maturity, T. Arbitrage pricing theory tells us that the price today, $\varphi(t)$, is the risk-neutral expectation of it's discounted payoff;

$$\varphi(t) = \hat{E}_t\left[\varphi(T)\exp\left(-\int_t^T r(\tau)d\tau\right)\right] \tag{6}$$

Monte Carlo simulation provides us with a means to evaluate this expectation. First introduced by Boyle [1977] the technique involves simulating the risk neutral paths of the state variables and the payoff on the option for each path. To implement the technique we 'discretise' the stochastic differential equations in (1) and (2). The first step is to partition the life of the option into, say, n time steps $\{t = t_0 < t_1 < \ldots < t_n = T\}$ where $t_i = t + i\Delta t$ and the length of each time step is given by;

$$\Delta t = \frac{(T-t)}{n}$$

The discretised versions of (1) and (2) are given by the following;

$$\Delta r = \alpha(\bar{r} - r)\Delta t + \sqrt{v}\Delta z_1 \tag{7}$$

$$\Delta v = \gamma(\bar{v} - v)\Delta t + \xi\sqrt{v}\Delta z_2 \tag{8}$$

where $\Delta z_k = \sqrt{\Delta t}\varepsilon_k$ for $k = 1, 2$ and $(\varepsilon_1, \varepsilon_2)$ are random samples from a standard bivariate normal distribution with correlation ρ. For the FV model we need only simulate the state variables out until the end of the life of the option as we have 'closed-form' solutions for the discount bond price. Starting from time t we simulate r and v according to (7) and (8) until T at which point we evaluate the maturity value of

the option. Each simulation will generate a time series of pairs $\{r_0, v_0\}$, $\{r_1, v_1\}$, ...,
$\{r_n, v_n\}$ representing the level of the short rate and it's variance at each point in time.

For European discount bond options the maturity value is given by:

$$\varphi(T) = \begin{cases} \max\{0, P(T,s,r_n,v_n) - K\} & \text{for a call} \\ \max\{0, K - P(T,s,r_n,v_n)\} & \text{for a put} \end{cases} \tag{9}$$

where the exercise price of the option is given by K. The price of the pure discount bond maturing at time s, $P(T, s, r_n, v_n)$, is evaluated according to the series solution given in Selby and Strickland. The maturity value is then discounted back to today using the realised time series for r under that simulation:

$$\varphi_j(t) = \exp\left(-\sum_{i=0}^{n-1} r_i \Delta t\right)\varphi(T) \tag{10}$$

The subscript j denotes that this is discounted payoff claim from the jth simulation. This procedure is repeated many, say M, times with the resulting Monte Carlo estimate being given by the mean of the simulated discounted payoffs

$$\varphi(t) = \frac{1}{M}\sum_{j=1}^{M} \varphi_j(t) \tag{11}$$

This is the simple Monte Carlo estimation. However, in it's simple form Monte Carlo simulation is very inefficient, with typically 100,000's of simulations having to be performed to achieve a reasonable standard error in the estimate. In order to provide a technique for real-time practical implementation we now propose an extension of the recent work of Clewlow and Carverhill [1994] who outline a variance reduction technique for complex or exotic options.

This variance reduction technique involves generating simultaneously M independent random samples for the option price $\{\varphi_1, \varphi_2, \ldots, \varphi_M\}$ and control variates $\{x_1, x_2, \ldots, x_M\}$ where each x_j is a m-vector whose components have zero mean. The control variates are designed to have zero mean and to be negatively correlated with the option payoffs. The variance reduced Monte Carlo estimate is obtained as the y-intercept after regressing the simulated payoffs against the simulated control variates. The technique is closely related to the idea of hedging. The control variates play the role of the hedge, their changes being negatively correlated with the changes in the option price. The variance of the payoff of a hedged option position is smaller than that of an unhedged payoff.

For practical implementation, the control variates should be chosen to capture the variation in the value of the option created by random changes in the underlying factors. The functions that do this exactly are the partial derivatives of the option with respect to the underlying sources of uncertainty. The main state variable in the FV model is the short rate, We therefore choose a 'delta-based' (sensitivity to r) control variate to be given by the following

$$x_j^1 = \sum_{i=0}^{n-1} \frac{\partial \varphi(t_i, T, s)}{\partial r} (\Delta r_{t_i} - E[\Delta r_{t_i}]) \qquad (12)$$

where $(\Delta r_{t_i} - E[\Delta r_{t_i}])$ denotes the difference from the change in the short rate at time t_i and it's expected change and ensures that the control variate has zero mean. The subscript j denotes that we evaluate an x^1 for each simulation (j = 1, ..., M). We do not know explicitly what the delta is for an option in a Fong and Vasicek world, but we do know what it is in a similar world, that of Vasicek [1977][3]. In this single factor

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3 In order to efficiently implement the control variate technique, we require the partial derivatives of the option value to have closed form solutions. The control variate hedge based on the Vasicek [1977] delta will not be as good as one based on the correct Fong and Vasicek delta but if the volatility of the short rate volatility is not

model of the term structure Vasicek assumes that the short rate follows an Ornstein Uhlenbeck process;

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz \tag{13}$$

He derives prices of discount bonds under these assumptions to be the following;

$$P(t,s,r) = \exp\left[\tfrac{1}{\alpha}\left(1-e^{-\alpha(s-t)}\right)\left(R_\infty - r\right) - (s-t)R_\infty - \frac{v}{4\alpha^3}\left(1-e^{-\alpha(s-t)}\right)^2\right] \tag{14}$$

where $R_\infty = \gamma - \dfrac{1}{2}\dfrac{v}{\alpha^2}$

Using the same process Jamshidian [1989] derives prices of European options on discount bonds to be the following;

$$c(t,T,s) = P(t,s,r)N(h) - KP(t,T,r)N(h-\sigma_P) \tag{15}$$

$$p(t,T,s) = KP(t,T,r)N(-(h-\sigma_P)) - P(t,s,r)N(-h) \tag{16}$$

where

$$h = \frac{\ln\left(\dfrac{P(t,s,r)}{KP(t,T,r)}\right)}{\sigma_p} + \frac{\sigma_p}{2}, \text{ and}$$

$$\sigma_p = \sqrt{\frac{v(1-e^{-2\alpha(T-t)})}{2\alpha}\frac{(1-e^{-\alpha(s-T)})}{\alpha}}$$

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too high it should be almost as good. A slight improvement could be obtained by using the Hull and White extended Vasicek model, allowing the Vasicek model to fit the Fong and Vasicek term structure. But this would have entailed computing Fong and Vasicek pure discount bond prices at every time step with a significant increase in computational effort.

We can therefore use the following as a control variate[4]

$$x_j^1 = \begin{cases} \sum_{i=0}^{n-1} \frac{\partial c(t_i, T, s)}{\partial r} \sqrt{v_i \Delta t}\varepsilon_i & \text{for a call} \\ \sum_{i=0}^{n-1} \frac{\partial p(t_i, T, s)}{\partial r} \sqrt{v_i \Delta t}\varepsilon_i & \text{for a put} \end{cases} \tag{17}$$

here ε_i represents the sample for the standard normal for the increment in the short rate at the i'th time step. The idea is for each simulation to accumulate the control variate - at the end of each simulation we have a pair representing the option payoff and control variate, $\{\varphi_j(t), x_j^1\}$. The intercept of the best fit line through these points, with the y-axis where the control variate is zero, gives the mean payoff.

In order to illustrate the procedure we value a discount bond option using the FV model with the state variable characteristics given in table 1. Figure 1 shows the resulting FV yield curve based on these parameter values. The option we value is a 1 year at-the-money-forward call on a 5 year discount bond.

Figure 2 illustrates the least squares regression. Panels a, b, and c show plots of the pay off against the single, delta-based, control variate for monthly (n = 12), weekly (n = 52), and daily (n = 252) time increments (rebalancing of the hedge) respectively for 100 simulations. Perfect correlation between the control variate and the payoff to the option would cause all of these points to lie on a straight line. Obviously the results get better as we decrease the length of the time step (i.e. increase the number of time steps/year). This is the same as observing that the hedging of option positions improves as we increase the frequency of rehedging.

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4 See Appendix 1 for a proof and an explicit representation of the partials.

Table 2 and Table 3 display results detailing standard deviations for the estimate of the options value and run times taken for both the simple Monte Carlo method and the delta based control variate method respectively[5]. For both sets of simulations we show results for varying numbers of simulations ($M = 50$, 100, 200, 500, and 1000) and for varying numbers of time steps ($n = 52$, 85, 125, and 250).

The results confirm our observation from Figure 2, that the variance is reduced as we increase the number of steps per year. For 1000 simulations the variance reduction ranges from 8 times with monthly time steps to 223 times with daily time steps. To obtain the same degree of accuracy with daily time steps, using only the simple method, would require 223,000 simulations. Obviously, having to accumulate the control variate takes time; for 250 steps/year and 1000 simulations this is a factor of 4 greater than the simple method. Taking into account the time taken our variance reduced Monte Carlo simulation is therefore 56 times more efficient than simple Monte Carlo.

As a further illustration of the effectiveness of the control variate technique Figure 3 shows the convergence of the simple Monte Carlo and control variate estimates for increasing numbers of simulations. The 'true' value, "FV Value" is the value based on 10,000 simulations.

Using the delta based control variate enables us to obtain significant variance reduction. However, the presence of stochastic volatility implies that the hedge ratio (which is based on the partial derivative with respect to a formula with constant volatility) is not exactly correct. If we were hedging this option we would also want to be hedged against volatility risk. In order to reduce the variance of our estimate

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5 All run times throughout this section are applicable to a 386 Notebook running at 75 MHz.

even further therefore, we add a 'vega-based' (sensitivity to v) control variate, x^2, to be given by the following;

$$x_j^2 = \sum_{i=0}^{n-1} \frac{\partial \varphi(t_i, T, s)}{\partial v} (\Delta v_{t_i} - E[\Delta v_{t_i}]) \tag{18}$$

where the expression in brackets denotes the difference from the change in the variance of the short rate at time t_i and it's expected change. The partial derivative is again taken with respect to the Jamshidian bond option pricing formula[6]. The simulation of each price path will now be more time consuming - we have to calculate and accumulate an extra derivative - but should be small compared with the reduction in variance.

Table 4 presents sample standard deviations and run times when we employ both delta and vega based control variates for pricing the discount bond option. Our results show that although the standard deviation is reduced over the single control variate estimate, the reduction in variance is small when we take into account the increase in computation time involved. For 250 time steps the time taken is 7 times greater than the simple method (compared with 4 times for the one control variate estimate). One of the reasons for only achieving a slight improvement is that we have chosen a relatively small value for the volatility of volatility although we believe that this is close to market values[7].

In the next section we describe how the same techniques can be extended to apply in a straightforward way to more complex and realistic interest rate derivative problems.

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6 See Appendix 2 for an explicit representation of the partials.

7 Results from more extensive sets of simulations show that this qualitative result is robust across varying parameter values.

4 Pricing Other Interest Rate Derivatives

Most traded interest rate derivatives are more complicated than discount bond options. We show in this section how to value other interest rate contingent claims by utilising our methodology. The methodology we propose is valid for any interest rate derivative which we can value as an option on a portfolio of discount bonds or, alternatively, a portfolio of discount bond options. Coupon bond options and swaptions, as well as interest rate caps, floors and collars are instruments which can be valued in this way.

Suppose we wish to value a T-maturity call option on a coupon bond which matures at time s with face value 1, and pays semi-annual coupon payments $\frac{c}{2}$ at times $\{s_k\}$ ($k = 1, ..., q$) where c is coupon of the bond[8]. We can think of a coupon bond as a portfolio of pure discount bonds whose maturities and face values match the timing and coupon payments of the bond. Let $B(t, \{s_k\}, s)$ represent the price of this coupon bond at time t.

As in section 3 we simulate the short rate and it's variance from the original time until the maturity of the option. At this time the coupon bond is worth;

$$B(T, \{s_k\}, s) = \sum_{s_k > T}^{s_{q-1}} P(T, s_k, r_n, v_n)\tfrac{c}{2} + P(T, s_q, r_n, v_n)(1 + \tfrac{c}{2}) \tag{20}$$

The payoff to a call option is then given by the maturity condition;

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8 The Jamshidian [1989] "trick" of decomposing an option on a coupon bond into a portfolio of options on discount bonds with adjusted strike prices only works for single factor models, and so cannot be used in this framework.

$$\varphi(T) = \max\{0, B(T, \{s_k\}, s) - K\} \qquad (21)$$

We then discount this payoff as in equation (10), with the resulting simple Monte Carlo estimate being given by equation (11). In order to use the control variate technique we need to choose a delta and a vega. In the one factor model of Vasicek an option on a coupon bond can be decomposed into a portfolio of options on pure discount bonds with adjusted strike prices (Jamshidian [1989]). The natural choices are therefore the delta's and vega's of these options. This implies that we need as many control variates as there are coupon payments after the maturity date of the option or to combine them into a single measure in some way. For the latter, to use the delta and vega-based control variate estimates we can apply the following control variates instead of (17) and (18).

$$x_j^1 = \sum_{i=1}^{n} \left(\sum_{s_k > t_i}^{s_{q-1}} \frac{c}{2} \frac{\partial c(t_i, T, s_k)}{\partial r} + \left(1 + \frac{c}{2}\right) \frac{\partial c(t_i, T, s)}{\partial r} \right) \sqrt{v_i \Delta t} \varepsilon_i \qquad (22)$$

$$x_j^2 = \sum_{i=1}^{n} \left(\sum_{s_k > t_i}^{s_{q-1}} \frac{c}{2} \frac{\partial c(t_i, T, s_k)}{\partial v} + \left(1 + \frac{c}{2}\right) \frac{\partial c(t_i, T, s)}{\partial v} \right) \sqrt{v_i \Delta t} \varepsilon_i \qquad (23)$$

However, all of the individual delta's (and vega's) within the inner summations are highly correlated and so we can just use one element of the summations, say that connected with the last payment;[9]

$$x_j^1 = \sum_{i=0}^{n-1} \frac{\partial c(t_i, T, s)}{\partial r} \sqrt{v_i \Delta t} \varepsilon_i \qquad (24)$$

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9 We can drop the constant $\left(1 + \frac{c}{2}\right)$ factor because this is accounted for by the least squares regression.

$$x_j^2 = \sum_{i=0}^{n-1} \frac{\partial c(t_i, T, s)}{\partial v} \sqrt{v_i \Delta t} \varepsilon_i \tag{25}$$

Because of the form of the control variates the computation time is similar to pricing discount bond options. Therefore, even though we are unable to use the Jamshidian decomposition, we can value coupon bond options as fast as discount bond options with carefully chosen control variates. Table 5 presents results for sample standard deviations and run times for the delta-based control variate for a one year option on a 10% coupon bond maturing in 4.7 years, where the strike price is the forward price of the bond. The price of the option is 2.62% of the face value[10].

One of the most popular interest rate derivatives is a swaption - an option on an interest rate swap. An interest rate swap can be regarded as an option to exchange a fixed rate (i.e. coupon) bond for a floating rate bond. At the start of the life of a swap, the value of the floating rate bond equals the principal amount of the swap. A swaption can therefore be regarded as an option to exchange a coupon bond for the principal amount of the swap. i.e. its value is the same as an option on a coupon bond with the strike of the option set equal to the principal of the swap. If the swaption gives the holder the right to pay fixed and receive floating, it is a put on the fixed rate bond. If the swaption gives the holder the right to pay floating and receive fixed, it is a call on the fixed rate bond. We can therefore use the preceding analysis for these instruments as well.

Coupon bond options and swaptions are examples of instruments that can be valued as an option on a portfolio of discount bonds. We now analyse instruments which can be valued as a portfolio of discount bond options. Consider an interest rate cap that, on a periodic basis, caps the interest rate on \$1 at the rate r_C until T_{cap}, the end of the life of the

10 We only report results for the delta-based control variate as the extra variance reduction gained by using the vega-based control variate is again only marginal.

cap. Let $\tau_k = t + k\Delta\tau$ ($k = 1, \ldots, u$ say) be the cap reset dates where $\Delta\tau$ represents the fraction of a year between resets (say 6 months).

Consider further the 'caplet' that caps the interest rate between times τ_1 and τ_2. A number of authors have shown[11] that an instrument which caps the interest rate at r_C between τ_1 and τ_2 is equivalent to $(1 + r_c\Delta\tau)$ European put options with exercise price $X_c = \dfrac{1}{1 + r_c\Delta\tau}$ and expiration date τ_1 on a $1 face value discount bond maturing at time τ_2. More generally an interest rate cap is a portfolio of European puts on a series of discount bonds;

$$Value\ Cap = (1 + r_c\Delta t)\sum_{k=1}^{u-1} p\left(t, T = \tau_k, s = \tau_{k+1}\right) \qquad (26)$$

In order to use our technique to price caps, we simply sum the estimates of the individual options.

Table 6 presents results for sample standard deviations and run times for the delta-based control variate for a three year cap with semi-annual resets (i.e. 5 individual caplets) struck at 9%. We performed two sets of simulations. In the first (panel A) we calculate separately the delta of each of the individual option positions, as in section 3. In the second (panel B) we use the same delta for each option in the portfolio (based on the longest maturing option) saving us having to calculate all of the individual remaining deltas at each time step. Our results show that the variance is smaller when we calculate the delta for each individual caplet, as we would expect, but the computation time is significantly increased. For example, for 1000 simulations and 250 steps/year, the standard deviation obtained by calculating the deltas for each individual caplet is 0.00141 (the cap price is 18.11% of the principal underlying the cap) and the time taken is 342

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11 See for example Hull [1993]

seconds. When we set the deltas from all the caplets to be equal the corresponding figures are 0.00179 and 144 seconds.

Interest rate floors and collars can be defined and valued analogously. An interest rate floor agreement places a lower limit on the interest rate that will be charged. For a floor with the same characteristics as the above cap, its value can be interpreted as a series of call options on zero coupon bonds, each with strike price $X_F = \frac{1}{1 + r_F \Delta\tau}$, where r_F is the floor level.

$$Value\ Floor = (1 + r_F \Delta t) \sum_{k=1}^{u-1} c\left(t, T = \tau_k, s = \tau_{k+1}\right) \tag{27}$$

A collar is just a long position on a cap and a short position on a floor with the same characteristics of settlement dates and reset intervals. This implies that the price of a collar is equal to the difference between the price of the put series with strike price X_C, and the price of the call series with strike X_F.

5 Summary and Conclusions

We have presented a methodology for efficient pricing of a wide range of interest rate derivatives in a two-factor stochastic volatility interest rate model, namely that of FV. In this respect we have extended their work beyond their original article where they derive only results for the term structure. Our methodology utilises Martingale Variance Reduction techniques to vastly improve the speed of Monte Carlo simulation.

We have shown that we are able to provide a variance reduction of over 220 times over the simple Monte Carlo estimate for pricing discount bond options. Even taking into account the extra time needed to accumulate our control variates the variance reduction is over 50 times. Also, by carefully choosing the control variates we are able to price coupon bonds, and hence swaptions, as fast as pricing a pure discount bond option, even though we can't use the Jamshidian decomposition of a coupon bond option into portfolios of discount bond options in a two-factor world.

Our final numerical result shows that, again by a careful choice of the control variates, we can greatly reduce the time to obtain accurate estimates for an interest rate cap agreement.

Our analysis is not restricted to only this interest rate model. It can be utilised for any multi-factor model which yields a closed form solution for the pure discount bond price, for example the Longstaff-Schwartz model - although to apply the control variate technique we need a 'close' one-factor model with closed form solution to the option pricing problem. Finally, Clewlow and Strickland [1996] show how the technique can also be applied to the Heath, Jarrow and Morton [1992] model.

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Table 1 : Parameter Values for Fong and Vasicek Simulation

α	2.0000
γ	2.0000
ξ	0.0001
λ	0.2000
η	0.1000
ρ	0.6000
r	0.0800
$\bar{r}$	0.0950
v	0.0150
$\bar{v}$	0.0150

Table 2: **Sample Standard Errors and Run Times for the Simple Monte Carlo Method for a 1 Year ATM Option on a 5 Year Discount Bond**

Standard Errors (x1000)				
	Steps/Year			
Simulations	52	85	125	250
50	1.85955	1.66213	1.62726	1.58364
100	1.31260	1.31990	1.28960	1.15310
200	0.75307	0.73640	0.72676	0.71125
500	0.54399	0.52489	0.56899	0.53388
1000	0.40208	0.39067	0.38333	0.37293
Time (Seconds)				
	Steps/Year			
Simulations	52	85	125	250
50	1	1	1	1
100	1	1	1	2
200	1	1	1	3
500	1	2	3	7
1000	2	4	7	12

Table 3: **Sample Standard Errors and Run Times for the Delta Based Control Variate Method for a 1 Year ATM Option on a 5 Year Discount Bond**

Standard Errors (x1000)				
	Steps/Year			
Simulations	52	85	125	250
50	0.07085	0.04950	0.01853	0.01683
100	0.02720	0.01300	0.00960	0.00860
200	0.01739	0.00728	0.00523	0.00354
500	0.00671	0.00295	0.00206	0.00174
1000	0.00300	0.00152	0.00104	0.00079
Time (seconds)				
	Steps/Year			
Simulations	52	85	125	250
50	1	1	1	3
100	1	1	3	5
200	1	3	4	10
500	2	5	13	24
1000	3	11	25	48

Table 4: **Sample Standard Errors and Run Times for the 2 Control Variate Method (Delta and Vega Based) for a 1 Year ATM Option on a 5 Year Discount Bond**

Standard Errors (x1000)				
	Steps/Year			
Simulations	52	85	125	250
50	0.07524	0.05049	0.01952	0.01612
100	0.02630	0.01360	0.00860	0.00850
200	0.01704	0.00728	0.00523	0.00354
500	0.00662	0.00300	0.00206	0.00170
1000	0.00294	0.00152	0.00108	0.00079
Time (seconds)				
	Steps/Year			
Simulations	52	85	125	250
50	1	1	2	4
100	1	1	5	7
200	2	3	8	16
500	3	9	20	39
1000	6	17	41	78

Table 5: Sample Standard Errors and Run Times for the Delta Based Control Variate Method For a Coupon Bond Option

Standard Errors				
	Steps/Year			
Simulations	52	85	125	250
50	0.00917	0.00717	0.00619	0.00540
100	0.00414	0.00412	0.00545	0.00419
200	0.00272	0.00284	0.00211	0.00247
500	0.00112	0.00094	0.00105	0.00083
1000	0.00052	0.00044	0.00049	0.00049
Time (Seconds)				
	Steps/Year			
Simulations	52	85	125	250
50	1	2	2	4
100	2	4	4	8
200	5	7	9	15
500	13	17	22	37
1000	27	34	44	75

Note: The state variable parameters are given in table 1. The option is a 1 year at-the-money forward on a 10% semi-annual coupon bond maturing in 4.7 years[12].

12 Results which are available from the authors show that by calculating the control variates using equations (22) and (23) reduces the standard errors only marginally, but with a significant increase in computation times. For example for 250 steps per year and 1000 simulations the standard error is 0.00048 taking 465 seconds.

Table 6: **Sample Standard Errors and Run Times for the Delta Based Control Variate Method for Pricing a 9% 3 Year Cap with Semi-Annual Resets**

Panel A: **Taking Into Account The Individual Caplet Deltas**

Standard Errors

	Steps/Year			
Simulations	52	85	125	250
50	0.00088	0.00091	0.00091	0.00090
100	0.00044	0.00043	0.00043	0.00046
200	0.00024	0.00022	0.00022	0.00022
500	0.00009	0.00009	0.00009	0.00009
1000	0.00005	0.00005	0.00005	0.00005

Time (Seconds)

	Steps/Year			
Simulations	52	85	125	250
50	4	6	9	17
100	7	12	18	34
200	16	24	34	69
500	39	60	87	171
1000	78	120	174	342

Panel B: **Setting All the Individual Caplet Deltas To Be Equal**

Standard Errors

	Steps/Year			
Simulations	52	85	125	250
50	0.00112	0.00111	0.00116	0.00116
100	0.00056	0.00054	0.00056	0.00058
200	0.00029	0.00028	0.00029	0.00028
500	0.00011	0.00012	0.00012	0.00011
1000	0.00006	0.00006	0.00006	0.00006

Time (Seconds)

	Steps/Year			
Simulations	52	85	125	250
50	2	3	4	7
100	4	5	8	14
200	7	11	15	29
500	19	27	38	72
1000	36	55	75	144

Figure 1: **Fong and Vasicek Simulated Term Structure**

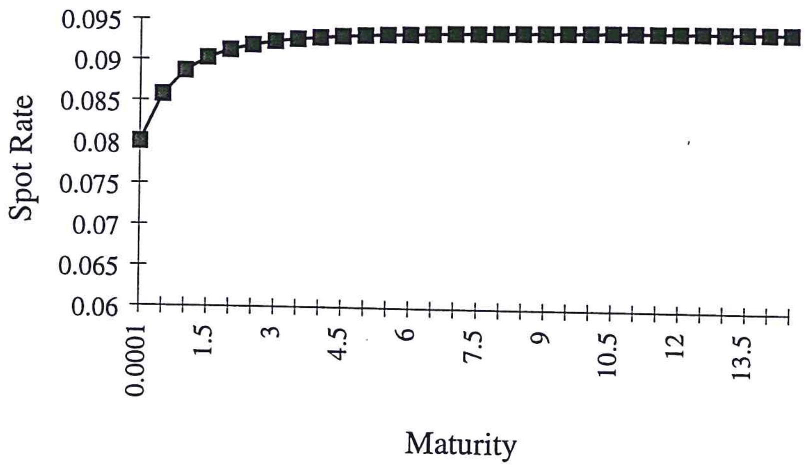
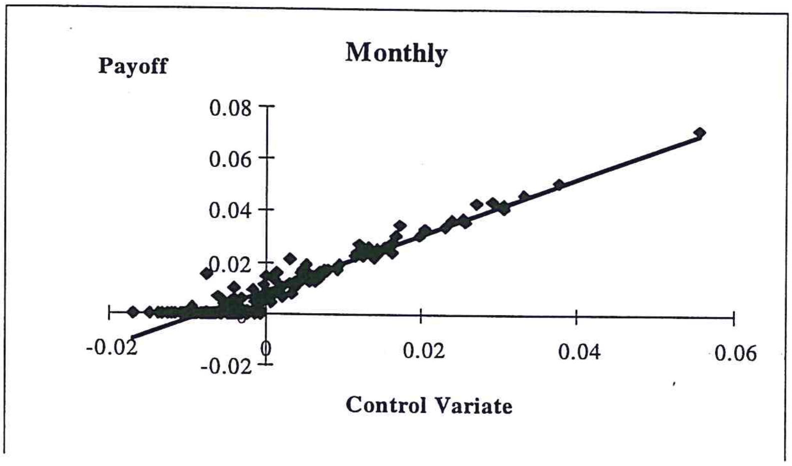
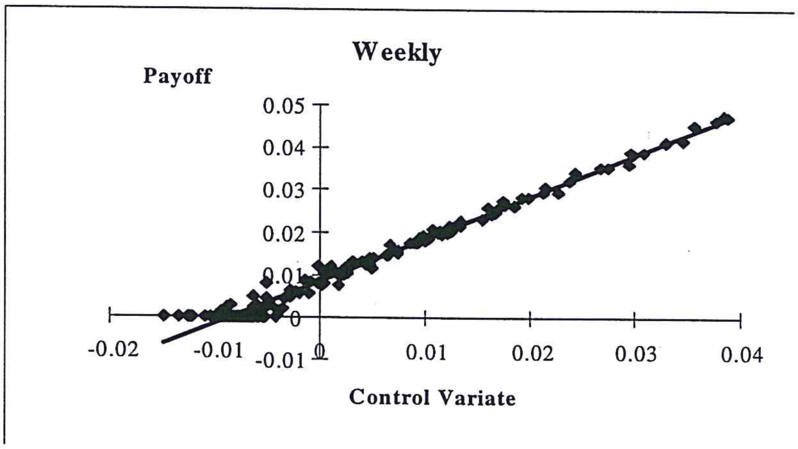


Figure 2: Least Squares Regression for Control Variate Technique

Panel (a)



Panel (b)



Panel (c)

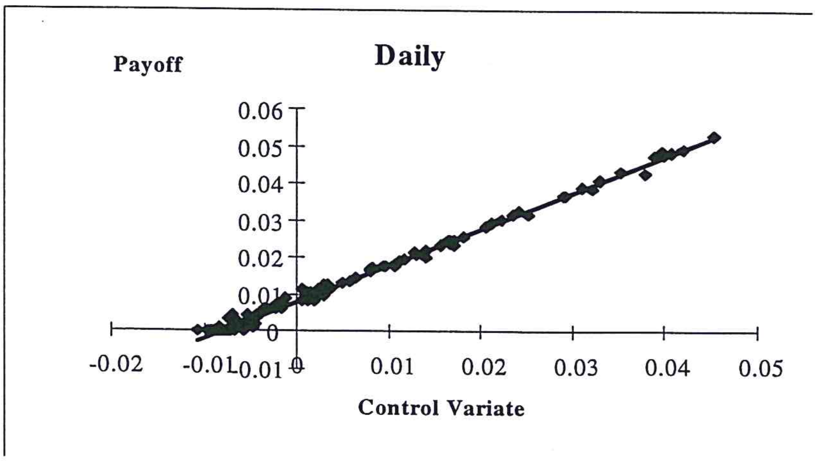
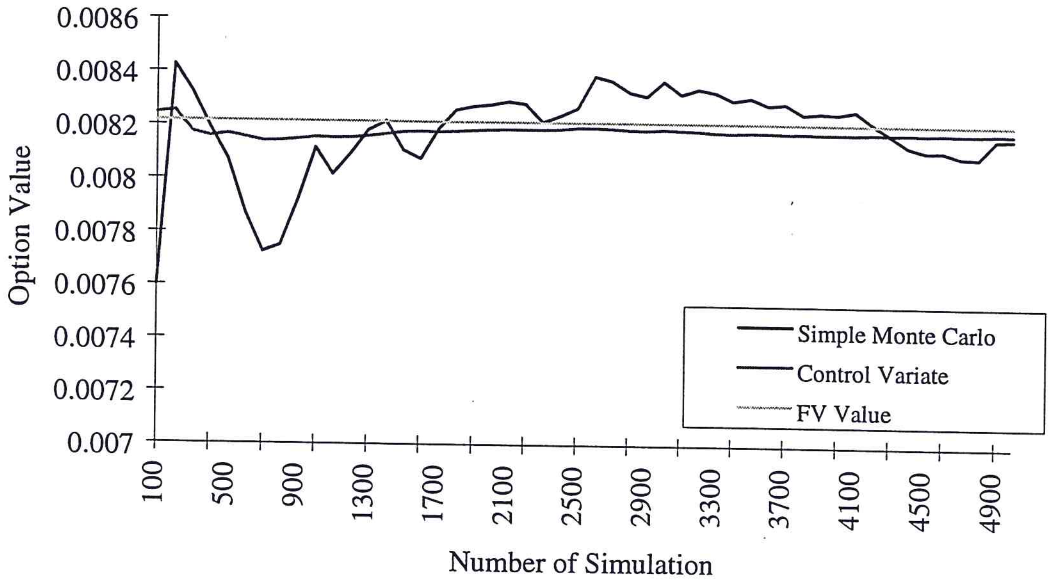


Figure 3 **Convergence of Simple Monte Carlo and Delta-Based Control Variate**



Appendix 1:

Delta-Based Control Variate for Pure Discount Bond Options

Explicit representations of the delta-based control variates are given by;

$$\frac{\partial c(t_i,T,s)}{\partial r} = P(t_i,s)f(t_i,s)N(h) + \frac{P(t_i,s)n(h)(f(t_i,s)-f(t_i,T))}{\sigma_P}$$
$$-K\left(P(t_i,T)f(t_i,T)N(h-\sigma_P) + \frac{P(t_i,T)n(h-\sigma_P)(f(t_i,s)-f(t_i,T))}{\sigma_P}\right)$$

$$\frac{\partial p(t_i,T,s)}{\partial r} = P(t_i,s)f(t_i,s)(N(h)-1) + \frac{P(t_i,s)n(h)(f(t_i,s)-f(t_i,T))}{\sigma_P}$$
$$-K\left(P(t_i,T)f(t_i,T)(N(h-\sigma_P)-1) + \frac{P(t_i,T)n(h-\sigma_P)(f(t_i,s)-f(t_i,T))}{\sigma_P}\right)$$

where

$$f(t,\tau) = -\frac{(1-e^{-\alpha(\tau-t)})}{\alpha}$$
$$n(h) = \frac{\exp\left(-\tfrac{1}{2}h^2\right)}{\sqrt{2\pi}}$$

Vega-Based Control Variate for Pure Discount Bond Options

Explicit representation of the vega-based control variate for a call option;

$$\frac{\partial c(t_i,T,s)}{\partial v} = \frac{1}{2\sqrt{v}}\left(\begin{array}{l} P(t_i,s)n(h)dh + N(h)P(t_i,s)b(s_i) \\ -K\left(P(t_i,T)n(h-\sigma_P)\left(dh - \frac{\sigma_P}{2\sqrt{v}}\right) + N(h-\sigma_P)P(t_i,T)b(T_i)\right)\end{array}\right)$$

where

$$dh = \frac{-h + \sigma_P}{\sqrt{v}} + \frac{b(s) - b(T)}{\sigma_P}$$

$$b(\tau) = -\frac{(1 - e^{-\alpha(\tau - t_i)})}{\alpha - (\tau - t_i)} \frac{\sqrt{v}}{\alpha} - \frac{\sqrt{v}\left(1 - e^{-\alpha(\tau - t_i)}\right)^2}{2\alpha^2}$$