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American Featured Options

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1 Introduction

In this paper we discuss the pricing and hedging of options in which the holder can make a decision at some point during the life of the option which can alter the payoff of the option. We consider as specific examples; standard American and Bermudan options, compound options, chooser options and shout options. We use these examples to show how more complex American featured options can be analysed to determine appropriate pricing and hedging methods.

2 American and Bermudan Options

A standard American option (so called because they first began trading on American exchanges) gives the holder the right to buy or sell (call or put) the underlying asset S for the predetermined strike (or exercise) price K at any time up to the maturity date T. In reality American options can only be exercised at their most frequent on a daily basis, that is, the underlying price against which they are settled is fixed once each business day, usually close to the end of trading. Bermudan options (or Mid-Atlantic options, so called because they are "half-way" between European and American) give the holder the right to buy or sell (call or put) the underlying asset S for the predetermined strike (or exercise) price K at a predetermined set of dates $t_i; i = 1,...,n$ up to the maturity date $T \equiv t_n$. Thus, in a sense, all American options are in fact Bermudan. However the term Bermudan is usually applied to options which are exercisable less frequently than daily. The relationship between the price of a Bermudan put option and the number of exercise dates is shown in 2.1. The price converges very quickly, adding more exercise dates beyond daily does not add much to the value of the option. Therefore, the standard exchange traded daily exercisable American options can be considered to be effectively true American options.

INSERT FIGURE 2.1 HERE

Notes: $K = 100$, $T - t = 1.0$ year, $S_t = 100$, $\sigma = 30\%$, $r = 5\%$.

A common example of a Bermudan option is a callable bond. For example a company may issue a five year semi-annual bond maturing in September 2001 which is callable in September of years 1998, 1999, and 2000. The callable feature is effectively an embedded Bermudan option with exercise dates in September of 1998, 1999, and 2000. Typically, an investor who wants to hedge against the bond being called would buy a Bermudan option on a synthetic bond with exercise dates on the call dates of the corporate bond. Then if the bond issuer does call the bond the investor can exercise the Bermudan option to recover the market value of the bond and purchase a substitute bond. The callable bond effectively has embedded in it a short position in a Bermudan call option on the bond. That is the bond purchaser will simultaneously sell the issuer a Bermudan call option on the bond. Therefore the price of the callable bond should be less than an otherwise identical non-callable bond. The valuation of the callable bond therefore require the use of Bermudan option pricing.

One of the most important factors affecting the pricing of American and Bermudan options is the nature of the cash flow stream associated with the underlying asset, for example the dividend stream in the case of an option on a share or equity index. Consider an American option on a share which pays a single known dividend at a known future date. On the ex-dividend date the share price will fall to reflect the fact that investors in the share will no longer receive the dividend. In reality the share price will usually fall by less than the cash amount of the dividend reflecting discounting of the cash amount back to the ex-dividend date and also tax effects. We define the dividend D_j to be the known cash amount by which the share price falls on the known ex-dividend date t_j. For example if the actual dividend is $1 and the share price falls by 90% of this on the ex-dividend date then $D_j = \$0.9$. Now imagine there are m dividends $D_j\,; j = 1,\ldots,m$ due between now and the maturity date of the option with ex-dividend dates $t_j\,; j = 1,\ldots,m; t_{m+1} \equiv T$. Hull shows that for an American call option it will not be optimal to exercise the option early at t_j if

$$D_j \le K\left(1 - e^{-r(t_{j+1} - t_j)}\right) \tag{2.1}$$

Therefore the dividend yield on the share would have to be close to the riskless rate of interest for early exercise to be optimal. Note, however, that early exercise is more likely for in-the-money options. American puts are more complicated, but dividends make it less likely that an American put will be exercised early and if $D_j \ge K\left(1 - e^{-r(t_{j+1} - t_j)}\right), \forall j$ then an American put should never be exercised early.

In some cases the asset can be considered to pay a continuous dividend stream δ. For example, an index on a large number of stocks whose dividend dates are spread uniformly throughout the year. In the case of a foreign exchange option the foreign interest rate can be considered a continuous dividend stream (see Garman and Kolhagen). In these cases the dividend stream can be taken into account by subtracting δ from the riskless rate of interest in the risk neutral drift of the asset price.

In general the pricing of American or Bermudan options is a free boundary problem. The free boundary is the level of the asset price at which it is optimal to exercise the option early at each date and this is unknown. Therefore there is no simple analytical solution for value of an American option and analytical approximations or numerical methods must be used[1].

The quadratic formula of Barone-Adesi and Whaley for American options on assets with a continuous dividend stream is a quasi-analytical approximation that is fast and accurate for typical parameters values. However, it cannot be used for American options on assets with

[1] See Clewlow and Strickland for details of the use of the numerical methods described here for pricing options.

discrete dividends or for Bermudan options. Other analytical approximations include Geske and Johnson based on compound options, Omberg with an exponential exercise boundary and the methods of lines of Carr and Faguet based on the perpetual American option solution.

Numerical methods can be divided into three categories: lattices or trees, finite difference methods and Monte Carlo simulation. The first lattice for pricing American options which appeared in the literature was the binomial method of Cox, Ross, and Rubinstein (CRR) which is in fact an example of dynamic programming. Amin and Khanna proved that the binomial tree is a convergent method and some generalisations to multinomial lattices are given by Boyle Lattices are very easy to implement and have the advantage of being easily adapted to value more complex American featured options.

2.1 The Binomial Method

The idea of the binomial method is as follows; Firstly, we imagine that over a small period of time Δt the asset price can only change from its current level S by going up to uS with a probability of p or down to dS with a probability $1-p$ where u, d and p are constants which depend on the drift and volatility of the asset price. Under this assumption we can construct a binomial tree which represents the entire future random evolution of the asset price up to the maturity date of the option, this is depicted in 2.1.1.

INSERT FIGURE 2.1.1 HERE

We refer to a state in the tree as a node and label the nodes (i,j) where i indicates the number of time steps from time zero and j indicates the number of upward movements the asset price has made since time zero and there are N time steps in total. Therefore the level of the asset price at node (i,j) is $Su^i d^{i-j}$.

It is straightforward to show (see CRR) that it is possible to set up a riskless hedge in the same way as for the Black-Scholes model. Therefore, at any node in the binomial tree, the value of the option $v_{i,j}$ must be its discounted expected future value

$$v_{i,j} = e^{-r\Delta t}\left(pv_{i+1,j+1} + (1-p)v_{i+1,j}\right) \tag{2.1.1}$$

The next observation is that the value of the option at the final maturity date is known, it is simply the payoff, for example for a call option ($i = N$)

$$v_{i,j} = \max(0, S_{i,j} - K) \tag{2.1.2}$$

Therefore using equation (2.1.2) we can compute the value of the option at every node at time step N. Then using equation (2.1.1) we can compute the value of the option at time step $N-1$. We can then reapply equation (2.1.1) at every node at every time step working backwards through the tree to compute the value of the option at every node. This procedure computes the value of a European option. In order to compute the value of an American or Bermudan option we simply compare, at every node at a time step corresponding to an exercise date, the value of the option if exercised (equation with the value if not exercised (equation and set the option value at that node equal to the greater of the two. In this way we simultaneously solve for the option price and the early exercise boundary. 2.1.2 shows the value of an American put option (with exercise dates at every time step) as a function of the number of time steps.

INSERT FIGURE 2.1.2 HERE

Notes: $K = 100$, $T - t = 0.5$ year, $S_t = 100$, $\sigma = 30\%$, $r = 6\%$, $\delta = 3\%$.

A very large number of time steps are needed for the value to be independent of the number of time steps. This is not surprising, our original assumption was that the time step was such that the asset price could only move up or down by a fixed amount. This most naturally corresponds to single tick moves which typically occur over time periods of a few minutes or less. There are many papers in the literature testing different variations of the binomial tree in order to optimise the method in terms of accuracy and speed. Among them Broadie and Detemple ( ) suggested a modified tree that reduces the oscillatory convergence of the binomial approximation. The Binomial Black-Scholes tree (BBS) is calculated in the same way as the CRR tree but using the Black-Sholes formula to set the value of the American option (if not exercised) at each node of the tree at the penultimate time step. Broadie and Detemple ( ) also showed that adding two-point Richardson extrapolation to the BBS tree (BBSR) produce a significant improvement over other binomial methods. 2.1.3 compares the convergence of the CRR, BBS and BBSR approximations for an American put as a function of the number of time steps.

INSERT FIGURE 2.1.3 HERE

Notes: $K = 95$, $T - t = 0.5$ year, $S_t = 100$, $\sigma = 30\%$, $r = 6\%$, $\delta = 3\%$.

2.2 Finite Difference Methods

A better method for pricing American or Bermudan options is to solve the Black-Scholes partial differential equation using a finite difference method (for early examples see Schwartz , Brennan and Schwartz and Courtadon). The Crank-Nicolson method, introduced into the option pricing literature by Courtadon , is the best of these methods. Finite difference methods are very similar to the binomial method in that the future evolution of the asset price is represented by discrete states or nodes. However, they allow the asset price to move to more than two possible values over a small time step. The Crank-Nicolson method effectively allows the asset price to move to any node at the next time step. This allows it to represent the behaviour of the asset price accurately over much larger time steps.

Strictly, to price a Bermudan option with for example weekly exercise dates, at least daily time steps should be used but with the early exercise test applied on the weekly dates only. However, since the convergence of the Crank-Nicolson method is very fast, it is possible to set the number of time steps equal to the number of exercise dates as long as this is greater than approximately 20 time steps. It is also straightforward to handle an underlying asset which generates cashflows (see Clewlow and Strickland for details). Basically we decompose the asset into two components a risky component which follows geometric Brownian motion S_t^* and a riskless part which is the present value of the future cash flows (divdends) $D_j\ ; j = 1, \ldots, m$. Therefore the asset price is given by

$$S_t = S_t^* + \sum_{j=j_t}^{m} D_j e^{-r(t_j - t)} \tag{2.2.1}$$

$$dS_t^* = rS_t^* dt + \sigma^* S_t^* dz(t)$$

where j_t is the index of the next dividend after t. The tree or lattice is built in terms of S_t^* in the usual way and then the actual asset price S_t can be obtained from equation (2.2.1).

Mention should be made of the Barone-Adesi and Whaley approximation for American options on assets with a continuous dividend stream. This approximation is almost analytical (and therefore fast) and accurate for typical parameters values. However, it cannot be used for American options on assets with discrete dividends or for Bermudan options.

2.3 Hedging Issues

Standard American or Bermudan options are very similar to European options in terms of hedging. Consider a Bermudan option with only one early exercise date, for example half way through its life. This option will be almost identical to a European option up to the early exercise date and after the early exercise date it will be exactly a European option. Obviously the crucial difference is the early exercise opportunity and the associated early exercise boundary (EEB). Now consider an American put option, which is exercisable at any time. There is an associated EEB which is a continuous function of time and which indicates the level of the asset price at which it is optimal the exercise the option early. We can obtain an approximation for the EEB using the binomial or a finite difference method. We simply record the asset value, associated with the highest node at which we find it optimal to exercise the option early, for each time step. 2.3.1 shows the EEB obtained in this way using the Crank-Nicolson finite difference method[2] for typical parameter values.

INSERT FIGURE 2.3.1 HERE

[2] The steps in the curve are due to the discrete levels of the asset price in the Crank-Nicolson lattice.

Notes: $K = 100$, $T - t = 1.0$ years, $S_t = 100$, $\sigma = 30\%$, $r = 5\%$, 256 time steps.

For a Bermudan put option the EEB becomes a set of discrete values, one for each early exercise date, below which it is optimal to exercise the option early. At the EEB, if the holder behaves rationally, they will exercise the option and will receive the payoff of the option. If the writer has been delta hedging then they will have a position in cash and the underlying asset which covers their liability. Transaction costs and other market imperfections will affect this simple scenario, but in a similar way to European options (see Clewlow and Hodges (1994, 1996) for methods of dealing with transaction costs and other market imperfections). If the holder behaves irrationally and exercises the option before the asset price reaches the optimal level then the writer will make profit (ignoring any profit or loss from differences between the realised volatility and the implied at which the option was sold).

American calls do not differ substantially in price or sensitivities from European calls but American puts can differ quite substantially from European puts. 2.3.2 shows the difference between the delta of an American put and a European put for the same parameters as 2.3.1.

INSERT FIGURE 2.3.2 HERE

Notes: $K = 100$, $T - t = 1.0$ years, $S_t = 100$, $\sigma = 30\%$, $r = 5\%$, 256 time steps.

The peak in the surface lies along the EEB where the delta is one for the American put. So, in the case of American puts, it is important to have a model which takes into account the early exercise feature. This type of visual analysis of the price and sensitivities of the option is a key tool in determining the risk management characteristics of the option. We will use this approach throughout this paper.

3 Compound Options

A compound option is an option on an option. The usual definition is a standard European put or call on a standard European put or call. This gives four basic compound options; a call on a call, a call on a put, a put on a call and a put on a put. A compound option is very similar to a Bermudan option with a single early exercise date at the maturity date of the compound option. In the case of the compound option if it is exercised the holder receives a standard European option in exchange for the strike price otherwise nothing, whereas in the case of the Bermudan option if the option is not exercised the holder receives a European option otherwise the difference between the asset price and the strike price.

Probably the most actively traded compound options are captions and floptions, that is options on caps and floors respectively. Caps and floors are call and put options on forward money market rates. Caps and floors are answers to a very common corporate risk management problem. For example, imagine a company needs to take out a loan to help finance a new project. The company is concerned that interest rates will rise and so they would like to "cap" or limit the maximum interest rate they will have to pay on the loan. An interest rate cap is a call option on an interest rate, that is it pays an interest rate, which is the maximum of zero and the difference between the market rate and the prespecified cap rate, applied to a prespecified notional principal. In this way if market rates rise above the specified cap rate then the company receives payments which compensate for the extra interest above the cap rate which they are paying on their loan. Now, if the company will only require the loan if, for example, it succeeds in a take-over bid, then it will want an option on the cap, i.e. a caption.

More generally, compound options arise when a company will require protection, for example against interest rates going above a certain level, but contingent on a some future event, for example a project being taken on at a future date.

As another example, consider a construction company that is bidding for a contract to redevelop a dockland area into a residential area. If they are awarded the contract in twelve months time they will be required to complete the development at the bidded price. The company is exposed to the price of the raw materials and therefore needs a call option on the price of the raw materials struck at the bidded price (presumably close to the current market price), but only if the contract is awarded. A bank would not normally sell an option contingent on the granting of a contract due to the difficulty of hedging that risk. The alternative is a compound option (call on a call) struck at the money forward with a maturity date of twelve months. This allows the company to pay a small premium now to lock into the raw material price with the option of paying an additional premium (the strike price of the compound option) in twelve months time to obtain the call option protection on the raw material price.

If the contract is awarded the company should compare the market price of the required call option protection with the strike price of the compound option. If the strike price is less then the compound option should be exercised, but if the market price is less then it is cheaper to forgo the initial premium and buy the required protection directly in the market.

Similarly, even if the contract is not awarded the company should compare the cost of exercising the option (the initial premium inflated to today plus the strike price) with the market price of the underlying option. If the market price is higher then it may be possible to exercise the compound option and sell the underlying call back into the market for a profit.

3.1 Analytical Formulae and Numerical Methods

The first published solution for compound options was Geske for the pricing of a call on a call. Rubinstein (1991b) generalised this result to all four of the standard combinations. The derivation of the general formula is straightforward and we give details here as an example of how analytical formulae for American featured options can sometimes be derived. Firstly, the payoff of a compound option can be written as

$$\max\left(0, \psi c(S_\tau, K, T) - \psi k\right) \qquad (3.1.1)$$

where the compound option has strike price k and maturity date τ ($\psi = 1,-1$ for call, put) and the underlying option $c(S_\tau, K, T)$ has striking price K, maturity date $T > \tau$ and S_τ is the value of the underlying asset at time τ.

The formula for the value at time τ of the underlying option is

$$c(S_\tau, K, T) = \phi S_\tau e^{-d(T-\tau)} N(\phi z) - \phi K e^{-r(T-\tau)} N\left(\phi z - \phi\sigma\sqrt{T-\tau}\right) \qquad (3.1.2)$$

$$z = \frac{\ln\left(\frac{S_\tau e^{-d(T-\tau)}}{K e^{-r(T-\tau)}}\right)}{\sigma\sqrt{T-\tau}} + \frac{1}{2}\sigma\sqrt{T-\tau}$$

This is just the Black-Scholes formula with $\phi = 1,-1$ for a call, put. Therefore the current value (date t) of the compound option, $Compound(t)$, is the discounted expectation of the payoff

$$Compound(t) = e^{-r(\tau-t)} E_t\left[\max\left(0, \psi c(S_\tau, K, T) - \psi k\right)\right] \qquad (3.1.3)$$

this can be written as the integral of the payoff over the probability density of the asset price at date τ

$$Compound(t) = e^{-r(\tau-t)} \int_{-\infty}^{+\infty} \max\left(0, \psi c\left(Se^{u}, K, T\right) - \psi k\right) f(u)\,du \qquad (3.1.4)$$

where

$$f(u) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-v^2/2}, \quad u = \ln\left(\frac{S_\tau}{S}\right), \quad v = \frac{u - \mu(\tau-t)}{\sigma\sqrt{\tau-t}}, \text{ and } \mu = (r-d) - \frac{\sigma^2}{2}$$

To evaluate the integral we first note that the payoff is only positive when $\psi c\left(Se^{u}, K, T\right) > \psi k$ since the underlying option price is monotonic in the asset price S_τ. If X is the value of S_τ at which $\psi c\left(Se^{u}, K, T\right) = \psi k$ then we have

$$Compound(t) =$$

$$e^{-r(\tau-t)} \int_{\ln(X/S)}^{+\infty} \left[\psi\left(\phi S_\tau e^{-d(T-\tau)} N(\phi z) - \phi K e^{-r(T-\tau)} N\left(\phi z - \phi\sigma\sqrt{T-\tau}\right)\right) - \psi k\right] f(u)\,du \qquad (3.1.5)$$

We can then integrate the three terms inside the square brackets separately as follows

$$\psi\phi Se^{-d(T-\tau)}\int_{\ln\left(X/S\right)}^{\psi\infty} e^{u}N(\phi z)f(u)du = \psi\phi Se^{(r-d)(\tau-t)-d(T-\tau)}N_2(\psi\phi x,\phi y;\psi\rho)$$

$$\psi\phi Ke^{-r(T-\tau)}\int_{\ln\left(X/S\right)}^{\psi\infty} N\left(\phi z-\phi\sigma\sqrt{T-\tau}\right)f(u)du =$$
$$\psi\phi Ke^{-r(T-\tau)}N_2\left(\psi\phi x-\psi\phi\sigma\sqrt{\tau-t},\phi y-\phi\sigma\sqrt{T-t};\psi\rho\right)$$

$$\psi k\int_{\ln\left(X/S\right)}^{\psi\infty} f(u)du = \psi kN\left(\psi\phi x-\psi\phi\sigma\sqrt{\tau-t}\right)$$

where $N_2(.,.;\rho)$ is the standard bivariate normal distribution function with correlation coefficient ρ. Putting these results together we obtain

$$\begin{aligned}&Compound(t) = \\ &\psi\phi Se^{-d(T-t)}N_2(\psi\phi x,\phi y;\psi\rho)-\psi\phi Ke^{-r(T-t)}N_2\left(\psi\phi x-\psi\phi\sigma\sqrt{\tau-t},\phi y-\phi\sigma\sqrt{T-t};\psi\rho\right) \\ &-\psi ke^{-r(\tau-t)}N\left(\psi\phi x-\psi\phi\sigma\sqrt{\tau-t}\right)\end{aligned} \quad (3.1.6)$$

where

$$x=\frac{\ln\left(Se^{-d(\tau-t)}/Xe^{-r(\tau-t)}\right)}{\sigma\sqrt{\tau-t}}+\frac{1}{2}\sigma\sqrt{\tau-t},\quad y=\frac{\ln\left(Se^{-d(T-t)}/Ke^{-r(T-t)}\right)}{\sigma\sqrt{T-t}}+\frac{1}{2}\sigma\sqrt{T-t},$$
$$\rho=\sqrt{(\tau-t)/(T-t)}$$

and we have made use of standard results for the integral of the product of a normal density and a cumulative normal distribution for the first two integrals. X is obtained as the solution to the equation

$$\phi X e^{-d(T-\tau)} N(\phi z') - \phi K e^{-r(T-\tau)} N\left(\phi z' - \phi\sigma\sqrt{T-\tau}\right) - k = 0$$

where

$$z' = \frac{\ln\left(Xe^{-d(T-\tau)} / Ke^{-r(T-\tau)}\right)}{\sigma\sqrt{T-\tau}} + \frac{1}{2}\sigma\sqrt{T-\tau}$$

which can be solved easily using the standard Newton-Raphson algorithm.

The formula for the compound option (3.1.6) involves the bivariate cumulative normal distribution. This comes from the fact that the option price depends on the joint distribution of the asset price at the maturity dates of compound and underlying options. In general a compound (or Bermudan) option with n exercise dates will depend on the n-variate density of the asset price at these dates and the price will involve up to n-variate cumulative normal distributions distribution (see for example Selby and Hodges). The compound option can be viewed as a standard option which is paid for in two instalments but where the second instalment is optional. This idea can be generalised to the instalment option in which the option is paid for in n instalments where all but the first are optional (Karsenty and Sikorav). An instalment option price which is paid for in n instalments involves the n-variate cumulative normal. However, these options can be valued in a similar way the Bermudan options using a binomial tree or finite difference grid. At each instalment date the value of the option at every node is set equal to the maximum of zero and the difference between the current value and the instalment payment. That is, if the instalment payment is more than the value of the option then the instalment should not be paid and value of the compound/instalment option is zero.

3.1 Hedging Issues

The first point to note is that, for compound option calls, the higher the second optional payment (the strike price of the compound option k) the lower the initial compulsory payment and vice versa for puts (see 3.1.1). In particular if the strike price is zero the compound option call is equivalent to the underlying option.

INSERT FIGURE 3.1.1 HERE

Notes: $k = 6$, $\tau - t = 0.5$, $K = 100$, $T - t = 1.0$ years, $S_t = 100$, $\sigma = 20\%$, $r = 5\%$, $\delta = 3\%$.

The underlying asset of a compound option can be considered to be the underlying option. So the compound option can be hedged in the same way as a normal option but with a position in the underlying option instead of the underlying asset. However, in reality, transaction costs are much higher for options and so hedging this way is likely to be expensive. Alternatively the compound option can be hedged using the asset underlying the underlying option. Analysis of the sensitivities of a call on a call and a call on a put show that they have similar sensitivities to the underlying call and put respectively. Puts on calls and puts on puts are more complex. 3.1.2 and 3.1.3 show the price. delta, gamma and vega surfaces with respect to the underlying asset of a put on a call and a put on a put.

INSERT FIGURE 3.1.2 AND 3.1.3 HERE

Notes: $k = 6$, $\tau - t = 0.5$, $K = 100$, $T - t = 1.0$ years, $S_t = 100$, $\sigma = 20\%$, $r = 5\%$, $\delta = 3\%$.

Firstly the deltas quite different to standard calls or puts, they are peaked near the strike price like the delta of a short and long straddle respectively. Consequently the gammas of these options can be positive or negative depending on the level of the underlying asset relative to the strike price. The most critical aspect of these options is the speed at which their vega or their sensitivity to volatility changes. The expected costs of managing this exposure should certainly be taken into account when pricing these types of compound options.

4 Chooser Options

Chooser options allow the holder to choose at some predetermined future date whether the option will be a call or a put with predetermined strike price and maturity date. This option is similar to a straddle, i.e. a portfolio of a call and a put, but is cheaper because the holder must choose between a call or put at the predetermined date.

A chooser, in a similar way to a straddle, can be thought of as a way of speculating on an extreme move in the market. A straddle is appropriate when the investor has no view on the likely direction of the move. A chooser would be appropriate when the investor believes information will become available in the future which will indicate the direction of the market move.

For example, imagine there is an election in three months time and it is not clear what the result will be and how the markets will react. The investor therefore buys an at-the-money chooser option on the market index with a choice date three months from now on options which expire six months from now. If, in three months time when the election results are announced, the index goes up the call option will be worth more than the put option. The investor would choose the call option and sell the option for an immediate profit. Conversely if the index falls in response to the election results then the put option would be worth more than the call and the investor would choose the put option and sell it for a profit.

The type of chooser described above is a simple chooser in which the strike price and maturity date of the call and put are identical. In a complex chooser the call and put have different strike prices and maturity dates. This complicates the pricing of the chooser slightly, but an analytical formula (similar to the compound option) can still be obtained.

The choice date is obviously the key parameter. If the choice date is today then the value of the chooser is simply the maximum of the values of the call and put. For a simple chooser, if the choice date is equal to the maturity dates of the call and put then the value of the chooser is the sum of the values of the call and put since the option has become a straddle.

Interestingly, in the Black-Scholes world, allowing the investor to choose at any time up to some date does not add any value to the option. This can be seen by recognising that the value of the chooser is an increasing function of the time to the choice date so it is optimal for the investor to wait as long as possible. Of course in the real world, being able to choose at any time is valuable in the sense that it allows an immediate profit from a move in the market at any time. Rubinstein (1991a) showed that analytical formulae for both simple and complex chooser could be derived. We give details of the derivations of the formulae in the following sections.

4.1 Analytical Pricing of Simple Chooser Options

A simple chooser option allows the holder to choose at some predetermined future date τ whether the option is a standard call or put with predetermined strike price (K) and maturity date T. Therefore the payoff from a simple chooser option at the choice date can be written as

$$\max\left(c(S_\tau, K, T), p(S_\tau, K, T)\right) \tag{4.1.1}$$

Using put-call parity this can be written as

$$\begin{aligned} &\max\left(c(S_\tau, K, T), \left(c(S_\tau, K, T) - Se^{-d(T-\tau)} + Ke^{-r(T-\tau)}\right)\right) \\ &= c(S_\tau, K, T) + \max\left(0, Ke^{-r(T-\tau)} - Se^{-d(T-\tau)}\right) \end{aligned} \tag{4.1.2}$$

So the payoff from a simple chooser option today will be the same as the payoff from,

buying a call with underlying asset price S, striking price K and maturity date T.

buying a put with underlying asset price $Se^{-d(T-\tau)}$, striking price $Ke^{-r(T-\tau)}$ and maturity date τ.

We can therefore write the formula for a simple chooser as

$$Simple\_chooser(t) = \left(Se^{-d(T-t)}N(x) - Ke^{-r(T-t)}N\left(x - \sigma\sqrt{T-t}\right)\right) \qquad (4.1.3)$$

$$+\left(Ke^{-r(T-t)}N\left(-y + \sigma\sqrt{\tau - t}\right) - Se^{-d(T-t)}N(-y)\right)$$

where

$$x = \frac{\ln\left(Se^{-d(T-t)} / Ke^{-r(T-t)}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, \quad y = \frac{\ln\left(Se^{-d(T-t)} / Ke^{-r(T-t)}\right)}{\sigma\sqrt{\tau - t}} + \frac{1}{2}\sigma\sqrt{\tau - t}$$

4.2 Analytical Pricing of Complex Chooser Options

The complex chooser generalises the simple chooser by allowing the underlying standard call and put to have different strikes and maturity dates. The payoff from a complex chooser option can be written as,

$$\max\left(c(S_\tau, K_1, T_1 - \tau), p(S_\tau, K_2, T_2 - \tau)\right) \qquad (4.2.1)$$

where the chosen call (put) has striking price K_1 (K_2) and maturity dates T_1 (T_2) on the choice date τ.

Therefore the current value of a complex chooser option is the discounted expectation of its payoff

$$Complex\_chooser(t) = e^{-r(\tau-t)}E_t\left[\max\left(c(S_\tau, K_1, T_1), p(S_\tau, K_2, T_2)\right)\right] \qquad (4.2.2)$$

this can be written as the integral of the payoff over the probability density of the asset price at the choice date

$$Complex\_chooser(t) = e^{-r(\tau-t)}\int_{-\infty}^{+\infty}\max\left(c\left(Se^u, K_1, T_1\right), p\left(Se^u, K_2, T_2\right)\right)f(u)du \qquad (4.2.3)$$

where

$$f(u) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-v^2/2}, \; u = \log\left(\frac{S_\tau}{S}\right), \; v = \frac{u - \mu(\tau - t)}{\sigma\sqrt{\tau - t}} \text{ and } \mu = (r-d) - \frac{\sigma^2}{2}$$

To evaluate this integral we note that since the underlying call and put are monotonic functions of the asset price then the integration space can be divided into two regions. In the lower region we integrate over the put price and in the upper region we integrate over the call price. The regions are divided at the value of the asset price which makes the call and put prices equal. We therefore obtain

$$Complex\_chooser(t) =$$
$$e^{-r(\tau-t)}\left[\int_{-\infty}^{\ln(X/S)} p(Se^u, K_2, T_2) f(u)du + \int_{\ln(X/S)}^{\infty} c(Se^u, K_1, T_1) f(u)du\right] \quad (4.2.4)$$

where X is the solution to

$$c(X, K_1, T_1) = p(X, K_2, T_2)$$

This can be evaluated in a similar way to the compound option to give

$$Complex\_chooser(t) =$$
$$Se^{-d(T_1-t)}N_2(x, y_1; \rho_1) - K_1 e^{-r(T_1-t)}N_2\left(x - \sigma\sqrt{\tau - t}, y_1 - \sigma\sqrt{T_1 - t}; \rho_1\right) \quad (4.2.5)$$
$$+K_2 e^{-r(T_2-t)}N_2\left(-x + \sigma\sqrt{\tau - t}, -y_2 + \sigma\sqrt{T_2 - t}; \rho_2\right) - Se^{-d(T_2-t)}N_2(-x, -y_2; \rho_2)$$

where

$$x = \frac{\ln\left(Se^{-d(\tau-t)} / Xe^{-r(\tau-t)}\right)}{\sigma\sqrt{\tau - t}} + \frac{1}{2}\sigma\sqrt{\tau - t}, \quad y_i = \frac{\ln\left(Se^{-d(T_i-t)} / K_i e^{-r(T_i-t)}\right)}{\sigma\sqrt{T_i - t}} + \frac{1}{2}\sigma\sqrt{T_i - t},$$

$$\rho_i = \sqrt{\frac{\tau - t}{T_i - t}}$$

and X solves the following equation

$$Xe^{-d(T_1-\tau)}N(z_1) - K_1e^{-r(T_1-\tau)}N\left(z_1 - \sigma\sqrt{T_1-\tau}\right) +$$
$$Xe^{-d(T_2-\tau)}N(-z_2) - K_2e^{-r(T_2-\tau)}N\left(-z_2 + \sigma\sqrt{T_2-\tau}\right) = 0$$

where

$$z_i = \frac{\ln\left(Xe^{-d(T_i-\tau)} / K_ie^{-r(T_i-\tau)}\right)}{\sigma\sqrt{T_i-\tau}} + \frac{1}{2}\sigma\sqrt{T_i-\tau}$$

4.3 Hedging Issues

In the case of the simple chooser hedging is straight forward. Since the simple chooser decomposes exactly into a portfolio of a call and put option then, if the required strike and maturity are available "cheaply" in the market, these can be used to perfectly hedge the chooser. Often, the precise strike and maturity are not available, and in the case of the complex chooser there is not an exact decomposition. However, a combination of a quasi-static hedge in the underlying standard options and a delta hedge of the residual risk works well for these options. 4.3.1 shows the residual price risk of a chooser option after taking positions in the underlying options chosen to minimise the overall squared difference.

INSERT FIGURE 4.3.1 HERE

Notes: $\tau - t = 0.5$, $K = 100$, $T - t = 1.0$ years, $S_t = 100$, $\sigma = 20\%$, $r = 5\%$, $\delta = 3\%$.

The positions in the underlying options give a good overall hedge which wouldn't need to be adjusted until quite close to the choice date.

5 Shout Options

Shout options were first described by Thomas . They are standard European options except that the holder can "shout" to the writer once at any time during the life of the option to set the minimum payoff to be equal to the current asset price less the strike price. It is helpful to consider the relationship of shout options to other similar options. Fixed strike lookback options are standard options except that level of the asset price which determines the payoff is the maximum (for a call) or minimum (for a put) price the asset achieves over the life of the option. So, shout options can thought of as fixed strike lookback options where the holder "shouts" the date at which the maximum or minimum price is set. Shout options are much cheaper than lookback options because the best price is not automatically set. 5.1 shows the price surfaces of a fixed strike lookback call and an equivalent shout call option.

INSERT FIGURE 5.1 HERE

Notes: $K = 100$, $T - t = 1.0$ years, $S_t = 100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 0\%$.

The lookback option is worth 28.17777 with one year to expiry and a current asset price of 100 whereas the shout option is only worth 17.953791. The lookback option becomes worth relatively more as the asset price increases because best price is automatically set whereas for the shout option it will have been set at the optimal shout level of the asset price (the optimal shout level is discussed in section 5.2). Shout options can also be thought of as a type of American option where the holder has the option to exercise into a European option struck at the current asset price plus a cash amount equal the difference between the current asset price and the predetermined strike price. This interpretation will become clear in the next section on pricing shout options.

The main use of a shout option is for an investor who has a view on the timing of a future temporary move in the market. For example, an investor might think that the FTSE 100 will undergo a temporary downward correction in the next six months will subsequently recover. The investor could wait until the downward correction happens and then buy an at-the-money call option anticipating the recovery in the index. But the investor may be worried about increased implied volatilities after the correction. Another alternative would be to buy a floating strike lookback call (payoff= $\max(0, S_T - \min(S_t; t \in [0, T]))$) or a fixed strike lookback put (payoff = $\max(0, K - \min(S_t; t \in [0,T]))$) but this would be expensive and wouldn't allow the investor to exploit their ability to predict the bottom of the correction. A shout put option would be the ideal instrument for the investor (payoff = $\max(0, K - S_T, K - S_{t_s})$ where S_{t_s} is the asset price when the holder shouts). It would allow them to obtain the option they require at todays, hopefully more favourable, implied volatilities, and it would be much cheaper than a lookback option, allowing them to call the timing of the bottom of the correction.

5.1 Pricing Shout Options

The payoff to a shout call option can be written as

$$\begin{array}{ll} \max(0, S_T - K) & \text{if not shouted} \\ \max(S_T - K, S_{t'} - K) & \text{if shouted} \end{array} \tag{5.1.1}$$

If we consider the case where the holder has shouted and S_{t_s} is a known constant, the payoff of the option can be written as

$$\max(0, S_T - S_{t_s}) + (S_{t_s} - K) \tag{5.1.2}$$

So the payoff is the same as a standard European call option with strike price equal to S_{t_s} plus a cash amount $S_{t_s} - K$ and we can value it analytically. Furthermore, since the option can only be shouted once we can use the same methods as for American/Bermudan options to value the shout option. That is at every node in the tree or lattice we set the value equal to the maximum of the discounted expectation and the value if shouted.

5.2 Hedging Issues

As we can see from 5.1 a shout call option is very similar to a European call option. In the same way as for American options the key factor in shout options is the optimal shout level for the underlying asset. We can obtain an approximation for this in the same way as for the EEB of an American option by locating the level at which it is first optimal to shout within an binomial or finite difference lattice. 5.2.1 shows the optimal shout boundary of the shout call option in 5.1[3].

INSERT FIGURE 5.2.1 HERE

Notes: $K = 100$, $T - t = 1.0$ years, $S_t = 100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 0\%$.

It is interesting to compare the EEB for the American put in 2.3.1 with the optimal shout boundary for the shout call option in . The two boundaries clearly have different functional forms but both exhibit the characteristic dependence on the square root of the time to maturity. This is because the fundamental determinant of the boundary is the probability that the asset price will reach a particular level before the maturity date and this is directly related to the square root of time to the maturity date.

6 Conclusions

In this paper we have discussed the pricing and hedging of options which have an American feature. That is, the holder can make a decision at some point during the life of the option which alters the payoff of the option. The specific examples we described were standard American and Bermudan options, compound options, chooser options and shout options. We used these examples to show how complex American featured options can be analysed to determine appropriate pricing and hedging methods. Although it is often possible to obtain analytical formulae for complex or exotic options these can take time to be developed and the window of opportunity in the market is often very short. Therefore, numerical or approximation methods are usually more appropriate as they allow solutions to be developed quickly and their flexibility allows the incorporation of special features which the buyer may require. Finally it is important to realise that the search for the exact price for an option under a particular model is misguided. Since we know that any model will be an approximation to reality the exact model price will not be the "correct" market price. The key issues are the

[3] The steps in the curve are due to the discrete levels of the asset price in the Crank-Nicolson lattice.

risks which the option is exposed to and the cost of the actual hedging strategy in the market. The hedging strategy can and should be analysed by simulation. The cost of the hedge is a lower bound on the price you should be willing to sell the option for in the market.

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AMERFEAT.XLS, Fig 2.1

Figure 2.1: The Price of a Bermudan Put Option as a Function of the Number of Early Exercise Dates

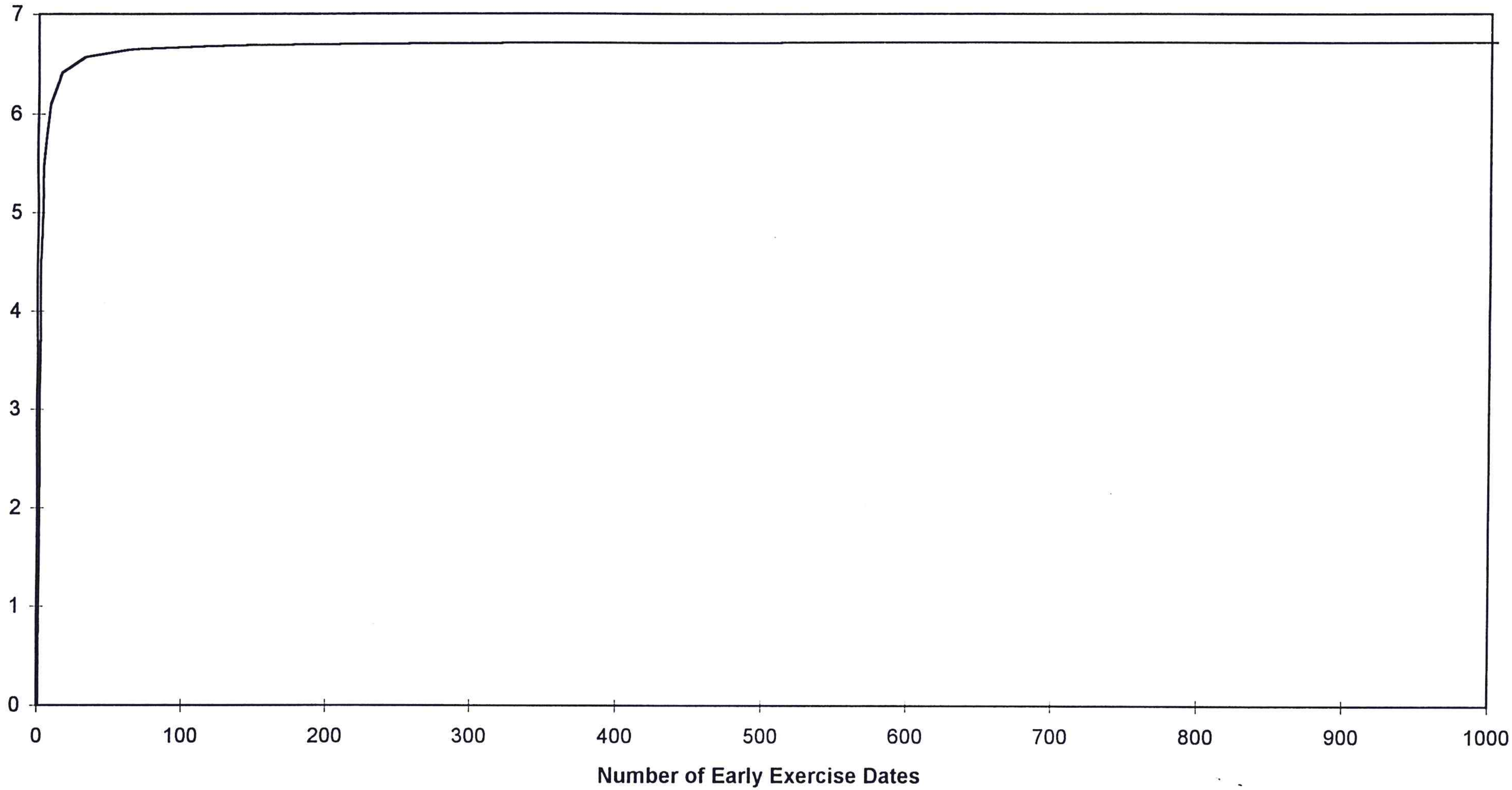


Figure 2.1.1: A Binomial Tree

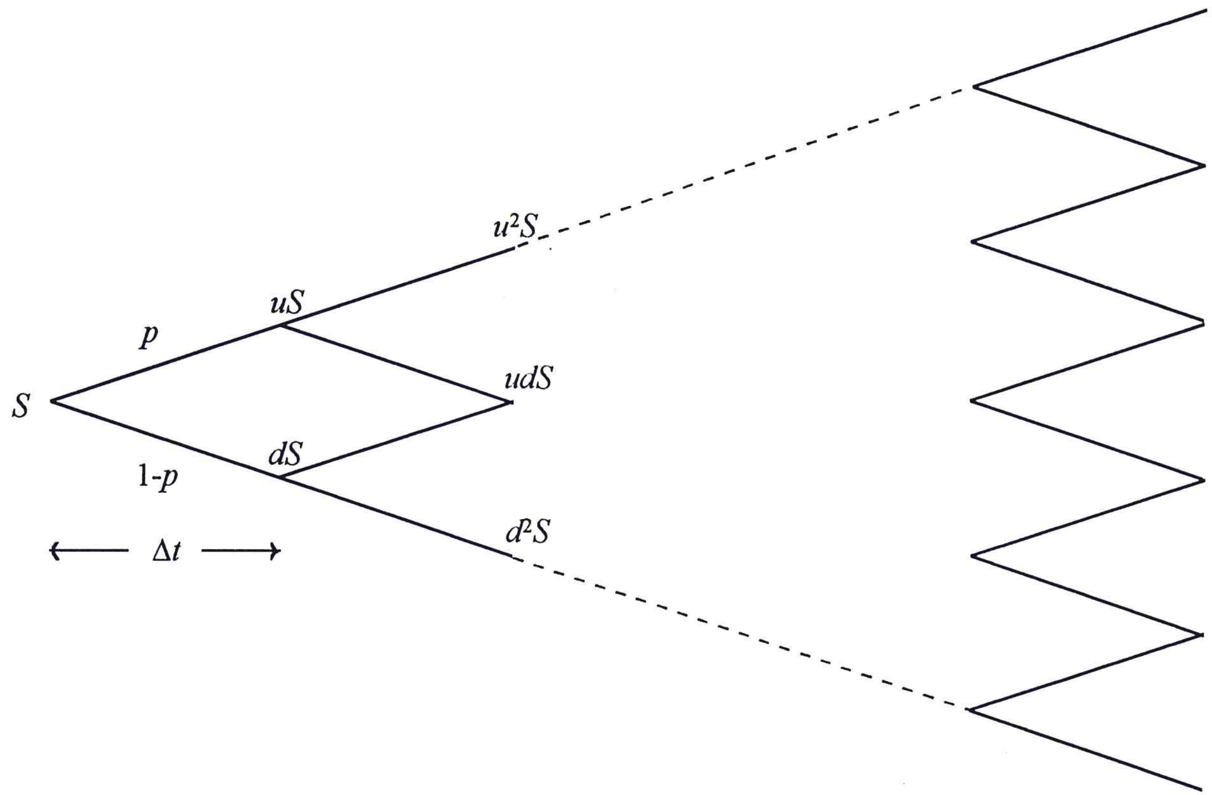


Figure 2.1.2: The Price of an American Put Option Computed by the Binomial Method as a Function of the Number of Time Steps

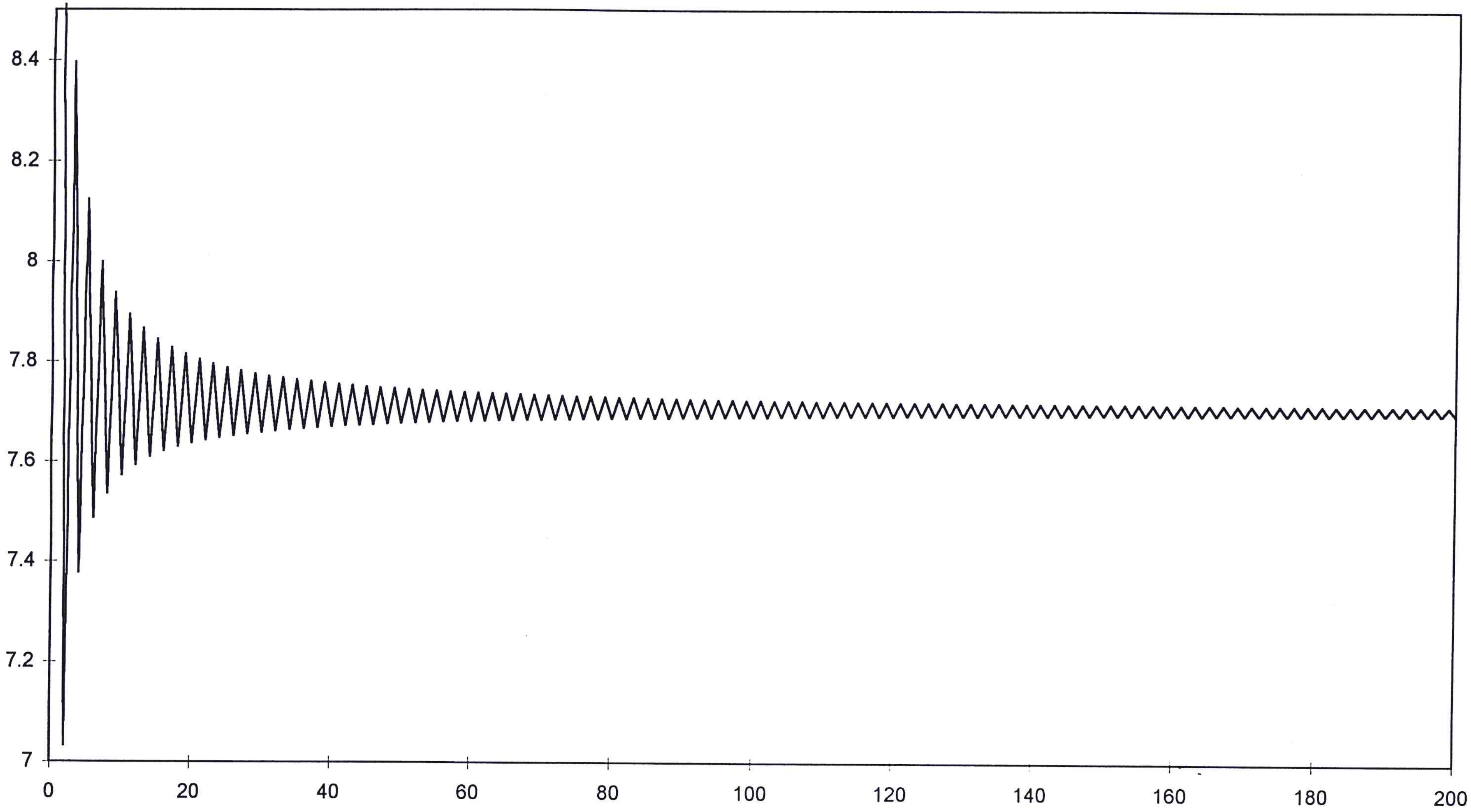
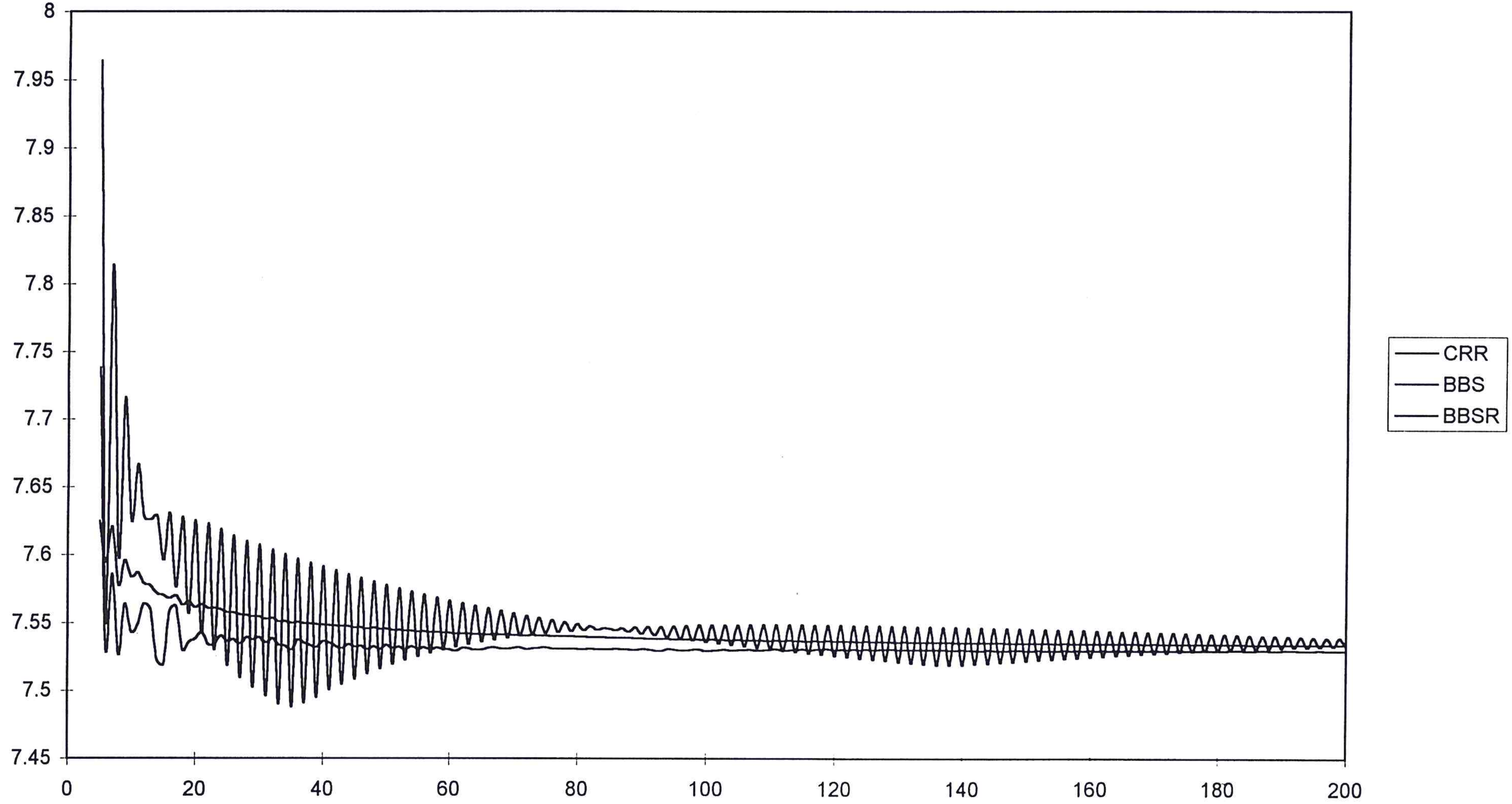


Figure 2.1.3: Oscillatory Convergence of the American Put Option Computed by the Binomial Method as a Function of the Number of Time Steps



AMERFEAT.XLS, Fig 2.3.1

Figure 2.3.1: The Early Exercise Boundary of an American Put Computed by the Crank-Nicolson Finite Difference Method

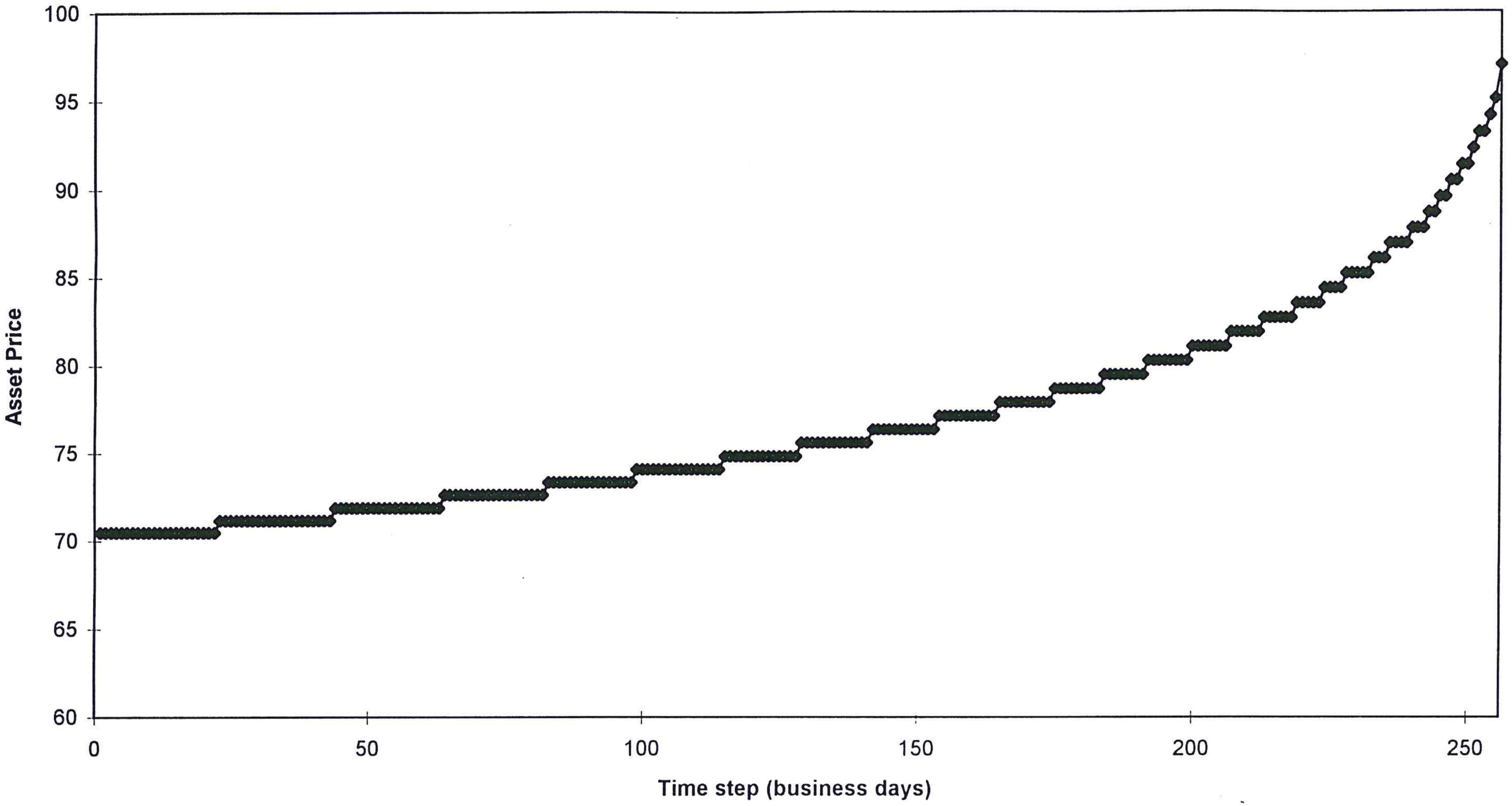
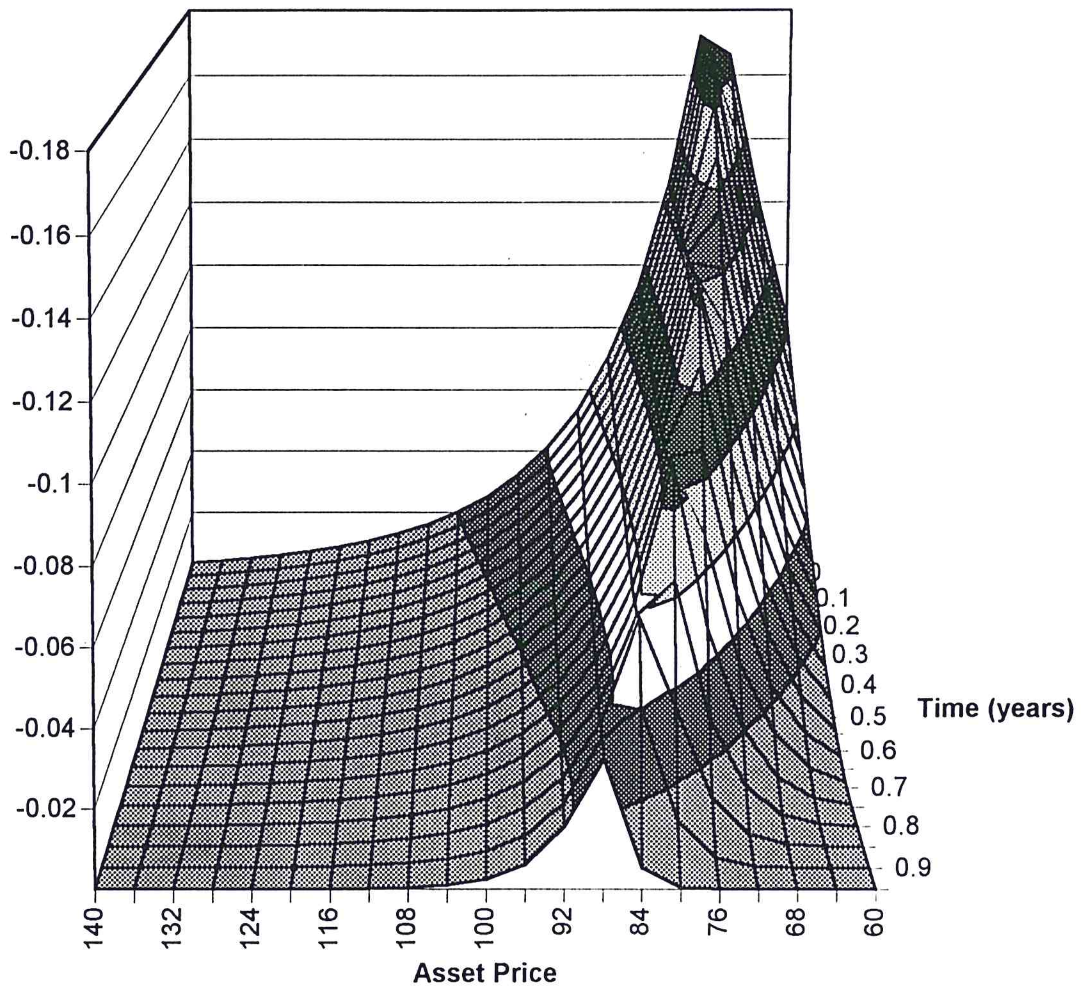


Figure 2.3.2: The Difference Between the Delta for an American Put and a European Put



AMERFEAT.XLS, Fig 3.1.1

Figure 3.1.1: The Prices of Compound Options as a Function of the Strike Price

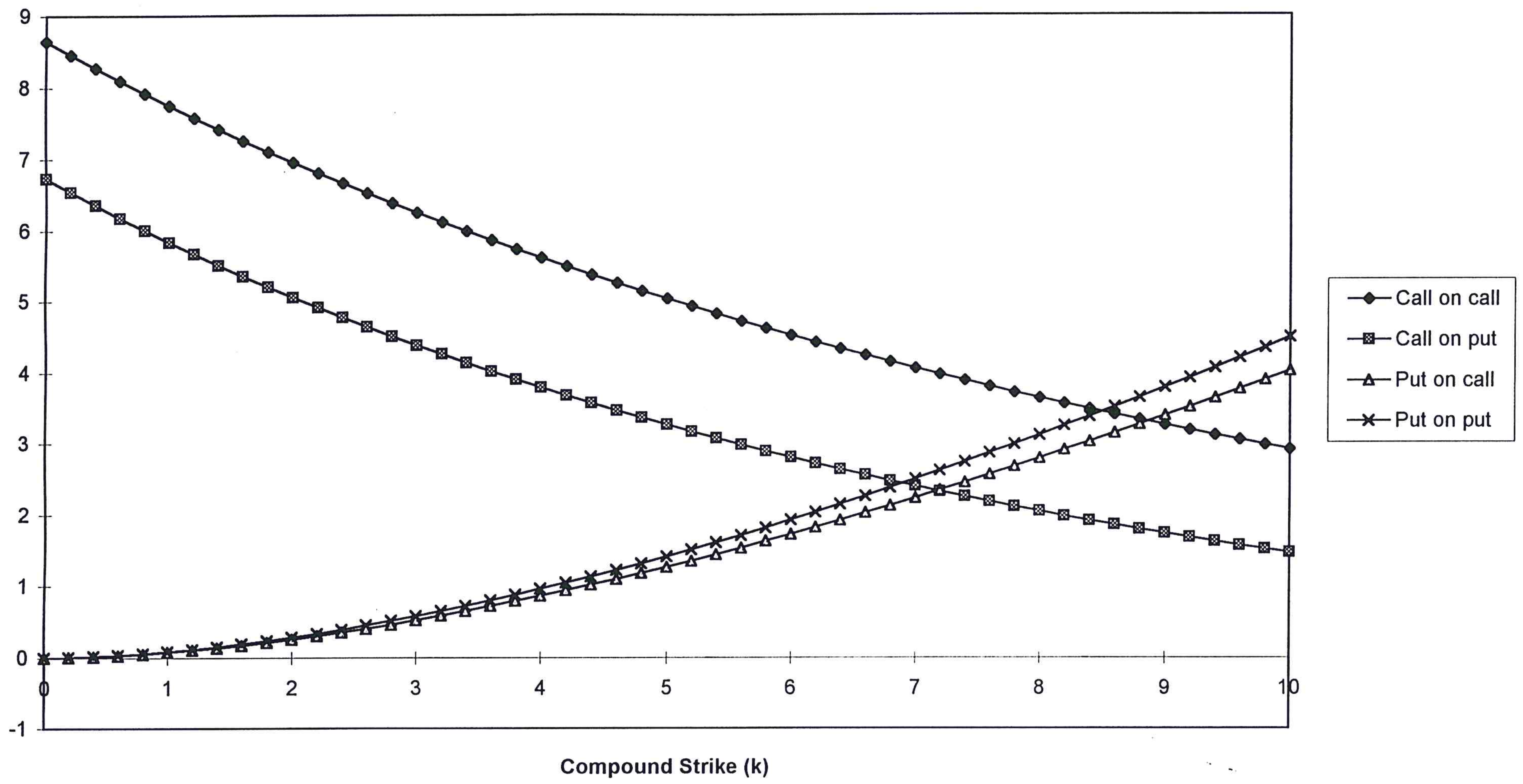
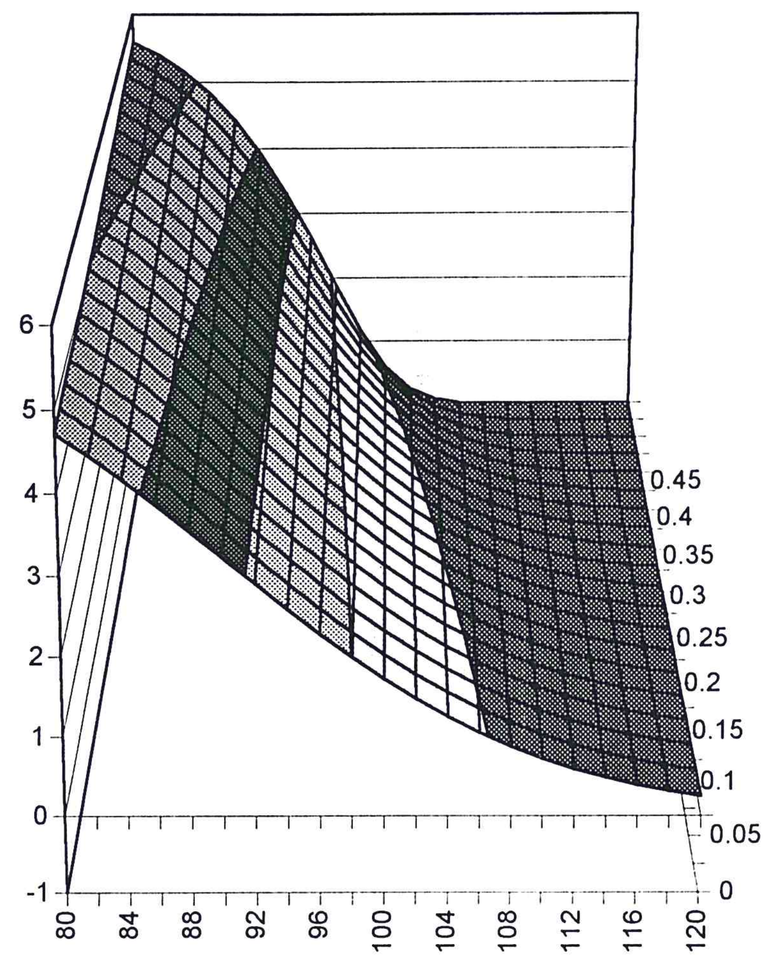


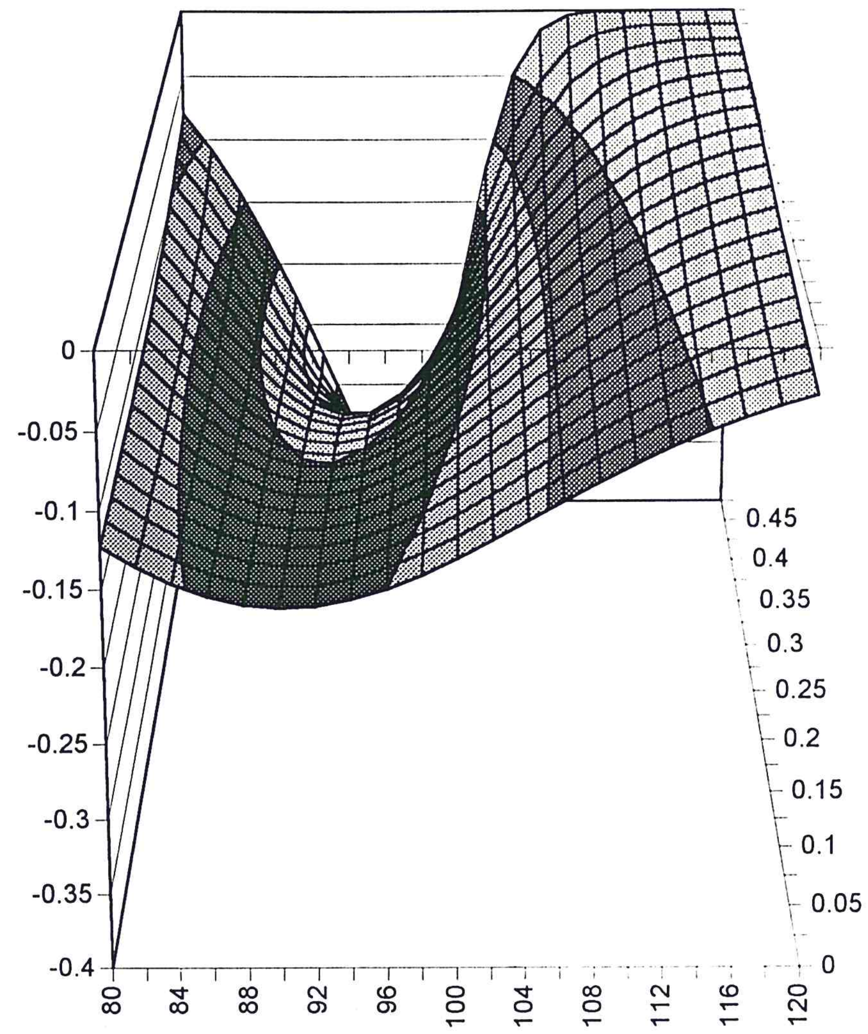
Figure 3.1.2 : Price, delta, gamma and vega of a put on a call

put on call price



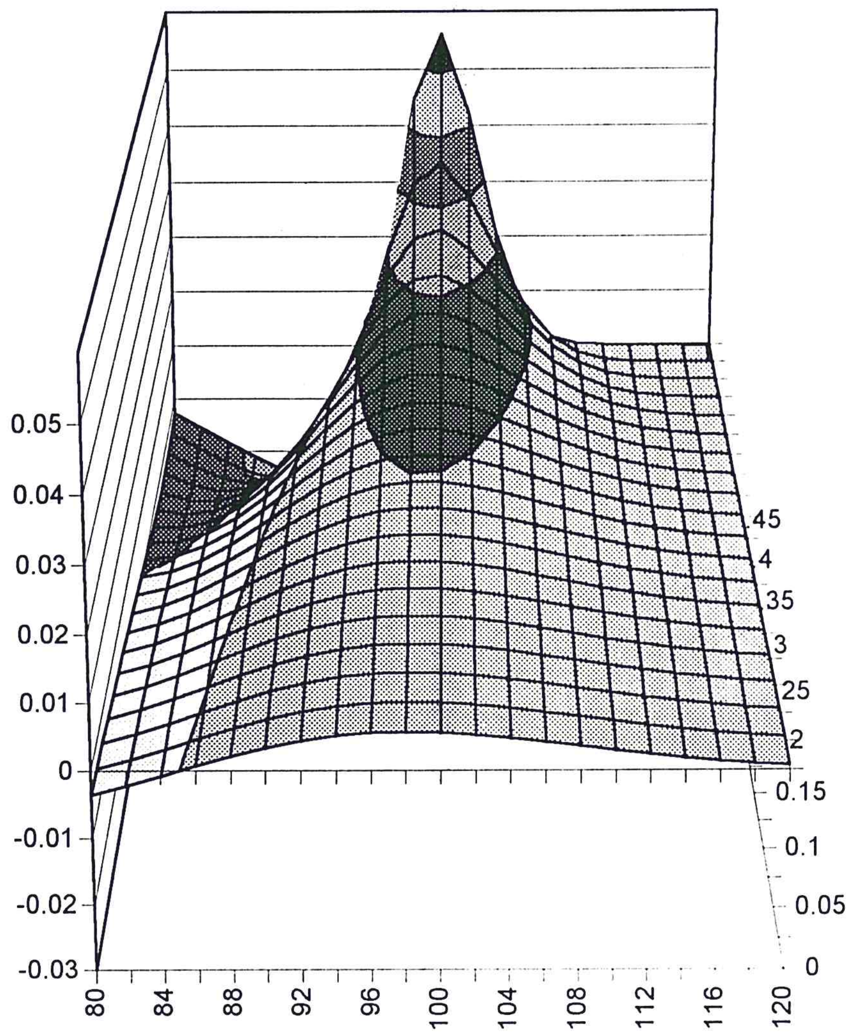
AMERFEA2.XLS, Fig 3.1.2 (b)

put on call delta

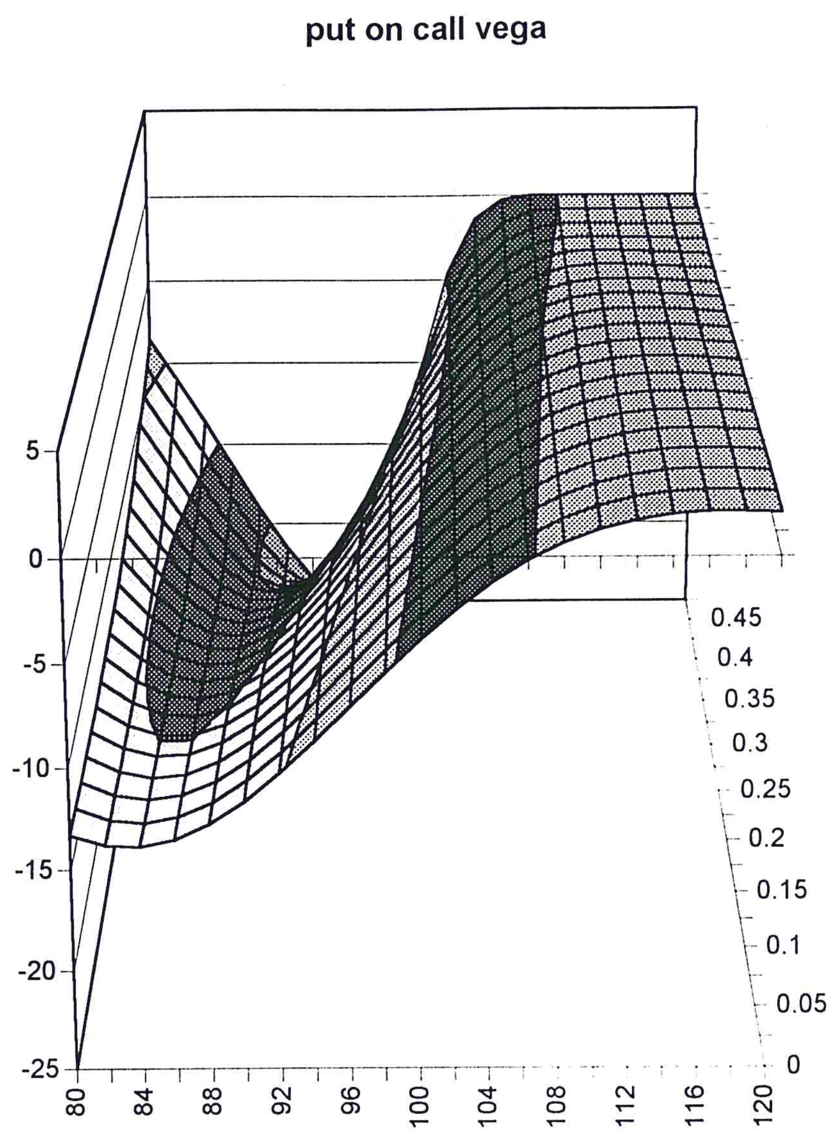


AMERFEA2.XLS, Fig 3.1.2 (c)

put on call gamma



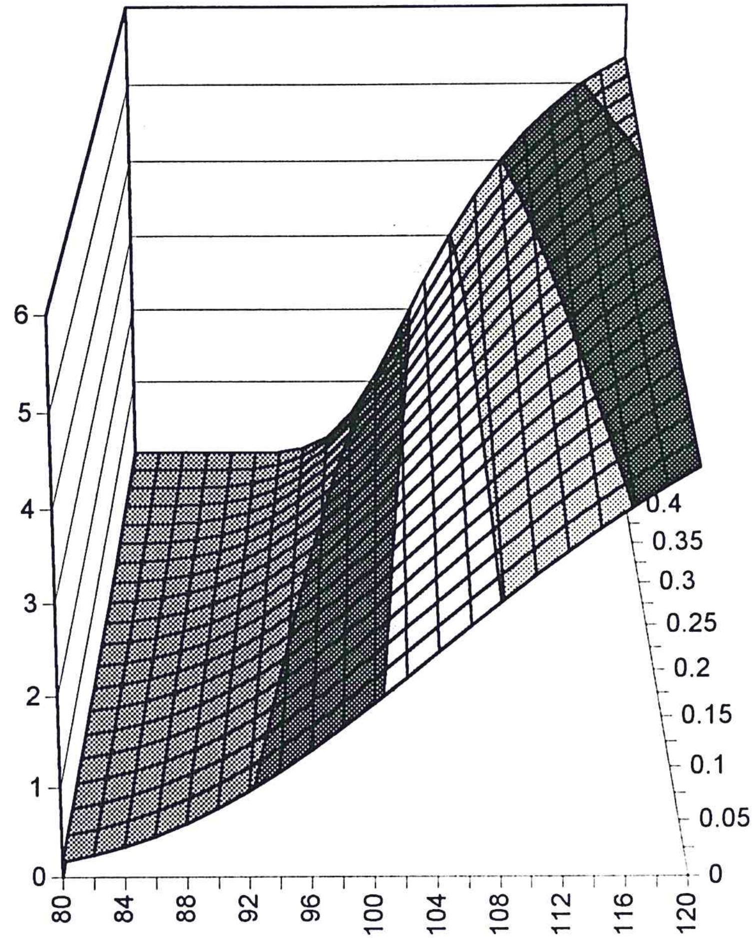
AMERFEA2.XLS, Fig 3.1.2 (d)



AMERFEA2.XLS, Fig 3.1.3 (a)

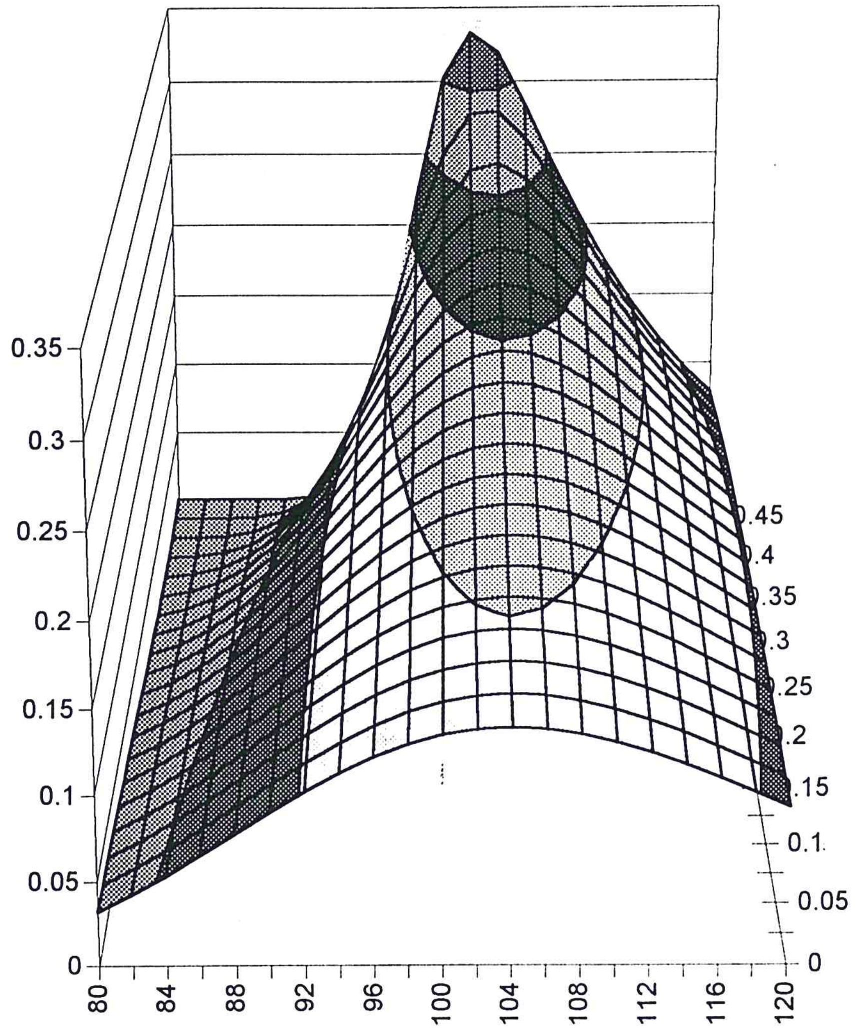
Figure 3.1.3 : Price, delta, gamma and vega of a put on a call

put on put price

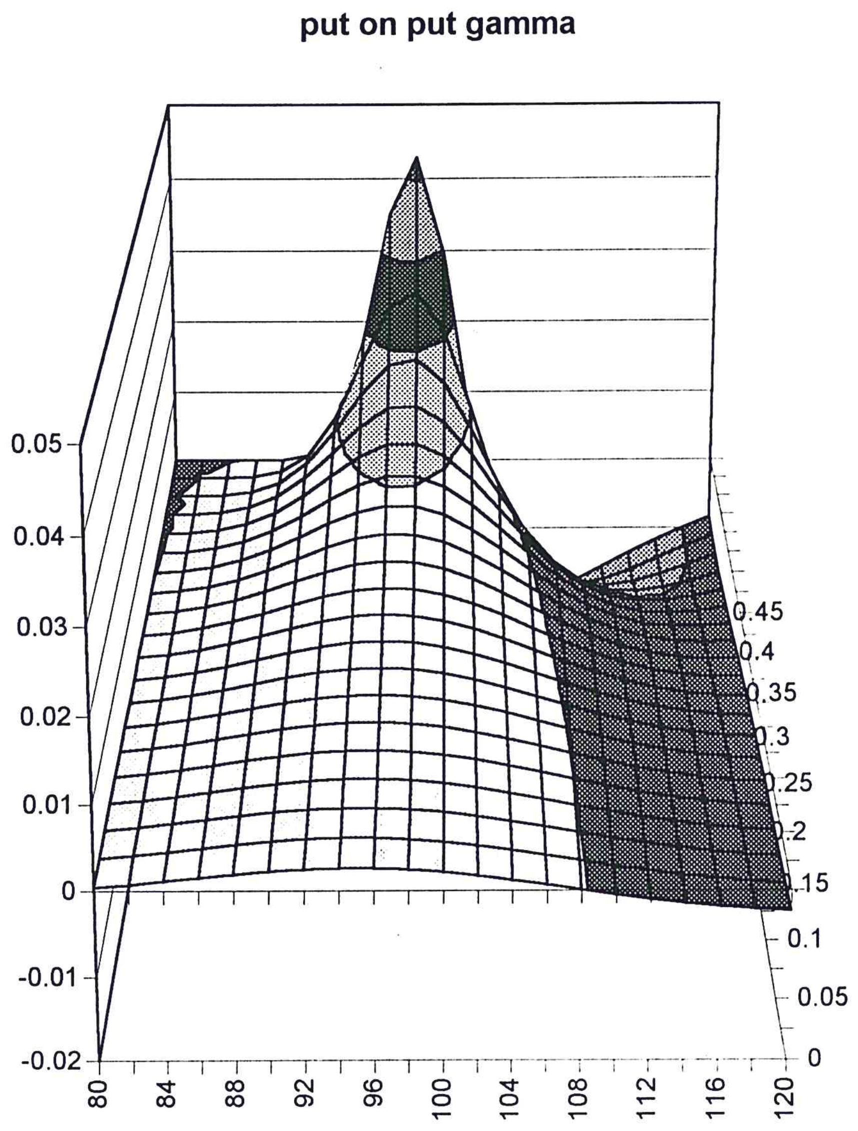


AMERFEA2.XLS, Fig 3.1.3 (b)

put on put delta



AMERFEA2.XLS, Fig 3.1.3 (c)



AMERFEA2.XLS, Fig 3.1.3 (d)

put on put vega

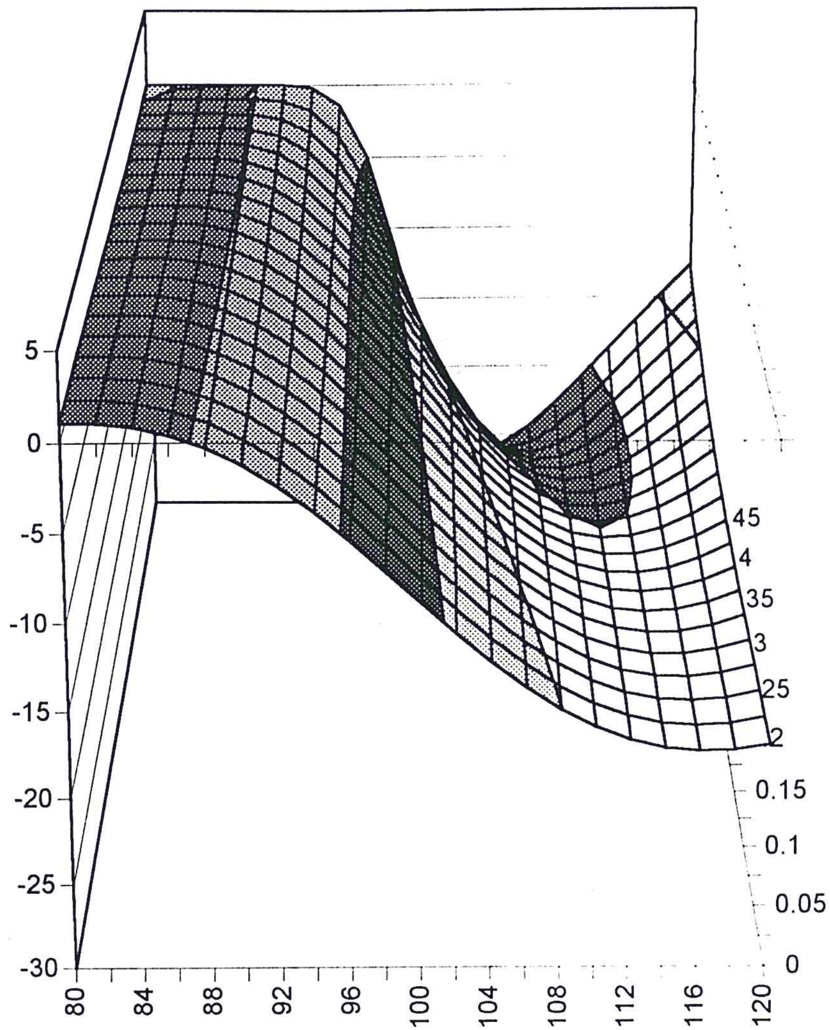


Figure 4.3.1: The Residual Price Risk a Chooser Option After Taking Positions in the Underlying Options

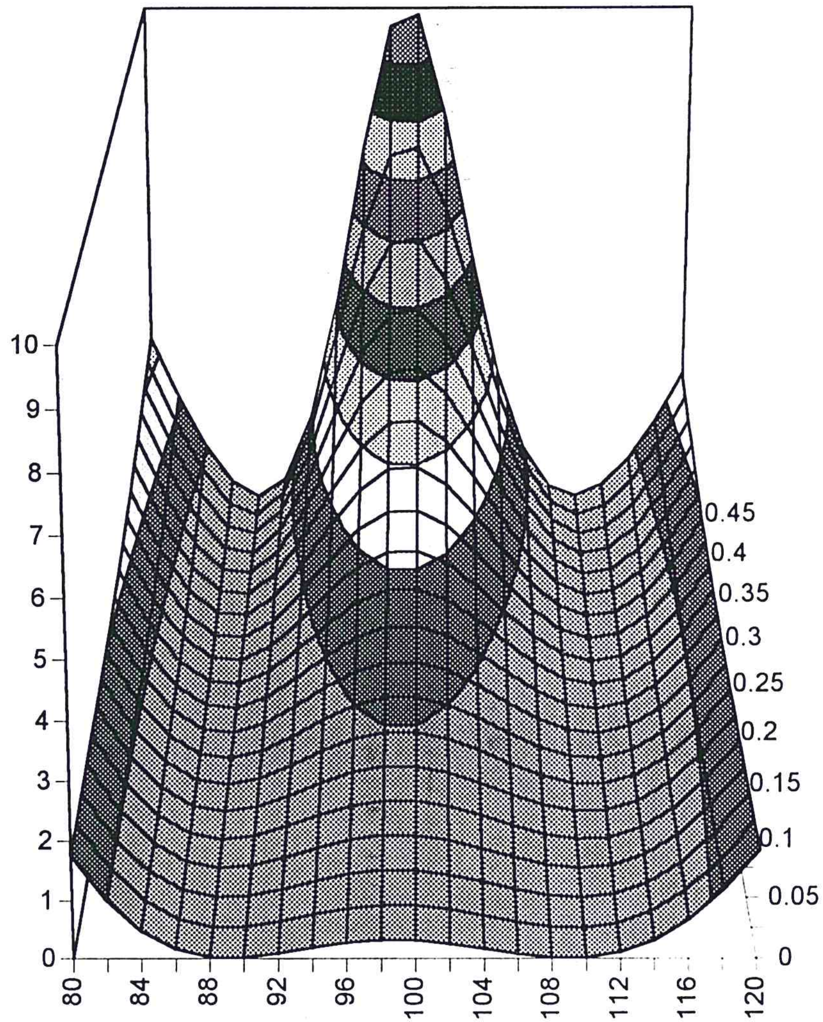


Figure 5.1: The Price Surfaces of a Fixed Strike Lookback Call and an Equivalent Shout Call Option

Fixed Strike Lookback Call

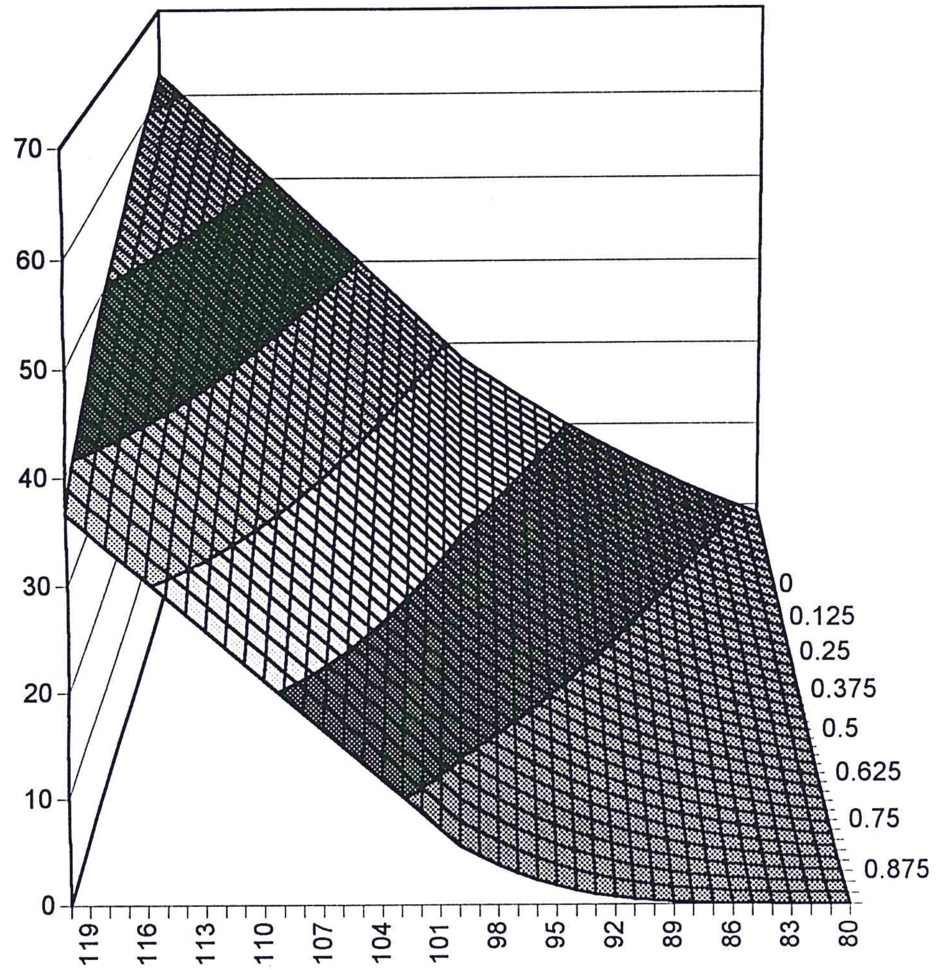


Figure 5.1: The Price Surfaces of a Fixed Strike Lookback Call and an Equivalent Shout Call Option

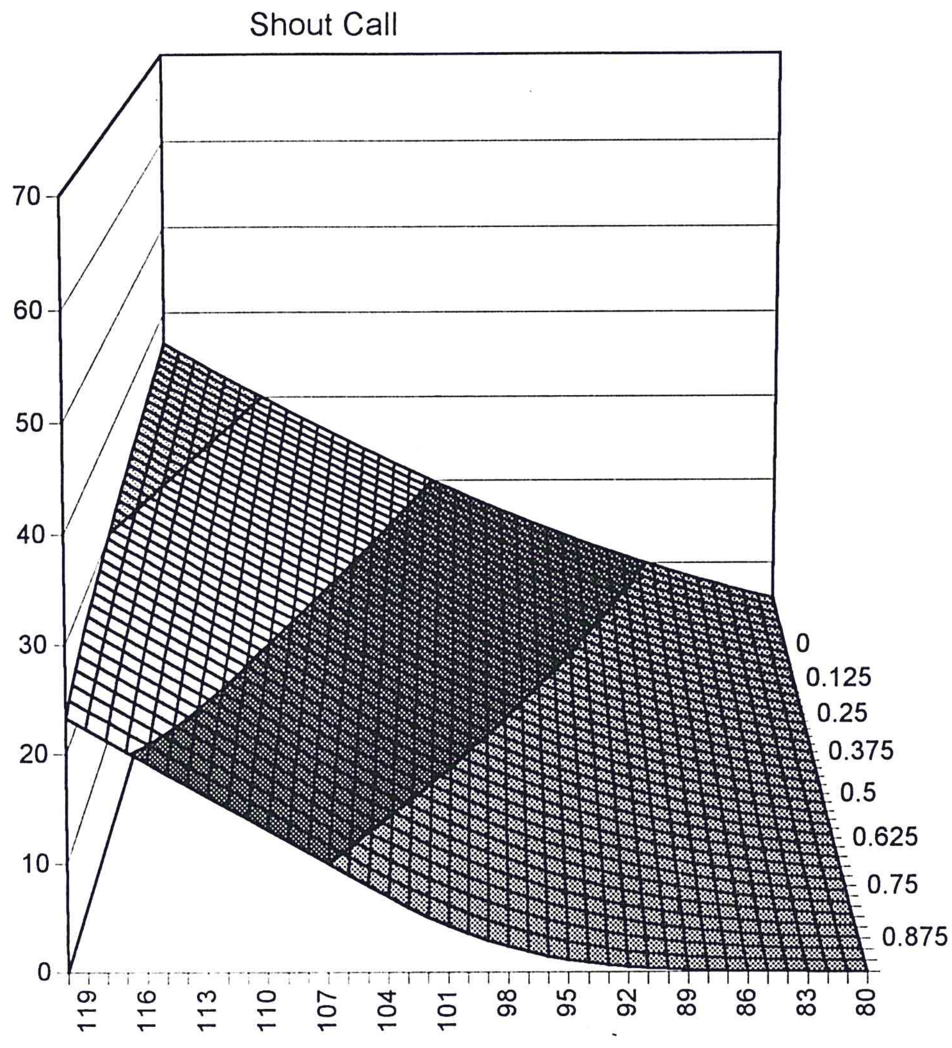


Figure 5.2.1: The Optimal Shout Level of the Asset Price for a Shout Call Option Computed by the Crank-Nicolson Finite Difference Method