Pricing by Arbitrage in Incomplete Markets

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PRICING BY ARBITRAGE IN INCOMPLETE MARKETS[1]

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A substantial literature on pricing by arbitrage has restricted information, so that it arises solely from securities markets, and return distributions are governed by past prices. We reconsider pricing by arbitrage in markets rendered incomplete by arbitrary information, which, moreover, influences required returns. Specific martingale representation results, playing a key role in existing analyses, depend on the restricted information, and so become inapplicable. We show that pricing by arbitrage alone, of claims depending on the market history, depends precisely on the agreement of all possible risk-adjusted probability laws of the primitive securities. Linear factor risk pricing turns out to hold under extremely mild conditions, permitting a characterisation of locally risk-adjusted probabilities. In particular, we rederive existing results in our generalized economy.

In a wider class of models, arbitrage arguments must be supplemented by specifying market price(s) of risk(s). We characterize viable models in terms of "risk pricing measures", which generalize Harrison and Kreps' [1979] (HK') "equivalent martingale measures", to handle primitive processes not representing securities prices - a situation not contemplated by HK. Our analysis leads us to conclude explicitly that the pricing of risks provides a more fundamental interpretation of such reassignments of probabilities than making securities 'fair bets'.

JEL CLASSIFICATIONS: general (G10); asset pricing (G12); contingent pricing (G13)

KEYWORDS: incomplete markets, information filtration, pricing by arbitrage, market prices of risks, risk pricing measure

1. INTRODUCTION

Traded securities cover only a tiny subset of the uncertain contingencies of concern to economic agents. By the same token, the information contained in the securities market history is but a part of the overall information available; moreover, other information has a bearing on the distribution of subsequent returns. We therefore submit that assertions that various contingent claims can be priced by arbitrage should be substantiated in an incomplete markets setting,

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under an arbitrary information filtration[2]. Moreover, securities price processes should be specified in such a way that the distribution of future returns need not depend on market histories alone. We use the phrase "priced by arbitrage" to cover not just pricing by means of arbitrage considerations alone, but also pricing in which arbitrage arguments are supplemented by assumptions as to one or more market prices of risk.

The setting demanded above - involving required returns being allowed to depend upon an arbitrarily large information filtration - was developed in the general theoretical frameworks of Harrison and Kreps [1979] and Harrison and Pliska [1981]. Nevertheless, commencing with those authors, a substantial applications literature has been devoted to pricing by arbitrage, under assumptions implying that available information arises only from the securities market history, and that, in consequence, the future return distribution is controlled by that history to date. Given the informational restriction, the payoffs of all possible contingent claims depend only on the market history.

The tasks entailed by dropping such restrictions turn out to be far from trivial, since allowing required returns to depend upon an arbitrarily large overall information filtration, makes the nature and size of even the sub-filtration representing the market history unclear. In consequence, the martingale representation results used by many existing analyses, which depend on a filtration generated by a finite dimensional vector of specified processes - usually Brownian or Poisson - are inapplicable in our more general framework. These issues are crucial for understanding much of this paper; we therefore provide an example:

Consider a Black-Scholes model with a zero interest rate. Only one (non-numeraire) price process - namely the stock price, S - is taken as primitive, and innovations in returns are driven by a Brownian motion, Z:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dZ$$

Existing analyses (eg Harrison and Kreps [1979] p388, Kreps [1981] p31, Harrison and Pliska [1981] pp244-50) assume that the required returns, μ, are adapted to the Brownian filtration, $\{\mathcal{F}_t^Z\}$.[3] Hence, the security market filtration, $\{\mathcal{G}_t\}$, generated by S,[4] is contained in the Brownian filtration. The subsequent

2 The information filtration provides a mathematical formalism for the increasing amount of information available to agents as uncertainty is resolved through time.

3 Harrison and Pliska suggest (p248) that their results on diffusion models could be generalized, allowing, *inter alia*, required returns to depend on more information, but make no attempt to justify this.

4 With a zero interest rate, no distinction need be made between the stock price and the normalized stock price, ie the stock price expressed in numeraire units rather than units of account.

arguments hinge on exploiting martingale representation results and a version of Girsanov's theorem, depending precisely on the filtration being Brownian. We, by contrast, demand that required returns, μ , must be allowed to depend on an arbitrarily large information filtration. Through μ , therefore, information from sources other than past Brownian innovations in returns becomes impounded into the stock price. In consequence, it is no longer clear what can be said about the nature of the history, $\{\mathcal{G}_t\}$.[5]

Prompted by these thoughts, this paper explores pricing by arbitrage, consistent with general equilibrium, where markets are necessarily incomplete as a result of the overall information filtration being arbitrarily larger than that generated by the securities market. In addition, required returns depend on the overall filtration; hence, their future distribution may not be determined by past market history. We provide a number of results to enable pricing by arbitrage to be carried out in this improved framework.

A particular concern of our paper is to test whether existing arbitrage-based pricing results, obtained under a restricted information filtration, with consequent restrictions on required returns, can be preserved in our more general framework.

While the existing applications literature has tended to be based on Brownian or Poisson processes, the generality of our results permits applications involving other processes, such as counting processes with stochastic intensities: see Babbs and Webber [1993].

The paper runs as follows. Section 2. elaborates certain points raised above; in particular, we counter appeals to pricing by replication as a way of sidestepping our challenge to existing results.

In Section 3. we begin our study of models in which each stochastic process taken as primitive is the price process of some security, and in which pricing results are sought by means of arbitrage arguments alone. We focus especially upon contingent claims whose payoffs - normalized by being expressed in units of some numeraire security - depend only on the restricted filtration generated by normalized securities prices. This is motivated by three considerations. Firstly, as noted above, under the restricted filtration common in many previous analyses, all contingent claims are of this type. Secondly, in many contexts, this type of contingent claim coincides with the class of derivative securities.[6] Thirdly - and this point will come into sharper focus in Section 7. - the primitive processes provide an explicit model only of risks underlying this restricted filtration. It is

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[5] The reader may feel that the market history can be called Brownian, in that the log returns process becomes a Brownian motion with drift, under risk-adjusted probabilities. However, this is, of course, a conclusion of the analysis, and has yet to be confirmed under an arbitrarily large information filtration.

[6] In some cases, such as Merton's [1973] model with stochastic interest rates, derivatives such as options maturing part way through the economy are not measurable in this fashion.

therefore natural to enquire whether this modelling suffices to provide unique prices for payoffs depending only on that filtration.[7] We show that the ability to price all these claims by arbitrage alone (we call this "*full pricing by arbitrage alone*"), under an arbitrarily large overall information filtration, is equivalent to all "equivalent martingale measures" (EMMs)[8] coinciding on the market history, and to the finite-dimensional distributions[9] of normalized security price processes under EMMs also coinciding.

The first major result in Section 4. establishes that linear factor risk pricing holds under very mild regularity restrictions on normalized securities price processes. We go on to deduce a technical result of great value in the sequel, namely an exact characterization of locally risk-adjusted probabilities - "equivalent local martingale measures" (ELMMs), ie equivalent probability measures under which normalized securities price processes become local martingales.

In Section 5., we apply our characterization of ELMMs to show that full pricing by arbitrage alone is preserved for popular diffusion-based arbitrage pricing models in finance, when the overall information filtration is allowed to be arbitrarily large, and the specification of required returns totally general. By way of example, Section 6. considers Merton's [1973] extension of the Black and Scholes [1973] model to incorporate stochastic interest rates; we show that, even with an arbitrarily large filtration, full pricing by arbitrage alone holds in Merton's model; in particular, European call options are priced by arbitrage - a result that eluded Harrison and Kreps [1979].

Section 7. moves on to consider a wider class of models in which arbitrage arguments are supplemented by specifying the market price of one or more sources of risk. In such "market price of risk models" (MPR models), the stochastic processes assumed as primitive may consist not only - or even not at all - of those followed by securities prices, but of those followed by state variables such as interest rates, volatilities, or by more abstract variables such as technological capabilities. By contrast, Harrison and Kreps [1979] (HK) confine

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7 This, of course, does not exclude unique pricing of some other claims. For example, the outcome of a self-financing investment strategy based on the weather represents a payoff whose unique price is given by initial wealth, even in models whose primitive processes relate exclusively to securities prices.

8 An EMM is a reassignment of probabilities, under which, after normalization by dividing through by the price of a numeraire security, securities price processes become martingales. The concept was introduced by Harrison and Kreps [1979], who showed, under certain conditions, that a securities market model is viable if and only if EMM(s) exist. Moreover, there is a one to one correspondence between EMMs and viable pricing operators for contingent claims; the normalized price of a claim equals the expectation, under the corresponding EMM, of its normalized payoff.

9 The finite-dimensional distributions of a process are the distribution functions of the values taken by the process at arbitrarily chosen finite sets of fixed times; see eg Ikeda and Watanabe [1981], p17.

their attention (see their opening paragraph) to relative pricing problems in which all the primitive processes are securities prices. Their results can be applied indirectly, to test the viability of candidate solutions to pricing problems. We argue, however, that this is not wholly satisfactory.

Firstly, considerable effort may be required in some cases to show that the assumptions of a particular model suffice to make the candidate solution unique. Secondly, an indirect approach makes it hard to assess what alternative specifications of the market price(s) of risk(s) yield viable models. Thirdly, the economic emphasis natural to HK' equivalent martingale measure concept is on the idea that securities price processes should, in some sense, be "fair bets". The relationship between this and the pricing of risk needs further articulation.

To provide a direct analysis of MPR models we introduce a new class of equivalent probability measures. Our new "risk pricing measure" (RPM) concept subsumes HK' "equivalent martingale measure" (EMM) concept. Whereas the emphasis in the interpretation of EMMs has tended to lie with making securities "fair bets", our generalized RPM concept lays the fundamental emphasis upon the specification of the market prices of the sources of risk in the economy, both traded and untraded. We regard this as offering important clarification of the economic significance of the changes of probability measure which HK made a standard tool in finance.

Using results on RPMs, we discuss viability (ie compatibility with general equilibrium) and pricing by arbitrage in MPR models. Notable existing MPR models include Wiggins [1987] and Vasicek [1977]. We indicate how these models can be brought fully within our framework, and that the pricing results go through, even when the overall information filtration is allowed to be arbitrarily large.

Section 8. illustrates the power of our "risk pricing measure" (RPM) concept for constructing new viable models, without the effort and strong assumptions usually involved in constructing an explicit general equilibrium. By way of example, we consider term structure models based on the Cox, Ingersoll and Ross [1985] square root process for the instantaneous spot interest rate r.

Section 9. concludes the paper. Proofs are relegated to the Appendix.

2. PRELIMINARY DISCUSSION

Since the path-breaking option pricing paper by Black and Scholes [1973], a major corpus of financial economics has been devoted to pricing by arbitrage. We have in mind not just models in which contingent claims are priced by arbitrage considerations alone, but also work in which arbitrage arguments are supplemented by assumptions concerning the market price(s) of one or more sources of risk. This latter research includes the pricing of options in markets which are incomplete on account of risks not being spanned by available trading opportunities (examples include: stock and interest rate risk, eg Merton [1973]; jump and diffusion risk, eg Merton [1976]; stochastic volatility, eg Wiggins [1987]). It also includes models (eg Vasicek [1977]) in which the processes assumed as primitive are those of interest rates or other state variables not themselves securities prices, and in which fundamental securities (eg bonds) are priced. We shall refer to all this area as arbitrage-based, distinguishing where necessary the case where pricing is by arbitrage alone.

Much of the literature referred to above lacked links with general equilibrium. Theoretical underpinning was provided by HK (their Sections 2-3), who established conditions under which the price processes of traded securities can be supported in general equilibrium, under an arbitrarily large information filtration. However, in applying their general framework to the case of diffusion price processes (HK' Section 5) - and to the Black-Scholes model in particular - HK (p394, final paragraph) restricted the overall information filtration to be that generated by the Brownian motions underlying innovations in securities prices. This restriction enabled HK to use a martingale representation theorem to determine a unique rational pricing operator for all contingent claims, based on a unique EMM.

Harrison and Pliska [1981] (HP) developed a general security market model, with semimartingale price processes, and capital gains represented by general stochastic integrals. However, in seeking concrete results in diffusion and other examples (their Sections 5-6, pp 244-56), they restricted the overall information filtration, just as HK had done, in order to use particular martingale representation results.

Influenced by HK and HP, similar restrictions are made - with the same end in mind, and often without comment - throughout virtually all the modern martingale literature on pricing by arbitrage.[10] The restricted filtration may relate partly to non-traded processes governing volatility or interest rates, say, as well as to securities prices, and may involve jump processes or other non-Brownian processes. However, the restrictive assumptions still commonly entail that the (restrictively specified) overall information can be recovered from the market history - incorporating that of any additional non-traded variables. Thus, agents' information is restricted to the market history, and the distribution of future returns is consequently determined thereby.

While technical in their details, these restrictions are far from innocuous; bluntly expressed, they bear the straightforward interpretation that economic agents neither know nor care about anything other than the securities markets. Not only is this patently unrealistic, but also, it fails to accommodate evidence, such as

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10 The assumption is either made at the outset (see eg Heath, Jarrow and Morton [1992] p79) or made at the point of first use (see eg Bensoussan [1984] p145). Our own work is no exception (see eg Babbs [1990] p142, where the restriction was recognised as being sufficiently uncontroversial to be relegated to a footnote).

that discussed by Fama [1991], that information other than from securities market histories has a bearing on the distribution of prospective returns. This leads us to the following important question: do existing arbitrage-based pricing results remain valid, once it is conceded that both the knowledge and concerns of agents extend far beyond securities market histories? In technical terms, do arbitrage-based pricing results go through if the overall information filtration is allowed to be arbitrarily large, and the additional information is allowed to impinge on required returns?

One might respond that some essentially trivial embedding argument should ensure that arbitrage-based pricing results go through when the relevant models are nested, essentially unchanged, in some larger whole. Quite possibly. But it is not adequate to seek to embed existing arbitrage-based models *essentially unchanged* in some larger whole. Instead, the recognition that information not included in securities market histories affects prospective returns, must be allowed to *change* the specification of expected returns; and the recognition that agents' concerns extend far beyond those histories, must *change* the nature of the general equilibrium supporting the securities markets. The task of adapting existing arbitrage-based pricing results to cope with these changes appears to us far from trivial. This is precisely the issue we wish to address in this paper.

2.1 REPLICATION: THE 1970s OPTIONS LITERATURE

The challenge we have posed to existing arbitrage-based pricing results, centres on the nature of the information assumed to be available to agents, the range of uncertainties of concern to them, and the consequent specification of required returns. One might attempt to sidestep that challenge by appealing to the 1970s options literature, in which the pricing results hinge on the construction of trading strategies which replicate the contingent claims involved, apparently regardless of required returns, and which makes no recourse to the nature of the information filtration. Such a manoeuvre is subject to the serious limitations of that literature.

Firstly, the methodology centred on the derivation and solution of partial differential equations (PDEs) for the prices of contingent claims as functions of the prices of underlying securities and of time.[11] The ability to price claims contingent on the price paths of securities, rather than just on their prices at a fixed date, is limited by the scope of available existence and uniqueness results for solutions of PDEs. In particular, completeness results appear unachievable. Secondly, if diffusion price processes are replaced by jump processes, as in Cox and Ross [1976a], PDEs give way to even less wieldy partial differential-difference equations. Thirdly - and this is the telling point theoretically - the approach lacks links with general equilibrium.

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11 Merton [1977] showed how to meet objections to assuming the smoothness properties involved.

Grappling with the second of these issues led Cox and Ross [1976a][12] towards the idea of a reassignment of probabilities, which flowered fully in HK' EMM concept, when the latter authors provided the seminal link between contingent claims analysis and general equilibrium. Thus, we see that attempts to remove the shortcomings of the 1970s approach lead precisely to the modern martingale approach to which our question is chiefly addressed.

2.2 *REPLICATION: THE MODERN MARTINGALE APPROACH*

An appeal to replication might also be attempted, to validate arbitrage-based pricing results obtained under the modern martingale approach pioneered by HK and HP.[13] Replication strategies are established, for broad classes of contingent claims, by appealing to various martingale representation theorems. The same theorems are also used at a more fundamental stage in the analysis, to establish the existence and identity of a unique pricing operator, corresponding to a unique EMM. The most commonly used theorems depend, in turn, either on the existence of a unique EMM, or upon relevant variables being measurable with respect to a particular type of filtration - usually Brownian. If we admit an arbitrarily large information filtration, a multiplicity of EMMs usually results[14]. Moreover, we remind the reader of the point made in our introduction, that, if we allow required returns to depend on the full filtration, then even the sub-filtration corresponding to the market history can no longer be assumed to be of any particular type. Hence, martingale representation results no longer apply across the board. The technicality of the details must not obscure the fact that the difficulties must be resolved if the validity of the results is to be maintained.

Perhaps a more intuitive rejoinder to our challenge might be that enriching the information filtration endows agents with greater knowledge, thereby making it easier rather than harder to determine a unique rational price for a contingent claim. Unfortunately, enlarging the filtration also increases the range of uncertainties awaiting resolution in the future. It is reasonable to fear that the latter effect may be especially problematic for contingent claim pricing, once it is recognised that information not contained in securities market histories may have a bearing on subsequent returns, and the concerns of agents.

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12 Cox and Ross [1976a] themselves refer to unpublished 1975 work as first containing the idea. The most extensive published discussion by the originating authors is in Cox and Ross [1976b].

13 Indeed, Jacka [1992] has recently shown, under mild regularity assumptions, that a contingent claim is priced by arbitrage if and only if it can be replicated.

14 see Section 4, especially Theorem 4.8

3. FULL PRICING BY ARBITRAGE ALONE

The starting point for models leading to pricing by arbitrage alone is to adopt as primitive a set of stochastic processes for the prices of various securities. We presuppose an arbitrary filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, P)$,[15] over a fixed finite interval $[0,T]$, satisfying the "usual conditions"[16], and with $\mathcal{F} = \mathcal{F}_T$.

Assumption (A) *$n+1$ securities are traded, with semimartingale price processes* $S_0, \ldots, S_n$.

At least one asset has a strictly positive price process:
$S_0(t) > 0 \ \forall t, \ a.s.$

Adopting the zeroth security as numeraire, we define normalized price processes:

$$S_k^* \equiv S_k / S_0 \quad ; \qquad k = 0, \ldots, n$$

which, by the generalized Ito's lemma[17], are also semimartingales,[18] which we can therefore write as:

$$S_k^* = A_k + X_k ; \qquad k = 0, \ldots, n \tag{1}$$

where each A_k is of finite variation, and each X_k is a local martingale. Let S^* denote the vector of these processes.

Remark 3.1 The choice of a security price process (or, more generally, of a portfolio value process) as numeraire - as opposed to some untraded process - is significant, since it ensures that the information content of normalized price histories coincides with that of the relative prices at which one security can be exchanged for another. Note that, like HK, we do not require the numeraire process to be continuous or of finite variation.[19]

To make our analysis precise, we require an explicit economic framework. We adopt that of HK, since all applications of existing arbitrage-based models, of which we are aware, can be embedded in HK' framework; moreover, HK' framework is attractive on account of the seminal nature of their work and the simplicity of their setup. We are confident that the insights in this paper can be applied in other settings. Following HK' general framework, therefore, we shall

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15 In this formalism, $\mathcal{F}_t$ represents the information available to agents at time t.

16 see eg Jacod and Shiryaev [1987] Definition 1.3 p2

17 see eg Elliott [1982] Theorem 12.19 p138

18 S_0^* is, of course, identically equal to unity

19 This requirement was introduced by Harrison and Pliska [1981], seemingly merely in order to establish their Proposition 3.24, and has become standard in the literature. Babbs [1990] (chapter 5, especially Appendix B) and [1991] has pointed out that the requirement was unnecessary.

consider a pure trading economy over $[0,T]$, with consumption space is $\Re \times L^2(\mathcal{F},P)$. We suppose that no dividends are payable until T, and that each $S_k(T) \in L^2(P)$.

Remark 3.2 The assumptions of absence of dividends until T, and that the number of securities is finite, are not as restrictive as may appear. Intermediate payouts can be accommodated by adjusting prices to reflect reinvestment in the numeraire security. The results of this Section go through unaltered if the number of securities is infinite. As regards subsequent Sections, although term structure models, for instance, may involve a continuum of pure discount bonds, they are usually so formulated that normalized prices have a finite 2-basis[20]. Essentially, therefore, (3) in Section 4 holds, and all the subsequent analysis applies.

A feature common to many contingent claims pricing models since the seminal work of HK and HP, has been to establish a unique pricing operator for all contingent claims, corresponding to the existence of a unique EMM. The price assigned to a contingent claim by the unique pricing operator is said to be determined "by arbitrage".

The uniqueness of the EMM in these models depends upon the dynamic spanning of all sources of uncertainty, by the relative prices at which one security can be traded for another, and thus - as long as we use a security as numeraire (see Remark 3.1 above) - by normalized securities prices.[21] The uniqueness therefore depends upon the information (or uncertainty) filtration being restricted to that generated by normalized securities price histories. Since the payoffs of contingent claims are, of necessity, measurable with respect to information at the terminal date of the economy, the same restriction artificially limits all contingent claims to depending solely on normalized securities price histories.

If we abandon the restricted information in favour of an arbitrarily large overall filtration, a multiplicity of EMMs usually ensues. Following HK, we can still say that a particular contingent claim is priced by arbitrage, if the pricing operators corresponding to all the EMMs assign it the same price.

Formally, EMMs, are defined as follows:

Definition 3.3 *An equivalent martingale measure (EMM) is a probability measure* P^* *on* $(\Omega,\mathcal{F})$, *equivalent to P, and such that each* S_k^* *is a martingale under* P^*

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20 see Lemma 4.4 below

21 Further elucidation of the role of dynamic spanning in contingent claims analysis can be found in Babbs and Selby [1992].

The following regularity condition plays a part in the analysis. Note that we do <u>not</u> assume it to hold for all EMMs.

<u>Condition (S)</u> *A probability measure* P^* *, equivalent to* P*, is said to satisfy Condition (S) if and only if:*

$$\frac{1}{S_0(T)}\frac{dP^*}{dP} \in L^2(P) \tag{2}$$

<u>Remark 3.4</u> The LHS of (2) represents the state-price density.

HK established that, if agents are restricted to following simple[22] trading strategies, and $\mathrm{E}[\{S_k^*(t)\}^2]<\infty, \quad \forall k,t$, then:

<u>Theorem 3.5</u> (Harrison and Kreps [1979]) *There is a one-to-one correspondence between EMMs satisfying Condition (S)*[23] *and continuous strictly positive linear pricing operators for all contingent claims. Namely, if* ψ *is a pricing operator, then we can define such an EMM,* P^* *, by:*

$$P^*(F) = \frac{\psi(S_0(T)1_F)}{S_0(0)}; \qquad \forall F \in \mathcal{F}$$

Conversely, if P^* *is such an EMM, we can define a continuous and strictly positive linear pricing operator,* ψ *, by:*

$$\psi(x) = S_0(0)\mathrm{E}^*\left[\frac{x}{S_0(T)}\right]; \qquad \forall x \in L^2(\mathcal{F},P)$$

HK had shown earlier (their Theorem 1, p386) that a model is viable if and only if a continuous strictly positive linear pricing operator for all claims exists, consistent with the prices of the primitive securities. Hence, they obtained:

<u>Corollary 3.6</u> *A model is viable if and only if there exists at least one EMM satisfying Condition (S).*

Duffie and Huang [1986] and Babbs [1990] (chapter 5) and [1991] have shown that Theorem 3.5 and its Corollary hold under less restrictive conditions, permitting *inter alia* continuous trading. For the purposes of the present paper, we shall simply assume that Theorem 3.5 and Corollary 3.6 hold.

<u>Remark 3.7</u> Subject to Remark 3.1 above, it can readily be verified that our results are independent of choice of numeraire; thus a change of numeraire is economically neutral. Briefly, if S_1 is also almost surely strictly positive, the filtration generated by securities prices normalized using S_1 as numeraire,

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[22] A simple trading strategy involves trading only at a finite number of fixed dates. (In addition, HK required the holding of each security to be adapted, and of square-integrable normalized value.)

[23] We have inserted Condition (S) into our statement of HK' result, rather than following HK by including it in the definition of an EMM.

coincides with that using S_0. Moreover, since (by Theorem 3.5) the set of EMMs corresponding to each numeraire is in one-to-one correspondence with the possible equilibrium pricing operators, they are in one-to-one correspondence with each other.

We now define the "securities market history", which plays a major part in our discussions:

Definition 3.8 *The "securities market history",* $\{\mathcal{G}_t : t \in [0,T]\}$ *(briefly* $\{\mathcal{G}_t\}$ *) is the completion of the sub-filtration generated by the normalized securities price processes,* S^* *. We let* $\mathcal{G} = \mathcal{G}_T$ *.*

Standard practice in the existing literature on pricing by arbitrage alone is to suppose that the overall information filtration in the economy, ie $\{\mathcal{F}_t\}$, is limited to that, $\{\mathcal{F}^M_t\}$, generated by a vector, *M*, of local martingales - usually standard Brownian or Poisson processes - of dimension equal to *n*, the number of non-numeraire securities. The cumulative innovations, *X*, are usually precisely specified as stochastic integrals with respect to *M*, where the integrands commonly depend only on time and on concurrent values of S^* . The required return component, *A*, by contrast, is left as an essentially arbitrary $\{\mathcal{F}^M_t\}$ -adapted finite variation process, though it is usually made absolutely continuous[24]. In such a setup, the overall information filtration is clearly restrictively specified. Moreover, the standard setup commonly satisfies regularity restrictions sufficient to ensure that the evolution of *M* can be recovered from observing normalized prices, ie that $\{\mathcal{G}_t\} = \{\mathcal{F}_t\}$.[25] This gives rise to the twofold situation to which we have objected, namely that all the information in the economy is to be inferred from the securities market history, and that the distribution of future returns is, in consequence, dependent only upon that history.

We treat any restriction, made in an existing analysis, on the size or nature of the overall information filtration, together with any restriction on the filtration upon which the required returns component, *A*, depends, as a distinct assumption, Restriction (R) say. We emphasise that Restriction (R) is a feature common to existing analyses; we do <u>not</u> assume it.

With Restriction (R), existing analyses are able to price, by arbitrage considerations alone, all contingent claims whose normalized payoffs are $\mathcal{G}$ -measurable; for brevity's sake, call this "*full pricing by arbitrage alone*". We wish to know how the pricing of these claims is affected if Restriction (R) is dropped,

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24 see Remark 4.2

25 Merton [1973] provides an exception, in that the history of the normalized stock price does not enable the separate recovery of the Brownian motions driving the unnormalized prices.

ie if the overall filtration is allowed to be arbitrary large, and no restrictions are placed upon the filtration upon which the required returns component, A, depends.

Our first result relates "full pricing by arbitrage alone" to the behaviour on $\mathcal{G}$ of the EMMs, and to the probability law of the normalized security price processes under EMMs:

Theorem 3.9 *The following are equivalent, under an arbitrary information filtration:*

(a) all EMMs yield the same finite-dimensional distributions for S^* *;*

(b) all EMMs coincide on $\mathcal{G}$ *;*

(c) full pricing by arbitrage alone.

This result affirms that the risk-adjusted joint distribution of increments in normalized prices (ie the finite-dimensional distribution of S^* under the EMM) suffices to determine the probabilities of all events whose occurrence depends solely on the paths of those prices (ie the probabilities under the EMM of all events in $\mathcal{G}$). These probabilities determine the risk-adjusted expectations of payoffs contingent on those paths; hence, we have full pricing by arbitrage alone. Conversely, a unique set of prices for (in particular) Arrow-Debreu claims for events determined by normalized price histories, determines the risk-adjusted probabilities of those events. Since such events include the occurrence of all possible patterns of increments in normalized prices, the probabilities of these patterns (ie the risk-adjusted finite distributions of S^*) are determined, in turn.

At a more general level, one could interpret the equivalence of parts (b) and (c) of Theorem 3.9 as confirming the unsurprising fact that "a unique EMM on $\mathcal{G}$ " is necessary and sufficient for "unique pricing for $\mathcal{G}$ -measurable claims". However, with an arbitrarily large overall information filtration, and no restrictions on the information influencing required returns, it is not clear that the securities market history, $\{\mathcal{G}_t\}$, is a filtration of any particular type or size. Therefore, the martingale representation results relied upon in existing analyses become inapplicable, and it is not clear how to demonstrate that there is a unique EMM on $\mathcal{G}$. Part (a) of Theorem 3.9 prompts us to tackle the problem by looking at the risk-adjusted dynamics of S^* under the multiple EMMs on $\{\mathcal{F}_t\}$.

4. EQUIVALENT LOCAL MARTINGALE MEASURES

Theorem 3.9 focuses attention upon the behaviour of normalized securities prices under the multiple EMMs existing under an arbitrary information filtration. This Section, however, introduces and characterizes a looser family of measures, that of locally risk-adjusted probabilities, or "equivalent local martingale measures" (ELMMs), under which the vector S^* becomes a local martingale, not necessarily a martingale. This has a twofold rationale: the prime stochastic calculus result on changes of measure (ie Girsanov's theorem) permits crisper analysis of ELMMs than appears possible for EMMs; more importantly, the looser concept turns out to be precisely what we need in the sequel in considering models requiring the specification of market price(s) of risk.

To further our analysis, we supplement Assumption (A) by two further assumptions, which simply place some localized moderation on the violence of changes in normalized securities price processes.

Assumption (B) *Each S_k^* is a special semimartingale, ie there exists a unique decomposition:*

$$S_k^* = A_k + X_k$$

in which A_k is a predictable finite variation process of locally integrable variation, and X_k is a local martingale with $X_k(0) = 0$.

Assumption (C) *Each $X_k \in \mathcal{H}^2_{0,loc}$ (ie is locally a square-integrable P -martingale with $X_k(0) = 0$).*

It is well known (see eg Elliott [1982] Theorem 12.38 p148) that Assumption (B) is equivalent to various other properties; noteworthy among these are a localized integrability property on jumps, and nice behaviour of A_k :

Lemma 4.1 *For each k, Assumption (B) is equivalent to either of the following:*

(a) $\sum_{0<s\leq t} \{\Delta S_k^*\}^2$

is locally integrable;

(b) there is a unique decomposition (1) in which A_k is predictable and locally of integrable variation.

Remark 4.2 If, as in most applications, A_k is assumed to be absolutely continuous - ie of the form:

$$A_k = \int_0^{\cdot} \mu_k \, du$$

then A_k is predictable and locally of integrable variation; thus Assumption (B) holds.

Remark 4.3 Ansel and Stricker [1992] have shown that Assumptions (B) and (C) must hold under absence of (a particular kind of) arbitrage, if normalized asset prices are *cadlag*[26] and their suprema satisfy a square-integrability regularity condition. We prefer to take Assumptions (B) and (C) as primitive, both to avoid such a strong regularity condition, and because we shall wish, in the sequel, to apply these assumptions to processes other than price ones. Nevertheless, Ansel and Stricker's result leads us to the view that only an extremely pathological model would violate Assumptions (B) and (C). Note that -

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26 *cadlag* is a French abbreviation signifying right continuous with left limits.

by part (a) of Lemma 4.1 - these assumptions cover processes such as those commonly employed in applications, which are continuous, or have, at worst, locally well-behaved jumps.

Assumption (C) enables us to go beyond Lemma 4.1, and express the random components of price changes in terms of orthogonal risk factors, $M_1, \ldots, M_n$. At one level, this can be thought of as merely a change of basis. However, it will enable us, as our analysis progresses, to identify distinctly the market prices attaching to the various risks.

Lemma 4.4 *We can choose a pairwise orthogonal 2-basis* $\{M_1, \ldots, M_n\}$ *for the stable subspace generated by* $\{X_1, \ldots, X_n\}$ *; in particular, for each k, we can write:*

$$S_k^* = A_k + \sum_{j=1}^{n} \int_0^{\cdot} \sigma_{jk} \, dM_j \tag{3}$$

where, for each j and k, $\sigma_{jk} \in L^2_{loc}(M_j)$.[27]

Remark 4.5 One can readily verify that our results are unaffected by the choice of 2-basis.

We now define our ELMM concept. Essentially, the definition follows our definition of an EMM, save for a loosening of the requirement that normalized securities prices should become martingales. One can therefore think of ELMMs as "candidate" EMMs.

Definition 4.6 *An equivalent local martingale measure (ELMM) is a probability measure* P^* *on* $(\Omega, \mathcal{F})$ *, equivalent to P, such that each* S_k^* *is a local martingale under* P^* .

A subclass of ELMMs, satisfying the following purely technical regularity condition, will play a special part in our analysis. Note that we do not assume that all ELMMs have this property.

Condition (L) $\mathrm{E}\left[\dfrac{dP^*}{dP} \;\middle|\; \mathcal{F}_t \right]$

is a locally square integrable process under P

Our first major result in this Section is stated below. It indicates that the existence of an ELMM is a significant property for normalized securities prices, in that (4) carries the interpretation that linear factor risk pricing holds. We

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27 $L^2_{loc}(M_j) = \left\{ \text{predictable } H : \int_0^{\cdot} H^2 \, d\langle M_j, M_j \rangle \in \mathcal{A}_{loc} \right\}$

where $\mathcal{A}_{loc}$ is the space of adapted processes of locally integrable variation.

consider the result remarkable, in that it enables us to conclude (using Theorem 4.7) that essentially <u>any</u> viable model with locally moderate price behaviour will exhibit linear factor risk pricing.

<u>Theorem 4.7</u> *If an ELMM satisfying Condition (L) exists, then each* A_k *must be of the form:*[28]

$$A_k \;=\; S_k^*(0) \;+\; \sum_{j=1}^{n}\int_0^{\cdot}\theta_j\sigma_{jk}\,d\langle M_j, M_j\rangle \tag{4}$$

where each "price of risk" process θ_j *is predictable.*

Changes of measure under which semimartingales become local martingales are often thought of as "changes of drift". Now that Theorem 4.7 has given an expression for the "drift" processes A_k, our next result shows that we can now characterize the changes of measure (ie the ELMMs) which accomplish it. A further interpretation of this result is that our "candidate" EMMs are determined by the factor risk prices, up to a more or less arbitrary factor, *N*, orthogonal to the sources of risk underlying the primitive processes. We shall see in Section 7. that *N* also can be thought of in terms of market prices of risk, namely of these orthogonal risks.

<u>Theorem 4.8</u> *Suppose that (4) holds. Then any ELMM,* P^**, satisfying Condition (L), has conditional expectations process*

$$\eta(t) \;\equiv\; \mathrm{E}\left[\frac{dP^*}{dP}\,\middle|\,\mathcal{F}_t\right] \tag{5}$$

satisfying:

$$\eta \;=\; \mathcal{E}\left\{\, N \;-\; \sum_{j=1}^{n}\int_0^{\cdot}\theta_j\,dM_j \,\right\} \tag{6}$$

for some $N \in \mathcal{H}^2_{0,loc}$ *orthogonal to each* M_j*, where* $\mathcal{E}\{\ \}$ *denotes the exponential semimartingale*[29]*. Conversely, if a probability measure* P^**, equivalent to P, is defined by*

$$\frac{dP^*}{dP} \;=\; \eta(T) \tag{7}$$

where η *is a strictly positive, and locally square-integrable, martingale under P, satisfying (6), then* P^* *is an ELMM satisfying Condition (L).*

Theorem 4.8 is significant in two ways. Firstly, in applications, it will enable us to show that the risk-adjusted probability law of the normalized securities price vector, S^*, under the various EMMs, is unique. We can then use Theorem 3.9 to obtain full pricing by arbitrage alone. We illustrate this in Section 5., by

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28 Throughout this paper, the notation $\langle X, Y\rangle$ denotes the predictable quadratic covariation of *X* and *Y* <u>under the actual probabilities,</u> *P*.

29 see eg Jacod [1979] pp190-1

analysing models in which normalized securities prices are diffusions. Secondly, it will be useful in Section 7. when we widen our analysis to models requiring the specification of market price of risk(s).

Note also that Theorem 4.8 generalizes the results for Ito process securities prices under Brownian filtrations, obtained by Pages [1987] and Karatzas, Lehoczky, Shreve and Xu [1991], to very general semimartingale securities prices, and to an arbitrary information filtration.

Remark 4.9 It has come to our attention that very similar results to those in Theorems 4.7 and 4.8, have been independently obtained by Schweizer [1992] (Propositions 4 and 5). Our results differ from those of Schweizer chiefly in that we orthogonalize the sources of risk underlying normalized securities prices. While this difference is of little moment mathematically, our approach permits additional economic insight, in that we have been able to make a linear decomposition of the overall risk premia on securities, into components attributable to separate sources of risk. Moreover, our orthogonalization of risk will enable us, in Section 7, when we consider models requiring the specification of market prices of risk(s), to distinguish clearly between those market prices of risk which are implied by taking certain securities price processes as primitive, and those which need to be specified by way of additional assumption.

5. SECURITY PRICE DIFFUSION MODELS

Our primary aim in this Section - fulfilled in Theorem 5.3 - is to show that full pricing by arbitrage alone holds for any well-behaved "*security price diffusion*" model. By way of corollary, existing pricing results, obtained under a restricted information filtration, remain valid under an arbitrary filtration.

Subject to the restriction that volatility be Markovian in the normalized asset prices, security price diffusion models include all those for pricing by arbitrage alone, in which n normalized asset prices are driven by n Brownian motions. Examples include the popular models for derivative securities, eg Black and Scholes [1973], Cox [1975], Margrabe [1978] and Garman and Kohlhagen [1983].

Definition 5.1 *We shall use "security price diffusion" to describe any model for which our basic normalized price dynamics equation (3) can be expressed in the form:*

$$S_k^* = A_k + \sum_{j=1}^{n} \int_0^{\cdot} \sigma_{jk}(S^*(u), u)\, dZ_j(u) \qquad (8)$$

where the argument list of each σ_{jk} *is intended to signify dependence upon the stated variables only, and* $(Z_1, \ldots, Z_n)$ *is a vector standard Brownian motion under* P.

Note that our use of the term "diffusion" in the above Definition is motivated solely by the form of the second term on the RHS; we place no restrictions upon A_k beyond those already imposed in Section 4..

Our first step is to characterize the dynamics of the normalized securities price vector S^* under ELMMs. Our main tool is Theorem 4.8, which provided a characterisation of ELMMs in terms of exponential semimartingales. Applying Girsanov's theorem yields the following:

Theorem 5.2 *Consider a security price diffusion model, with arbitrary information filtration, $\{\mathcal{F}_t\}$. Let P^* be any ELMM satisfying Condition (L). Then we can rewrite (8) as:*

$$S_k^* = S_k^*(0) + \sum_{j=1}^{n} \int_0^{\cdot} \sigma_{jk}(S^*(u), u)\, dZ_j^*(u) \qquad (9)$$

where $(Z_1^*, \ldots, Z_n^*)$ is a vector standard Brownian motion under P^*.

Note that the form of (9) is independent of the choice of P^*; this provides the key for further analysis.

We know, from HK' result reiterated as our Corollary 3.6, that a model is viable if it possesses an EMM satisfying Condition (S). An ELMM under which (9) has a martingale solution - ie one in which each S_k^* is a martingale - is an EMM. We deduce that a model is viable if it has such an ELMM satisfying Condition (S). If (9) has a unique weak solution[30] (ie determines the probability law of each S_k^*) then the finite-dimensional distributions of S^* coincide under all ELMMs. Combining these observations, provided that S^* is a martingale in this solution, it follows both that any ELMM is actually an EMM, and also that we can use Theorem 3.9 to obtain full pricing by arbitrage alone. This argument establishes:

Theorem 5.3 *A security price diffusion model which possesses an ELMM under which (9) has a martingale solution, and which satisfies Condition (S), is viable. If (9) has a unique weak solution in which S^* is a martingale, then:*

(a) every ELMM is an EMM; and

(b) full pricing by arbitrage alone holds.

As discussed in previous Sections, existing security price diffusion models have frequently restricted the information on which required returns depend, and the overall information filtration. We now consider the effect of enlarging the information filtration to be arbitrary. Since the information restrictions in existing

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30 See Karatzas and Shreve [1988] pp300-1 for the concept of a unique weak solution. Roughly, a weak solution concentrates on the probability law of the solution process, without regard to the properties of individual paths. The surrounding chapter offers various sufficient conditions ensuring such a solution.

Since we know of course that S^* satisfies (9), on our original probability space, it provides a strong solution (see Karatzas and Shreve pp284-6). Our purpose in speaking of weak solutions is to direct attention to the sufficient conditions for a unique weak solution, which are less onerous than for a strong solution.

models commonly have the effect that all contingent claims depend solely on the market history, it is natural to say that existing pricing results are *preserved*, if full pricing by arbitrage alone holds in the enlarged model.

By Theorem 5.2 above, the existing model will satisfy (9), regardless of the filtration employed. Our Theorem 5.3 now yields a Corollary giving a straightforward sufficient condition for existing pricing results to be preserved:

Corollary 5.4 *Suppose that in the existing model, (9) has a unique weak solution in which* S^* *is a vector martingale. Then existing pricing results are preserved.*

The requirement of a unique weak solution seems invariably to be met in existing models, as it tends to form part of their analysis under a restricted filtration. HK' treatment of the diffusion case is a case in point, where the requirement is explicitly assumed on p395.

6. EXAMPLE: MERTON [1973]

Merton's [1973] extension of the Black and Scholes [1973] model, to incorporate stochastic interest rates, provides a simple illustration of a number of the points discussed in this paper so far.

Merton's model involves two primitive securities, a stock and a pure discount bond, whose price processes involve two imperfectly correlated Brownian motions, Z_0 and Z_1. However, it turns out (see below) that the local martingale component of the normalized stock price process is driven by a single source of risk, a new standard Brownian motion, Z, formed by a time-varying combination of the two original motions.

In applying their general framework to the diffusion case, HK had assumed that the information filtration was that generated by normalized securities price processes. In the context of Merton's model, this meant that the information filtration could support only a single Brownian motion[31], contradicting the fact that it clearly supports at least the two motions Z_0 and Z_1. HK therefore concluded that their results for the diffusion case "do not enable us to claim that, say, European call options can be priced by arbitrage". We now proceed to show that, even under an arbitrary information filtration, our results enable us to obtain the result that eluded HK.

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31 This follows from the fact that the normalized stock price is represented in terms of a single Brownian motion with deterministic coefficients under any ELMM.

Merton's model replaced the Black-Scholes assumption of riskless borrowing/lending at a common constant interest rate, by allowing frictionless trading in a pure discount bond maturing at the expiry date, T, of the option, and whose price process can be written as:[32]

$$S_0 = S_0(0)\exp\left\{ \int_0^{\cdot} \mu_0 - \frac{1}{2}\sigma_0^2 \, du + \int_0^{\cdot} \sigma_0 \, dZ_0 \right\}$$

where μ_0 is predictable, σ_0 is a deterministic process, and where Z_0 is a standard Brownian motion whose innovations bear a deterministic imperfect correlation process ρ to those of the standard Brownian motion Z_1 featuring in the stock price process:

$$S_1 = S_1(0)\exp\left\{ \int_0^{\cdot} \mu_1 - \frac{1}{2}\sigma_1^2 \, du + \int_0^{\cdot} \sigma_1 \, dZ_1 \right\}$$

where μ_1, σ_1 are predictable and strictly positive deterministic processes, respectively.

Using the bond as numeraire, and applying Ito's lemma, the normalized stock price process is:

$$S_1^* = A_1 + \int_0^{\cdot} S_1^*(u)\sigma(u)\, dZ \tag{10}$$

with

$$A_1 = S_1^*(0) + \int_0^{\cdot} S_1^*(u)\mu(u)\, du$$

where: $\mu = \mu_1 - \mu_0 - \sigma_1\rho\sigma_0 + \sigma_0^2$; $\sigma = \sqrt{\sigma_1^2 - 2\sigma_1\rho\sigma_0 + \sigma_0^2} > 0$; and Z, defined by:

$$Z \equiv \int_0^{\cdot} \frac{\sigma_1}{\sigma} dZ_1 - \int_0^{\cdot} \frac{\sigma_0}{\sigma} dZ_0$$

is a new standard Brownian motion under P.

Comparing (10) with (8), we see that Merton's model falls into the category of security price diffusion models defined in Definition 5.1, with $n = 1$ and

$$\sigma_{11}(S_1^*(u), u) = S_1^*(u)\sigma(u)$$

If - as we shall verify below - Merton's model possesses an ELMM, P^* say, satisfying Condition (L), then, by Theorem 5.2, we can rewrite (10) as

$$S_1^* = S_1^*(0) + \int_0^{\cdot} S_1^*(u)\sigma(u)\, dZ^*$$

which has the unique weak solution

$$S_1^* = S_1^*(0)\exp\left\{ -\frac{1}{2}\int_0^{\cdot} \sigma^2(u)\, du + \int_0^{\cdot} \sigma(u)\, dZ^* \right\}$$

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[32] Several authors have subsequently shown that processes of the specified form can be constructed in such a way that the bond price converges to par at maturity with probability one.

which is a martingale, since σ is deterministic. It follows, by Corollary 5.4, that full pricing by arbitrage alone holds, ie every contingent claim whose normalized payoff depends solely on normalized stock price histories, is priced by arbitrage alone. This category of contingent claims includes, in particular, the payoff on a call option on the stock, expiring at *T*, since the bond price equals par at that date, and thus the payoff:

$$\max\{\ S_1(T)-K,\ 0\ \} = \max\left\{\ \frac{S_1(T)}{S_0(T)}-K,\ 0\ \right\} = \max\{\ S_1^*(T)-K,\ 0\ \}$$

(where *K* is the exercise price) is a function of the normalized stock price.

It remains to establish that Merton's model does indeed have an ELMM satisfying Condition (L), and that it is viable. By Theorem 5.3, it will suffice to exhibit an ELMM satisfying Conditions (L) and (S). From our assumptions, (4) holds with price of risk process $\theta = \mu / \sigma$; hence, by Theorem 4.8, ELMMs satisfying Condition (L) correspond to strictly positive, and locally square integrable, martingales η which can be written in the form:

$$\eta = \mathcal{E}\left\{\ N - \int_0^{\cdot} \theta\, dZ\ \right\} \qquad (11)$$

for some *N* orthogonal to *Z*. Setting $N \equiv 0$ provides, trivially, a special case of (11). We can ensure that this special case defines a square integrable martingale by applying any one of a number of well-known sufficient conditions on θ .[33] Since the bond matures at par at *T*, the corresponding ELMM satisfies Condition (S).

7. MODELS REQUIRING MARKET PRICE OF RISK(S)

We have concentrated so far on models in which contingent claims are priced by arbitrage alone. As indicated in Section 1, however, there are many models in which arbitrage arguments are supplemented by explicit assumptions about the market prices of one or more sources of risk ("*MPR models*"). These prices of risk enter the models by either (or both) of two routes.

The first route is illustrated by Merton [1976]. To price the option, he decomposed the overall risk of the stock into Brownian-risk and jump-risk parts, and made an explicit assumption as to how the market prices one of them.[34] Translated into the modern martingale approach, the model has multiple EMMs, which assign different values to the option; Merton's approach to this difficulty can be viewed as dividing the local martingale component, of (normalized) stock price movements, into Brownian and jump sub-components, and specifying the change of drift which the EMM imposes on one of them.

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33 One of the simplest would be to suppose that θ is essentially bounded.

34 Merton selected the jump-risk component and imposed a zero price of that risk. Other choices are possible.

The second route is illustrated by Wiggins [1987] and by Vasicek [1977]. Wiggins required an assumption about the market price of unanticipated changes in volatility. In this case, unlike that of Merton [1976], the local martingale component of (normalized) stock prices is not decomposed; rather, the assumed market price of risk imposes a change of drift on the volatility process. Unlike Merton [1976] or Wiggins [1987], Vasicek [1977] makes no explicit assumptions at all about the form of security price processes; instead, his assumptions centre on taking as primitive a process for the instantaneous spot interest rate. He has to supplement arbitrage arguments by a market price of risk, which can be thought of as specifying a change of drift for the interest rate process under risk-adjusted probabilities. Vasicek does not demonstrate that his model can be supported in general equilibrium.

The models just discussed share a feature, which we take to be characteristic of MPR models. Under a probability measure used for pricing, the normalized price processes, S^* , of any primitive securities, must, of course, become martingales. MPR models have the feature that the pricing measure is further characterized, by imposing a specified change of drift upon other primitive processes.

As just indicated, MPR models involve sets of primitive processes which extend beyond securities prices - or even do not include them at all. By contrast, in the opening paragraph of their seminal paper, Harrison and Kreps [1979] restricted their attention to models in which "one takes as given [only] the price dynamics of certain securities". MPR models are, therefore, outside the scope of HK' results - most crucially on viability. Of course, one can make indirect use of HK' results, by obtaining - by whatever means - a candidate solution for the prices of whatever contingent claims are of interest, and then testing whether, treating the candidate price processes as if they were primitive, an EMM satisfying Condition (S) exists. This indirect approach is implicit in (eg) Hull and White [1990] (especially footnotes 1 and 3). We contend, however, that it suffers from significant limitations.

Firstly, it may not always be clear - at least without potentially significant effort - whether the assumptions of a particular model suffice to make the candidate solution unique. Secondly, so long as the viability of MPR models is handled only indirectly, it is hard to assess what mathematical properties the market price(s) of risk(s) must possess to yield a viable model. Thirdly, the economic interpretation of what is going on is not fully articulated. In particular, the relationship remains obscure between, on the one hand, the idea underlying HK' equivalent martingale measure concept, namely that securities price processes should, in some sense, be "fair bets", and, on the other hand, the specification of market prices of risk.

The foregoing discussion has signalled our view that a <u>direct</u> analysis of the viability of MPR models is required, akin to that which Harrison and Kreps [1979] provided for models whose primitive processes are all securities prices. By "direct", we mean that it should be possible to determine the viability of an MPR model at the level of the primitive processes and the specified market price(s) of risk(s), rather than having first to obtain candidate solutions for non-primitive securities prices.

To facilitate a direct analysis, we introduce a "risk pricing measure" (RPM) concept, which generalizes, to an MPR model context, the "equivalent martingale measure" (EMM) concept of HK. We establish a correspondence between the pricing operators in viable models, and expectations under RPMs, precisely analogous to the key results of HK.

Given the possibility - discussed above - of providing a partial analysis of MPR models by indirect use of HK' results, it is inevitable that our RPM concept and HK' EMM are closely related. Indeed, in some senses, they coincide (see Corollary 7.6 below). Our concept, however, can be distinguished as the more general, by three features. Firstly, the scope of our concept caters explicitly for primitive processes other than securities prices, which were excluded by HK. Secondly, we shall show that it permits a direct treatment of viability of MPR models, without any need for candidate solutions for securities prices. Thirdly, it provides more powerful economic insights. Our results, especially Corollary 7.7, will enable us to conclude explicitly that the fundamental significance of the changes of probability formally pioneered by HK lies in the specification of the market prices of all the sources of risk underlying innovations in the primitive processes, rather than in making securities "fair bets".

Having established key results on viability, therefore, we go on to provide tools for the pricing of contingent claims. The need to do so springs from our assertion that pricing by arbitrage in MPR models, like pricing by arbitrage alone, should be conducted against the background of an arbitrarily large overall information filtration. Given such a filtration, it is clear that the primitive processes of an MPR model, and the specified prices of risk, provide an explicit model of only a part of the economy. It is therefore natural to enquire whether the pricing of all the explicitly modelled risks suffices to price uniquely all contingent claims whose normalized payoffs depend solely on the evolution of the primitive processes. This motivates us to adapt the definition of the market history from Section 3., to relate to this evolution; we then extend the pricing results of that Section to an MPR model context. In our treatment of pricing of arbitrage alone, the key to applying these results lay in the characterization of (local) EMMs achieved in Theorem 4.8; we bring this result across as Theorem 7.12. We conclude the Section by sketching how the results on diffusion-based models in Section 5 extend to MPR models, and how the models discussed at the start of this Section can be formulated in our framework. Babbs and Webber [1993] exemplifies a new applied model which depends on our machinery.

We commence the above extensive programme by giving the idea of an MPR model an exact formulation:

Definition 7.1 *An MPR model takes as primitive a collection of processes* $Y_k : k = 0, \ldots, n+m$ *for some* $n \geq 0$ *and* $m \geq 0$ *, with* $n+m > 0$. *For* $k = 0,\ldots,n$, *these processes represent securities prices* S_k *; with* S_0 *satisfying Section 3. Assumption (A).*

The normalized securities price processes

$S_k^* \equiv S_k / S_0$; $k = 0, \ldots, n$

and (for $m > 0$ *) the additional primitive processes* $Y_{n+1}, \ldots, Y_{n+m}$ *satisfy Assumptions (B) and (C) of Section 4..*

By allowing for the possibility that $m=0$, our definition of MPR models includes, by way of special case, the models discussed in previous Sections. The requirement that the primitive processes include the price process of a numeraire security will be met if, for example, a process for the instantaneous spot interest rate r is assumed (or directly implied).[35]

By virtue of Assumptions (B) and (C), we have the following Lemma, which generalizes Lemma 4.4. It shows that, for $m>0$,[36] we can extend the set of orthogonal risk factors, which underly innovations in the normalized prices of primitive securities (if any), to embrace also the additional primitive processes $Y_{n+1}, \ldots, Y_{n+m}$.

Lemma 7.2 *We can choose a pairwise orthogonal 2-basis* $\{M_1, \ldots, M_n\}$ *for the stable subspace generated by the local martingale components* $\{X_1, \ldots, X_n\}$ *of the canonical decompositions of* $S_0^*, \ldots, S_n^*$. *If* $m > 0$, *we can extend this 2-basis to a pairwise orthogonal basis* $\{M_1, \ldots, M_{n+m}\}$ *spanning also the local martingale components of* $Y_{n+1}, \ldots, Y_{n+m}$. *Thus (3) holds, and we may also write, for* $k = n+1, \ldots, n+m$:

$$Y_k = A_k + \sum_{j=1}^{n+m} \int_0^{\cdot} \sigma_{jk} \, dM_j$$

where, for each j *and* k, $\sigma_{jk} \in L^2_{loc}(M_j)$.

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35 We have in mind using the compounded money market account as the zeroth asset, so that S_0 is given by (14a) in Section 8.

36 ie for MPR models not degenerate into models addressed in previous Sections

We now define our concept of risk-adjusted probabilities for MPR models: the "*risk pricing measure*" (RPM). Our new "risk pricing measure" concept generalizes HK' "equivalent martingale measure". It will be evident from Definition 7.4, that an RPM is an EMM; it might therefore seem that our concept is the more restrictive. However, as we shall see, every EMM (subject to a mild regularity condition) implies price of risk processes, and thus is an RPM. The greater generality of our new concept arises from its ability to handle any additional primitive processes explicitly; in particular, it provides the framework necessary for analysing the pricing of contingent claims when the only primitive security price process is that of the numeraire.

The considerations which led us to localise EMMs, lead us to localise our new concept also. For brevity, we state the localised concept first:

Definition 7.3 *A "local risk pricing measure" (LRPM) is a probability measure* P^* *on* $(\Omega, \mathcal{F})$ *, equivalent to P, such that each* S_k^* *is a local martingale under* P^* *and such that (if* $m > 0$ *) there exist predictable "price of risk" processes* θ_j ; $j = n+1, \ldots, n+m$ *such that, for* $j = n+1, \ldots, n+m$,

$$M_j^* \equiv \int_0^\cdot \theta_j \, d\langle M_j, M_j \rangle + M_j \tag{12}$$

is a local martingale under P^* .

Definition 7.4 *A "risk pricing measure" (RPM) is an LRPM such that each* S_k^* *is a martingale under* P^* .

Note that, if only the numeraire security process is given exogenously (ie $n=0$), any LRPM is an RPM.

It is vital to appreciate how the market prices of risk enter the model. We shall see (Corollary 7.7 below) that (local) risk pricing measures are associated with market price of risk processes, $\theta_1, \ldots, \theta_{n+m}$. Where applicable (ie for $n>0$), the first n market prices of risk, $\theta_1, \ldots, \theta_n$, relate to the sources of innovations in the normalized prices of the primitive securities; they are implied - just as in Section 4. - by taking the price processes of those securities as primitive. By contrast, the remaining m prices of risk, $\theta_{n+1}, \ldots, \theta_{n+m}$, relate to the further sources of innovations which enter the picture through the additional primitive processes, $Y_{n+1}, \ldots, Y_{n+m}$. Since these latter processes are not securities prices, we cannot infer prices of risk from them[37]; instead, $\theta_{n+1}, \ldots, \theta_{n+m}$ are supplied as additional assumptions of the model.

Our next Theorem and its first Corollary show that - as intimated above - the apparent restrictiveness of our "risk pricing measure" (RPM) concept *vis a vis* HK' "equivalent martingale measure" (EMM) is illusory. This is because, under a mild regularity condition, (local) EMMs imply the additional prices of risk

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37 see Babbs and Selby [1992] for further discussion of this point

required for them to be (local) RPMs. These results prepare the ground for a second Corollary, whereby we shall show that the fundamental significance of these changes of measure is the specification of the market prices of risk.

Theorem 7.5 *Suppose that an MPR model possesses an ELMM,* P^* *say, satisfying Condition (L). Then there exist predictable "price of risk" processes* $\theta_1, \ldots, \theta_{n+m}$ *such that (4) holds for k = 0, ..., n, and that (if m>0), the processes* M_j^* *, defined by (12) for* $j = n+1, \ldots, n+m$ *, are local martingales under* P^* *.*

By way of Corollary, we deduce the basic equivalence of (local) RPMs to (local) EMMs.

Corollary 7.6 *Any (L)RPM is an (ELMM) EMM. Conversely, any (ELMM) EMM, satisfying Condition (L), is an (L)RPM.*

From Definition 7.3, an LRPM is associated with market price of risk processes, $\theta_{n+1}, \ldots, \theta_{n+m}$, corresponding to the sources of risk $M_{n+1}, \ldots, M_{n+m}$ which enter the picture only through the non-price primitive processes, $Y_{n+1}, \ldots, Y_{n+m}$. But, from Corollary 7.6, every LRPM, satisfying a mild regularity condition, is an ELMM and - by Theorem 7.5 - is associated with market price of risk processes $\theta_1, \ldots, \theta_{n+m}$ corresponding to all the sources of risk, $M_1, \ldots, M_{n+m}$, underlying all the primitive processes, $S_0^*, \ldots, S_n^*, Y_{n+1}, \ldots, Y_{n+m}$. Hence we have:

Corollary 7.7 *Any (local) "risk pricing measure" (ie (L)RPM) satisfying Condition (L) is associated with price of risk processes* $\theta_1, \ldots, \theta_{n+m}$ *for all the risk factors* $M_1, \ldots, M_{n+m}$ *underlying the innovations in all the primitive processes.*

Theorem 7.12 below will provide a converse to Corollary 7.7, by showing that the prices of risk determine a unique LRPM, up to an essentially arbitrary process relating to the pricing of risks orthogonal to those explicitly modelled.

We regard Corollary 7.7 as affording an important clarification of the economic significance of the change of probability measures pioneered by HK' "equivalent martingale measures" (EMMs), and now generalized by our "risk pricing measures" (RPMs).

A long-standing economic intuition is that securities should offer "fair bets" after allowing for the cost of funds - see eg Samuelson [1965] - and for a consistent set of risk premia. In HK, all the primitive processes are those of securities prices. Their key general results[38] confirmed the intuition, by showing that viable price systems are associated with changes of probabilities (ie EMMs), under which normalised securities prices[39] are martingales - ie fair bets. Certainly, EMMs have been thought of as being concerned with risk-adjustment of probabilities. The main emphasis, however, has tended to be laid, not on the pricing of risk, but on EMMs making securities "fair bets".

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[38] re-iterated in this paper as Theorem 3.5 and Corollary 3.6

[39] ie allowing for cost of funds in numeraire units

By contrast, Corollary 7.7 shows that the corresponding generalization of changes of probabilities - ie our concept of "risk pricing measures" (RPMs) - lays the fundamental emphasis on specifying the market prices of all of the sources of risk which underly innovations in the primitive processes.

HK' fundamental links between viability, pricing operators, and EMMs, generalize to our RPM concept. Indeed, our results above enable us immediately to establish results for RPMs identical in form to Theorem 3.5 and Corollary 3.6:

Theorem 7.8 *There is a one-to-one correspondence between RPMs satisfying Condition (S), and continuous and strictly positive linear pricing operators for all contingent claims. Namely, if* ψ *is a pricing operator, then we can define such an RPM,* P^* *, by:*

$$P^*(F) = \frac{\psi(S_0(T)\,1_F)}{S_0(0)}; \qquad \forall F \in \mathcal{F}$$

Conversely, if P^* *is such an RPM, we can define a continuous and strictly positive linear pricing operator,* ψ *, by:*

$$\psi(x) = S_0(0)\,\mathrm{E}^*\left[\frac{x}{S_0(T)}\right]; \qquad \forall x \in L^2(\mathcal{F}, P)$$

Corollary 7.9 *A model is viable if and only if there exists at least one RPM satisfying Condition (S).*

In previous Sections, we paid especial attention to contingent claims whose normalized payoffs were $\mathcal{G}$ -measurable, ie depended solely on the history of the normalized prices of the primitive securities. In an MPR model, the set of primitive processes includes - even only includes - processes other than securities prices. We have seen that our RPM concept is associated with market prices of the sources of risk underlying all the primitive processes. It is therefore natural to ask whether the pricing of all these risks suffices to price uniquely all contingent claims whose normalized payoffs depend solely on the evolution of the extended set of primitive processes, $S_0^*, \ldots, S_n^*, Y_{n+1}, \ldots, Y_{n+m}$. The first step is to extend Definition 3.8:

Definition 7.10 *The "MPR market history"* $\{\mathcal{G}_t : t \in [0,T]\}$ *(briefly* $\{\mathcal{G}_t\}$ *) is the completion of the sub-filtration generated by* $S_0^*, \ldots, S_n^*, Y_{n+1}, \ldots, Y_{n+m}$. *We let* $\mathcal{G} = \mathcal{G}_T$.

We say that "*full pricing by arbitrage*" holds if all $\{\mathcal{G}_t\}$ -measurable claims are priced by arbitrage. Again by analogy, here with Theorem 3.9, we have:

Theorem 7.11 *The following are equivalent, under an arbitrary information filtration:*

(a) *all RPMs yield the same finite-dimensional distributions for* S^* *and* $(Y_{n+1},\ldots,Y_{n+m})$ *;*

(b) *all RPMs coincide on* $\mathcal{G}$ *;*

(c) *full pricing by arbitrage.*

As was the case with the analogous result in Section 3, Theorem 7.11 focuses attention upon the multiple pricing measures existing under an arbitrarily large overall filtration. These arise, of course, because while, by Corollary 7.7, an LRPM is associated with market prices of risk, $\theta_1,\ldots,\theta_{n+m}$, for all the sources of risk underlying the primitive processes, that is not to say that $\theta_1,\ldots,\theta_{n+m}$ determine an LRPM exactly. Specifically, specifying the prices of the risk factors $M_1,\ldots,M_{n+m}$ leaves undetermined the market prices of other sources of risk in the economy. One would, however, hope that the question of these other prices of risk would be a separate issue, in the sense that $\theta_1,\ldots,\theta_{n+m}$ would determine an LRPM uniquely up to an essentially arbitrary process relating to the pricing of risks orthogonal to those explicitly modelled. That this is indeed the case is the import of the following theorem, which carries Theorem 4.8 over to the generalized setting of MPR models:

Theorem 7.12 *Suppose that (4) holds. Then any LRPM,* P^* *, satisfying Condition (L), has conditional expectations process*

$$\eta(t) \equiv \mathrm{E}\left[\frac{dP^*}{dP} \,\middle|\, \mathcal{F}_t \right]$$

satisfying:

$$\eta = \mathcal{E}\left\{ N - \sum_{j=1}^{n+m} \int_0^{\cdot} \theta_j \, dM_j \right\} \tag{13}$$

for some $N \in \mathcal{H}^2_{0,loc}$ *orthogonal to each* M_j *. Conversely, if a probability measure* P^* *, equivalent to* P*, is defined by*

$$\frac{dP^*}{dP} = \eta(T)$$

where η *is a strictly positive, and locally square-integrable, martingale under* P*, satisfying (13), then* P^* *is an LRPM satisfying Condition (L).*

We can interpret any particular N as specifying - in aggregate - the market prices of these risks orthogonal to those explicitly modelled in our primitive processes. Subject to a technical regularity condition, we can go further and disaggregate N in terms of the prices attaching to a full enumeration of these additional risks.[40]

The converse part of Theorem 7.12 is a key result for applications, as it shows how a particular choice of $\theta_{n+1},\dots,\theta_{n+m}$ fits into a "recipe" for an LRPM.

In the context of pricing by arbitrage alone, we showed in Sections 5 and 6 how key results from Sections 3 and 4 - notably Theorem 3.9 and Theorem 4.8 - can be used to obtain full pricing by arbitrage alone in diffusion-based models. We leave it to the reader to see how the corresponding results in this Section - namely Theorems 7.11 and 7.12 - carry over to give full pricing by arbitrage for diffusion-based MPR models.

It is straightforward to recast existing MPR models in the framework established in this Section. For example, Wiggins' [1987] stochastic volatility model has three primitive processes representing: a bond price corresponding to a constant riskless interest rate; a stock price; and the volatility of the stock price. This model is obviously an MPR diffusion model. The results of Section 5., extended as indicated in our last paragraph, ensure that full pricing by arbitrage holds, once the market price of volatility risk has been specified, even if the information filtration is allowed to be arbitrary. Moreover, subject to mild regularity conditions on the price of volatility risk, the model is viable. An analogous treatment can be applied to Vasicek [1977]; further discussion of term structure models based on the instantaneous spot interest rate r is given in Section 8. below.

The primitive processes in Merton [1976] both represent securities prices: a bond as in Wiggins [1987]; and a stock. Thus all the primitive processes refer to securities - in the notation of our definition of MPR models (Definition 7.1), $m=0$. However, the stock price process is sufficiently complex that, as noted earlier, the model has multiple EMMs. In this Section, our "risk pricing measure" (RPM) concept has generalized the EMM idea, and given us the insight that such changes of measure correspond to specifying the market prices of all the sources of risk. If therefore, following Merton, we assign zero prices to jump risks, then - if we suppose that the filtration is in fact restricted to that generated by the diffusion and pure jump components of the stock price innovations - only a single RPM remains, and contingent claims are uniquely priced. With a more general filtration, further assumptions appear to be required

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40 If the terminal sigma-algebra $\mathcal{F}$ is separable, then we can enumerate the additional risks by extending the 2-basis $\{M_1,\dots,M_{n+m}\}$ to an at most countable 2-basis $\{M_1,\dots,M_{n+m},\dots\}$ and express N in the form

$$N = -\sum_{j=n+m+1}^{\infty}\int_0^{\cdot}\theta_j\,dM_j$$

(see Jacod pp125-132, especially Corollaire 4.42).

to handle possible interdependence between jump risks on the stock and non-primitive risks elsewhere in the economy. Babbs and Webber [1993] approach this latter issue by assuming that any non-primitive risks which bear a non-zero price, do not charge a common jump time[41] with the jump risks they model explicitly.

8. ILLUSTRATION

The power of our results in Section 7. is probably most evident in models where only the numeraire security price process is given by the primitive assumptions. In particular, the analogue of Theorem 4.8, coupled with Corollary 7.9, provide that such models are viable, subject only to easily satisfied regularity conditions on the market price(s) of risk. We illustrate this below by reference to term structure models based solely on an exogenous process for the instantaneous spot interest rate r satisfying Assumptions (B) and (C). Cox, Ingersoll and Ross [1985] (CIR) pointed out that, in such a model, an inappropriately specified price of risk could guarantee rather than eliminate arbitrage opportunities. Their suggested solution was to rely solely on term structure models depending, like their own celebrated CIR term structure model ("the CIR model"), upon an explicitly constructed general equilibrium. Our results provide an alternative solution, by identifying viable prices of risk for an exogenously specified process for r.

We regard this kind of alternative solution - and indeed the features of our "risk pricing measure" (RPM) which it illustrates - as an extremely powerful tool for constructing new models. To obtain concrete pricing results for contingent claims, based on an explicit general equilibrium, requires very strong assumptions. The CIR model is a case in point. It is assumed that the evolution of the entire economy can be expressed in terms of a single state variable, following an exogenously specified square root diffusion (their (13) p390). In particular, the information filtration is limited to that generated by the sole state variable. All agents are assumed to have logarithmic utility (p389). By contrast, as we shall see, an RPM-based approach can operate under an arbitrary filtration - not necessarily expressible in terms of any finite number of state variables; only r is required to follow an exogenously specified process. Agents' preferences need not be specified beyond the market price of the source of risk underlying innovations in r. In particular, we need not specify agents' attitudes to other sources of risk.

41 Two processes are said to charge a common jump time if there is a positive probability of jumps occurring in the two processes simultaneously.

In the notation of Definition 7.1, the type of term structure model with which we are concerned is an MPR model with $n=0$, $m=1$, and we can write

$$S_0 = \exp\left\{ \int_0^{\cdot} r(u)\,du \right\} \tag{14a}$$

$$Y_1 = r = A_1 + \int_0^{\cdot} \sigma_{11}\,dM_1 \tag{14b}$$

where: A_1 is a predictable and locally integrable process of finite variation, $\sigma_{11} \in L^2_{loc}(M_1)$, and $M_1 \in \mathcal{H}^2_{0,\,loc}$. Suppose that we wish to know whether a predictable price of risk process θ leads to a viable model. Our results establish that θ is viable, provided there exists a probability measure P^*, equivalent to P, and satisfying Conditions (L) and (S), such that

$$M_1^* = \int_0^{\cdot} \theta\, d\langle M_1, M_1\rangle + M_1 \tag{15}$$

is a local martingale under P^*.

Let us consider an example, inspired by our recent discussion of the CIR model. Suppose that the sole process assumed as primitive concerns the instantaneous spot interest rate, r, and takes the CIR form:

$$dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}\,dZ \tag{16}$$

where κ, μ, σ are strictly positive constants, and Z is a standard Brownian motion. Relating (16) to (14b), we set:

$$A_1 = \int_0^{\cdot} \kappa(\mu - r)\,du$$

$$\sigma_{11} = \sigma\sqrt{r}$$

and

$$M_1 = Z$$

With these choices, (15) becomes:

$$M_1^* = \int_0^{\cdot} \theta\,du + Z$$

and to obtain a viable model all we need is to choose θ in such a way that M_1^* becomes a local martingale under some RPM P^* satisfying Conditions (L) and (S). As at the end of Section 6., we note that a number of well-known sufficient conditions exist, including essential boundedness of θ. Moreover, under these conditions M_1^* is a standard Brownian motion under P^*. Under the RPM, the dynamics of r can be written:

$$dr = \{\kappa(\mu - r) - \theta\sigma\sqrt{r}\}\,dt + \sigma\sqrt{r}\,dM_1^*$$

9. SUMMARY AND CONCLUSIONS

We have submitted that assertions that various contingent claims can be priced by arbitrage should be in a setting in which markets are necessarily incomplete markets, as a result of the overall information filtration being arbitrarily large.

Moreover, required returns must be allowed to depend upon the overall filtration, and not just upon the past market history. We noted that a substantial literature on pricing by arbitrage alone has violated these principles.

This paper has provided a number of results to permit pricing by arbitrage alone to be carried out in our preferred setting. A special concern has been to show that existing results, obtained under a restricted information filtration, and with restrictions on required returns, remain valid.

We said that "full pricing by arbitrage alone" holds if all possible equilibrium pricing systems assign identical prices to those contingent claims whose normalized payoffs depend solely on the paths of the normalized prices of primitive securities. Under an arbitrary information filtration, full pricing by arbitrage alone is equivalent to all equivalent martingale measures (EMMs) coinciding on the sigma-algebra generated by those paths, and to all EMMs agreeing the finite-dimensional distributions of normalized prices. We have shown that linear factor risk pricing holds, under mild regularity conditions, and deduced a characterization of ELMMs - equivalent measures under which normalized prices become local martingales. These results enabled us to show that existing pricing results, obtained by arbitrage alone in diffusion-based models under a restricted filtration, remain valid under an arbitrary filtration. An illustration showed that options are priced by arbitrage alone in Merton [1973] - a result that eluded Harrison and Kreps [1979] (HK).

We then turned our attention to a wider class of models, in which arbitrage arguments have to be supplemented by specifying the market prices of one or more sources of risk. We noted that such "market price of risk" (MPR) models involve primitive processes other than securities prices. As a consequence, they therefore fall outside the direct scope of the results on viability due to HK.

Our results enabled us to conclude explicitly that the fundamental significance of the change of probability measure introduced by HK' "equivalent martingale measure" (EMM) and generalized in our "risk pricing measure" (RPM) concept, consists of identifying the market prices of risks in the economy, rather than in making securities martingales - "fair bets".

An illustration outlined how our new RPM concept can be used to check that suggested price of risk processes give rise to viable models, rather than - as in a cautionary tale told by Cox, Ingersoll and Ross [1985] - give rise to arbitrage anomalies.

APPENDIX

__Proof of Theorem 3.9__ $(c) \Rightarrow (b)$ Suppose that the EMMs do not all coincide on $\mathcal{G}_T$. Then there exist $A \in \mathcal{G}$ and EMMs P_1, P_2, such that $P_1(A) \neq P_2(A)$. Consider the contingent claim $X = 1_A S_0(T)$, where 1_A denotes the indicator function of A. By construction, the normalized payoff of X is $\mathcal{G}$-measurable. However, by Theorem 3.5, the prices assigned to X

under P_j, $j = 1,2$ are given (with obvious notation) by: $V_j = S_0(0)\ \mathrm{E}^{(j)}[1_A] = P_j(A)$ (where $\mathrm{E}^{(j)}$ denotes the expectation operator under P_j), which differ according to the choice of j. The result follows by contradiction.

$(b) \Rightarrow (c)$ Suppose all EMMs coincide on $\mathcal{G}$. Let X be a contingent claim whose normalized payoff is $\mathcal{G}$-measurable, and P^* any EMM. Then, by Theorem 3.5, the value of X under P^* is:

$$V = S_0(0)\ \mathrm{E}^*\left[\frac{X}{S_0(T)}\right] \qquad (A1)$$

where E^* is the expectation operator for P^*. Now, *ex hypothesi*, $X / S_0(T)$ is $\mathcal{G}_T$-measurable, and all the EMMs coincide on $\mathcal{G}$. Thus the RHS of (A1) is independent of the choice of P^*.

$(b) \Rightarrow (a)$ is trivial.

$(a) \Rightarrow (b)$ is an immediate application of the following lemma. ∎

<u>Technical Lemma</u> *Let S^* be any $\mathfrak{R}^n$-valued process defined on a measurable space $(\Omega, \mathcal{F})$. Suppose that $\mathcal{P}$ is a family of probability measures on $(\Omega, \mathcal{F})$ for which the finite-dimensional distributions of S^* coincide. Then the members of $\mathcal{P}$ coincide on $\{\mathcal{G}_t\}$, the filtration generated by S^*.*

<u>Proof of Technical Lemma</u> Let $\mathcal{J}^{n \times k}$ denote the $(n \times k)$-fold product of $\mathcal{J}$, the collection of subsets of $\mathfrak{R}$ of the forms: $\varnothing$, $(-\infty, b]$, $(a,b]$ (a,∞), $\mathfrak{R}$, where $a, b \in \mathfrak{R}$. Define $\mathcal{D}$ as the collection of elements of $\mathcal{G}$ of the form:

$$\{\omega \in \Omega \ :\ (S^*(t_1), \ldots, S^*(t_k)) \in J\} \qquad (A2)$$

where $t_1 < \ldots < t_k$ and $J \in \mathcal{J}^{n \times k}$. It is elementary to verify that $\mathcal{D}$ is a semi-algebra[42].

Ex hypothesi, the members of $\mathcal{P}$ coincide on $\mathcal{D}$. Let μ denote their common restriction to $\mathcal{D}$. Clearly μ is countably additive and $\mu(\Omega) = 1$. Hence, by Caratheodory's Extension Theorem, μ has a unique extension to $\sigma(\mathcal{D})$, the sigma-algebra generated by $\mathcal{D}$. Thus the members of $\mathcal{P}$ must agree on $\sigma(\mathcal{D})$.

But, for any t, $S^*(t)$ is $\sigma(\mathcal{D})$-measurable[43], whence $\mathcal{G} = \sigma(\mathcal{D})$. The result follows. ∎

<u>Proof of Lemma 4.4</u> See Jacod [1979] Proposition (4.41) pp131-2 (specialised to the case of a finite set), and Jacod [1979] pp129-130, especially (4.36). ∎

<u>Proof of Theorem 4.7</u> Let P^* be any ELMM satisfying Condition (L), and let η be defined by (5). Then η is a locally square-integrable martingale; hence[44], we can express η as:

$$\eta = 1 + \sum_{j=1}^{n} \int_0^{\cdot} \alpha_j dM_j + N^* \qquad (A3)$$

where $N^* \in \mathcal{H}^2_{0,\,loc}$ and, for each j, $\alpha_j \in L^2_{loc}(M_j)$ and M_j is orthogonal to N^*.

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42 ie a family of subsets of Ω, containing Ω, closed under finite intersections, and such that: $D \in \mathcal{D} \Rightarrow \Omega \setminus D$ is expressible as a finite disjoint union of members of $\mathcal{D}$.

43 Set $n = 1$ in (A2) and recall that $\sigma(\mathcal{J})$ is the Borel sigma-algebra on $\mathfrak{R}$.

44 localising Jacod [1979] Theoreme (4.27) p126 as described on p127, and exploiting the definition of 2-basis

By the definition of an ELMM, each S_k^* is a local martingale under P^*; whence[45]

$$\eta A_k + \sum_{j=1}^{n} \eta \int_0^{\cdot} \sigma_{jk} dM_j = \eta S_k^* \qquad (A4)$$

is a local martingale under *P*.

From the integration by parts formula,[46]

$$\eta A_k = \int_0^{\cdot} \eta_- dA_k + \int_0^{\cdot} A_k d\eta \qquad (A5)$$

Now, η_- is *caglad*[47] and thus predictable[48]; hence[49], since A_k is predictable by Assumption (B), the first integral on the RHS of (A5) is predictable and of finite variation. Since, by Assumption (B), A_k is predictable and of finite variation, it is also locally bounded[50]; hence[51], the second integral on the RHS is a local martingale.

Since

$$\eta, \int_0^{\cdot} \sigma_{jk} dM_j \in \mathcal{H}^2_{loc}$$

the predictable quadratic (co-)variation

$$<\eta, \int_0^{\cdot} \sigma_{jk} dM_j>$$

is well-defined, and by "polarisation"[52],

$$\eta \int_0^{\cdot} \sigma_{jk} dM_j - <\eta, \int_0^{\cdot} \sigma_{jk} dM_j>$$

is a local martingale.

We are now in a position to rearrange (A4) as

$$\int_0^{\cdot} \eta_- dA_k + <\eta, \sum_{j=1}^{n} \int_0^{\cdot} \sigma_{jk} dM_j>$$

$$= \eta S_k^* - \int_0^{\cdot} A_k d\eta - \sum_{j=1}^{n} \left(\eta \int_0^{\cdot} \sigma_{jk} dM_j - <\eta, \int_0^{\cdot} \sigma_{jk} dM_j> \right) \qquad (A6)$$

where the LHS is predictable and of finite variation, while the RHS is a local martingale. Thus each side is a predictable local martingale of finite variation and hence[53] constant (in fact zero).

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45 by Elliott [1982] Lemma 13.10 p161

46 see Jacod [1979] Theoreme (2.53) (b) p47

47 ie left continuous with right limits. As a semimartingale, η has *cadlag* paths. It is simple to prove from first principles that if a process is *cadlag*, its left limits process is *caglad*.

48 by Jacod and Shiryaev [1987] Proposition 2.6 p17

49 by localising Jacod and Shiryaev [1987] Proposition 3.5 p28

50 As a finite variation process, A_k is the difference of two increasing processes, *B* and *C*, say; moreover, since A_k is predictable, we can choose predictable *B* and *C* (see eg Jacod [1979] Lemme 1.35 p17). Applying Jacod [1979] Lemme 1.37 p17 to *B* and *C* separately, we find that each of them is locally bounded. It is then trivial to show that A_k is locally bounded also.

51 by Elliott [1982] Theorem 11.44 p121

52 *cf* Jacod [1979] pp33-4

53 by Elliott [1982] Lemma 11.39 p121

Now

$$<\eta, \sum_{j=1}^{n}\int_0^{\cdot}\sigma_{jk}dM_j> \;=\; <\sum_{j=1}^{n}\int_0^{\cdot}\alpha_j dM_j, \sum_{j=1}^{n}\int_0^{\cdot}\sigma_{jk}dM_j> \;+\; <N^*, \sum_{j=1}^{n}\int_0^{\cdot}\sigma_{jk}\,dM_j>$$

which, by the various orthogonality relations[54],

$$= \sum_{j=1}^{n}\int_0^{\cdot}\alpha_j\sigma_{jk}\,d<M_j, M_j> \qquad (A7)$$

Substituting (A7) into (A6), we obtain

$$\int_0^{\cdot}\eta_- dA_k \;+\; \sum_{j=1}^{n}\int_0^{\cdot}\alpha_j\sigma_{jk}\,d<M_j, M_j> \;=\; \text{constant}$$

which, since the integrands are predictable, implies[55]

$$A_k \;=\; S_k^* \;-\; \sum_{j=1}^{n}\int_0^{\cdot}\frac{\alpha_j}{\eta_-}\sigma_{jk}\,d<M_j, M_j> \qquad (A8)$$

In (A8), since P^* was an arbitrary ELMM,

$$\theta_j \;\equiv\; -\frac{\alpha_j}{\eta_-} \qquad (A9)$$

must be independent of the choice of P^*. The results follow immediately. ■

<u>**Proof of Theorem 4.8**</u> Let P^* be any ELMM. As was shown in the proof of Theorem 4.7 - see especially (A3) and (A9), we can write the conditional expectations process, η, of the Radon-Nikodym derivative of P^* with respect to P as:

$$\eta \;=\; 1 \;-\; \sum_{j=1}^{n}\int_0^{\cdot}\eta_-\theta_j\,dM_j \;+\; N^* \qquad (A10)$$

where $N^* \in \mathcal{H}^2_{0,\,loc}$ is orthogonal to each M_j.

From its definition, $\eta > 0$; so, by Ito's lemma, $1/\eta$ is a semimartingale and thus *cadlag*. Hence both η_- and $1/\eta_-$ are *caglad*. Now, trivially,

$$N^* \;=\; \int_0^{\cdot}dN^* \;=\; \int_0^{\cdot}\eta_-\frac{1}{\eta_-}dN^*$$

which, by the associativity of stochastic integration with *caglad* integrands[56],

$$= \int_0^{\cdot}\eta_-\,dN \qquad (A11)$$

where

$$N \;\equiv\; \int_0^{\cdot}\frac{1}{\eta_-}dN^*$$

Moreover,[57] $N \in \mathcal{H}^2_{0,\,loc}$ and is orthogonal[58] to each M_j.

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54 *cf* Jacod [1979] p125

55 we exploit the associativity of stochastic integration where the integrands are predictable; see Protter [1990] Theorem 21 p135

56 see Protter [1990] Theorem 19 p55

57 by Protter [1990] Theorem 20 p56

58 by Jacod [1979] Corollaire 2.29 pp36-7 and Proposition 2.51 p47, and Protter [1990] Theorem 17 p106

Substituting (A11), we can rewrite (A10) as

$$\eta = 1 + \int_0^{\cdot} \eta_- d\left\{ N - \sum_{j=1}^{n} \int_0^{\cdot} \theta_j \, dM_j \right\}$$

whose unique solution[59] is precisely (6), as required.

For the converse, let P^* and η be as specified. We need to prove that each S_k^* is a local martingale under P^*. Proceeding in reverse, as it were, through the arguments in the first part of the current proof, we can re-express η in the form:

$$\eta = 1 - \sum_{j=1}^{n} \int_0^{\cdot} \eta_- \theta_j \, dM_j + N^* \qquad (A12)$$

where

$$N^* = \int_0^{\cdot} \eta_- \, dN \in \mathcal{H}^2_{0,\,loc} \qquad (A13)$$

is orthogonal to each M_j.

Substituting (A12) into (A6)[60], and exploiting (4) and (A13), we find that the LHS of (A6) vanishes, leaving us with:

$$\eta S_k^* = \int_0^{\cdot} A_k \, d\eta + \sum_{j=1}^{n} \left(\eta \int_0^{\cdot} \sigma_{jk} \, dM_j - \langle \eta, \int_0^{\cdot} \sigma_{jk} \, dM_j \rangle \right) \qquad (A14)$$

where, as in the proof of Theorem 4.7, the terms on the RHS of (A14) are local martingales under *P*. Thus each ηS_k^* is a local martingale under *P*, whence[61] each S_k^* is a local martingale under P^*, as required. ▮

Proof of Theorem 5.2 Theorem 4.7 determines A_k. Given any ELMM, P^*, satisfying Condition (L), the Radon-Nikodym derivative with respect to *P* is determined by Theorem 4.8. We can now apply Girsanov's Theorem[62] to deduce that, for $j = 1,\ldots,n$:

$$Z_j^* = \int_0^{\cdot} \theta_j du + Z_j$$

is a continuous local martingale under P^*. Moreover the predictable quadratic (co)variation processes $\langle Z_i^*, Z_j^* \rangle^*$ under P^*, coincide with $\langle Z_i, Z_j \rangle$ under *P*, whence[63] Z^* is a vector standard Brownian motion under P^*. ▮

Proof of Lemma 7.2 The first part is simply Lemma 4.4. Further use of the arguments cited there[64] yields the second part. ▮

Proof of Theorem 7.5 For $n=0$, (4) holds trivially. If $n>0$, the existence of $\theta_1, \ldots, \theta_n$ such that (4) holds for $k = 0, \ldots, n$, is given by Theorem 4.7.

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[59] see eg Jacod [1979] Theoreme 6.2 p190

[60] Given our hypotheses, it is easily verified that (A6) holds.

[61] by Elliott [1982] Lemma 13.10 p161

[62] see eg Karatzas and Shreve [1988] Proposition 5.4 p194

[63] see eg Karatzas and Shreve [1988] Theorem 3.16 p157

[64] ie Jacod [1979] Proposition (4.41) pp131-2 (specialised to the case of a finite set), and Jacod [1979] pp129-130, especially (4.36)

Define η by (5). Theorem 4.8 now gives us (6). We can[65] decompose N as:

$$N = N_0 + \sum_{j=n+1}^{n+m} \int_0^{\cdot} \theta_j dM_j$$

where $N_0 \in \mathcal{H}^2_{0,loc}$ is orthogonal to each M_j, and, for each j, $\alpha_j \in L^2_{loc}(M_j)$. The result now follows by (eg) Elliott [1982] Theorem 13.19 p165. ▮

Proof of Corollary 7.6 The result follows immediately from Theorem 7.5 and the respective definitions. ▮

Proof of Corollary 7.7 By Corollary 7.6, any (L)RPM is an (ELMM) EMM. The result now follows immediately from Theorem 7.5. ▮

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[65] As in the proof of Theorem 4.7 we localise Jacod [1979] Theoreme (4.27) p126 as described on p127, and exploit the definition of 2-basis

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