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**Les Clewlow**

**Kin Pang**

**and**

**Chris Strickland**

*Financial Options Research Centre*
*Warwick Business School*
*University of Warwick*
*Coventry CV4 7AL*
*Phone: (01203) 524118*

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The Financial Options Research Centre
Warwick Business School
University of Warwick
England

Instituto de Estudios Superiores de Administración
Caracas
Venezuela

Centre for Financial Mathematics
Australian National University
Canberra
Australia

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# Numerical Procedures for Pricing Interest Rate Exotics Using Markovian Short Rate Models

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## Abstract

In this paper we discuss the pricing of exotic (American featured and path dependent) interest rate derivatives. We work within the framework of a discrete time and state lattice model or trinomial tree for the evolution of the short term interest rate. Examples of models in this class are Hull and White [1990], Black, Derman and Toy [1991], and Black and Karasinski [1992] in the one factor case and Hull and White [1994b] in the two factor case. We describe numerical procedures that can be used to efficiently price and hedging American featured and in particular path dependent interest rate derivatives. Examples we consider are American swaptions into new and existing swaps, down and out barrier caps, lookback caps, average rate caps, and index amortising rate swaps.

# 1. Introduction

In this paper we discuss the pricing of exotic (American featured and path dependent) interest rate derivatives. Unlike many assets which can for practical purposes be modeled as a geometric Brownian motion (GBM) this is not a realistic process for interest rates. Mean reversion in interest rates, the pull to par effect for bond prices, and the desire to model the entire (market observed) yield (spot rate) curve in an arbitrage free framework have led to the development of a wide range of interest rate models. Unlike the GBM world, under these yield curve consistent models it is not possible to obtain closed form solutions for the prices of most interest rate exotics. In this paper we describe numerical procedures which can be used for the pricing and hedging of these instruments.

We begin by outlining techniques for constructing short rate trees to represent certain one-factor short rate models which are designed to be consistent with market observed term structure data. In section 2 we show how plain vanilla European and American bond options can be valued in a short rate tree. In section 3 we extend our methodology to pricing interest rate exotics looking at both American featured and path dependent derivatives.

There are two basic ways of achieving consistency with the yield curve. One is to specify a process for the short rate, as in the traditional approach of Vasicek [1977], and Cox, Ingersoll, and Ross [1985], and then effectively increase the parameterisation of the model by using time dependent factors until all initial market data can be returned. These models can be characterised by Hull and White [1990, 1994b], Black, Derman and Toy [1991], and Black and Karasinski [1992]. The second starts by specifying the initial yield or forward rate curve and its volatility structure(s) and then determines a drift structure that makes the model arbitrage free. These models can be characterised by Heath, Jarrow, and Morton [1992], and Carverhill [1995]. In this paper we use the first approach for pricing interest rate derivative securities; the second approach is considered in a further paper (Clewlow and Strickland (1997)).

In recent years a number of authors have presented methods for building short rate trees to be consistent with observed term structure data. This data might include the observed term structure of interest rates as well as the volatility term structure. Amongst the most well known of these are the models of Ho and Lee [1986], Hull and White [1993, 1994a, 1996], Black, Derman and Toy [1991], and Black and Karasinski [1992]. All of these models are single factor. Hull and White [1994b] have also recently shown how to construct short rate trees for a two factor model for the short rate. Table 1 summarises these models by presenting their defining stochastic differential equations.

**Table 1: Markovian Short Rate Models with Trees**

| | |
|---|---|
| Ho-Lee | $dr = \theta(t)dt + \sigma dz$ |
| "Lognormal" Ho-Lee | $d\ln r = \theta(t)dt + \sigma dz$ |
| Hull-White 1 factor | $dr = [\theta(t) - \alpha(t)r]dt + \sigma(t)r^{\beta}dz$ |
| Black-Derman-Toy | $d\ln r = \left[\theta(t) - \frac{\sigma'(t)}{\sigma(t)}\ln r\right] dt + \sigma(t)dz$ |
| Black-Karasinski | $d\ln r = [\theta(t) - \alpha(t)\ln r]\ dt + \sigma(t)dz$ |
| | |
| Hull-White 2 Factor | $dr = [\theta(t) + u - \alpha r]dt + \sigma_1 dz_1$<br>$du = -budt + \sigma_2 dz_2$ |

We concentrate on the one time dependent function version of the Hull and White [1992] model with constant short rate volatility, although we stress that the procedures that we describe are applicable with only minor modifications to all of the models summarised in Table 1. This model has become known as the 'Hull-White' model:

$$dr = [\theta(t) - \alpha r]dt + \sigma dz \qquad (1)$$

The one time dependent function in the drift allows the model to be consistent with the currently observed yield curve. The constant parameters $\alpha$ and $\sigma$, the speed of mean reversion of the short rate and the volatility of the short rate respectively, determine the volatility structure of yields of different maturities. One of the reasons for the popularity of this model is that the resulting normal distributions for interest rates mean that it has a number of analytical properties. For example one explicit result relates the prices of pure discount bonds (the discount function) at some future time, $T$, to the initially observed term structure (at time $t$, $t \leq T$), the parameters of the short rate process and the level of the short rate at time $T$. If $P(t,s)$ is the price at time $t$ of a pure discount bond, which pays off one unit at time $s$ then

$$P(T,s) = A(T,s)e^{-B(T,s)r_T} \qquad (2)$$

where

$$B(T,s) = \frac{1}{a}\left(1 - e^{-a(s-T)}\right)$$

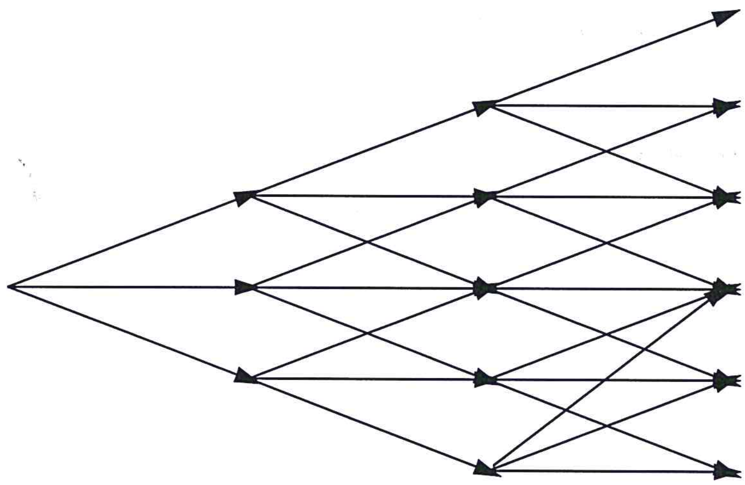
$$\ln A(T,s) = \ln\frac{P(t,s)}{P(t,T)} - B(T,s)\frac{\partial \ln P(t,T)}{\partial T} - \frac{1}{4a^3}\sigma^2\left(e^{-a(s-t)} - e^{-a(T-t)}\right)^2\left(e^{2a(T-t)} - 1\right)$$

and where ($t \leq T \leq s$). In many circumstances this result means that we only need to 'grow' the tree as far as the end of the life of the derivative instead of the end of the life of the security underlying the derivative.

There also exists closed-form solutions for prices of European options on pure discount bonds allowing us to calibrate the model efficiently to market interest rate caps and European swaptions data[1].

The first step to building a short rate tree is to divide the life of the interest rate derivative into, say, $i = 1,\ldots,N$ equal segments each of length $\Delta t$. The branching process in the trinomial tree and the associated transitional probabilities are determined so that the change in $r$ has the correct mean and standard deviation over each time interval $\Delta t$. Figure 1 depicts a typical trinomial tree after 3 time steps.

**Figure 1 : Typical Trinomial Tree After 3 Time Steps**



The value of $r$ on the tree at time 0 is the initial short rate, and is by definition the yield on a pure discount bond maturing at time $\Delta t$. At time $i = 0$ there is a single state

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1 See Strickland [1996] for the decomposition of caps and swaptions into portfolios of European discount bond options

$j = 0$. At time $i = 1$ there are three states $j = -1$, $j = 0$, and $j = 1$. At time $i = n$ there are at most $2n+1$ nodes $j = -n, -n+1, \ldots.., n-1, n$.

We refer to $r(i, j)$ as the short rate at the $(i, j)$ node. One of the by products of constructing the tree using the method of forward induction[2] to match the initial yield curve exactly is that we accumulate information about the risk neutral distribution in the form of pure security prices or state prices. Define $Q(i,j)$ as the value, at time 0, of a security that pays the following;

$1 if node $(i,j)$ is reached and

$0 otherwise

the $Q(i,j)$'s are therefore prices of Arrow-Debreu securities, pure securities or state prices. For European style derivatives we can use the prices of these pure securities to allow us to jump back to the origin of the tree directly from the option maturity date saving us the more usual repeated calculation of discounted expectations.

For the remainder of this paper we are going to assume that the short rate trees have been built and use the resulting trees for pricing interest rate derivatives. Hull and White [1994a] describe procedures for building trinomial trees for the Hull-White and Black-Karasinski models. Jamshidian [1991] provides an algorithm for a binomial implementation of a wide range of Brownian path independent models. Clewlow and Strickland [1997] provide detailed descriptions of building binomial and trinomial trees for all of these models.

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2 See Jamshidian [1991], Hull and White [1993a] and Clewlow and Strickland [1997].

## 2. Pricing Interest Rate Derivatives Using Short Rate Trees

Once the tree has been constructed we know the short rate at every time and every state of the world consistent with our original assumptions about the process and the initial interest rate term structure, and we can use it to price a wide range of interest rate derivatives via backwards induction. As an example of the steps that need to be taken we begin by pricing a $T$ maturity European call option on a $s$-maturity pure discount bond ($T \leq s$) with a strike price of $K$. Let $C(i,j)$ represent the value of the contingent claim at node $(i,j)$. Its value is related to the three connected nodes at time step $i$+1 by the usual discounted expectation:

$$C(i,j) = e^{-r(i,j)\Delta t}\left[p_u C(i+1, j_u) + p_m C(i+1, j_m) + p_d C(i+1, j_d)\right] \quad (3)$$

where $p_u$, $p_m$, and $p_d$ are the probabilities of moving to $C(i+1, j_u)$, $C(i+1, j_m)$, and $C(i+1, j_d)$, the up, middle and down nodes as seen from node $(i, j)$ respectively[3]. Assume that the short rate tree has been constructed out as far as time step $N$ ($T - t = N\Delta t$). Let $P(N, j, s)$ represent the value of the $s$-maturity bond at node $(N, j)$ for all nodes $j$. For the Hull-White model this can be calculated via equation (2)[4]. Next, evaluate the maturity condition for the option.

$$C(N, j) = \max\{0, P(N, j, s) - K\} \quad \text{for all } j \text{ at } N \quad (4)$$

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3 $j_u$, $j_m$, and $j_d$ will not necessarily be $j$+1, $j$ and $j$-1 because the branching may be "non-standard". See node (2,-2) in figure 1.

4 If the model we are using does not have this analytical property (for example any of the lognormal models) then the short rate tree has to be extended to the maturity time of the bond, the maturity condition of the bond evaluated and backwards induction performed back to time step $N$ to obtain the value of the bond at the option maturity date.

Today's value of the option can then be obtained by repeatedly applying equation (3) back to the origin of the tree ($i = N - 1$ down to 0), yielding $C(0,0)$ as the option value. A more efficient procedure for valuing European option prices utilises the fact that the pure securities are equivalent to discounted risk-neutral probabilities. The value of a European option can therefore be calculated directly from the tree as the sum of the product of the maturity condition of the option and the state price, for each node at the maturity time, i.e. for the call option of our example;

$$C(0,0) = \sum_{\forall j} Q(N,j)C(N,j) \qquad (5)$$

with the summation across all of the existing nodes $j$ at time step $N$.

## 3. Pricing Interest Rate Exotics

We now turn our attention to pricing more exotic interest rate derivatives looking at both American featured options and path dependent options.

### 3.1 American Featured Options

For American options we need to allow for the possibility of early exercise in the normal way by taking the maximum of the discounted expectation and the intrinsic value of the option at every node. For an American pure discount bond call option equation (3) therefore becomes;

$$C(i,j) = \max\left\{\begin{matrix} P(i,j,s) - K, \\ e^{-r(i,j)\Delta t}\left[p_u C(i+1,j_u) + p_m C(i+1,j_m) + p_d C(i+1,j_d)\right] \end{matrix}\right\} \qquad (6)$$

A currently popular American featured option is an American swaption. We look at two versions of this derivative, a swaption that when exercised produces a new swap and one which exercises into an already existing swap. We show that these are both essentially American options on coupon bonds but where the timing of the cashflows of the underlying bond differs.

### 3.1.1 American Payer Swaption Exercising Into New Swap

We price a $T$-maturity option which when exercised enters the holder of the option into a brand new swap. Upon exercise of the option, at date $T'$ ($T' \leq T$), the cashflows to the swap occur at dates $s_1, s_2, \ldots, s_n$ where $T' < s_1 < s_2 < \ldots < s_n$, $s_k = T' + k\Delta\tau$; $k = 1, \ldots n$, $\Delta\tau$ is the reset period on the swap, $s_n$ is the maturity date of the swap and $s_n - T' = n\Delta\tau$ is the life of the swap. We will assume that the swap has an underlying principal of $L$ and pays semi-annually, i.e. $\Delta\tau = 0.5$ (our approach generalises easily for other payment frequencies e.g. quarterly or annually).

For one-factor models Jamshidian [1989] has shown that European options on coupon bonds can be valued as portfolios of options on pure discount bonds. An interest rate swap is equivalent to exchanging a fixed rate bond for a floating rate bond, implying that a swaption is an option to exchange these two bonds. Because the floating rate bond will be trading at its face value immediately after a reset date, a swaption is an option to exchange a fixed rate bond for the principal of the swap. For a payer swaption (the right to pay fixed and receive floating) the option is therefore equivalent to a put on a fixed rate bond with strike equal to the principal underlying the swap. For a receiver swaption (the right to pay floating and receive fixed) the option is equivalent to a call on a fixed rate bond with strike equal to the principal. We concentrate here on payer swaptions.

Let $K$ represent the exercise rate of the swaption. For a semi-annual swap, if the option is held until expiry the payoff is equal to;

$$C(T) = L\max\left\{0,1-\left[\sum_{k=1}^{n}\frac{K}{2}P(T,s_k)+P(T,s_n)\right]\right\} \tag{7}$$

In terms of the tree, the payoff at node $j$ is determined as;

$$C(N,j) = L\max\left\{0,1-\left[\sum_{k=1}^{n}\frac{K}{2}P(N,j,s_k)+P(N,j,s_n)\right]\right\} \tag{8}$$

The value of European option can be evaluated via the pure security prices, equation (5). The value of American option is obtained via the usual backwards induction;

$$C(i,j)=\max\left\{\begin{matrix}EEV, \\ e^{-r(i,j)\Delta t}\left[p_u C(i+1,j_u)+p_m C(i+1,j_m)+p_d C(i+1,j_d)\right]\end{matrix}\right\} \tag{9}$$

where the early exercise value, $EEV$, is given by

$$EEV = L\left\{1-\left[\sum_{k=1}^{n}\frac{K}{2}P(i,j,i\Delta t+k\Delta\tau)+P(i,j,i\Delta t+n\Delta\tau)\right]\right\}$$

Table 2 shows the values for European and American swaption values under varying assumptions of option maturity and strike.. Figures in the table are produced under assumptions of a flat term structure (5% continuously compounded), with $\alpha = 0.10$ and $\sigma = 0.01$. The tenor of the swap is 3 years and the tree for the short rate is constructed with weekly time steps.

**Table 2 Swaption Into New Swap Prices**

| Strike Rate | Swaption Type | | Option Maturity | | |
|---|---|---|---|---|---|
| | | | 1.0 | 1.5 | 2.0 |
| **0.0475** | European | receiver | 0.0053 | 0.0068 | 0.0079 |
| | European | payer | 0.0135 | 0.0148 | 0.0157 |
| | American | receiver | 0.0055 | 0.0072 | 0.0086 |
| | American | payer | 0.0141 | 0.0158 | 0.0172 |
| **0.0500** | European | receiver | 0.0081 | 0.0096 | 0.0106 |
| | European | payer | 0.0097 | 0.0112 | 0.0122 |
| | American | receiver | 0.0084 | 0.0101 | 0.0116 |
| | American | payer | 0.0101 | 0.0119 | 0.0134 |
| **0.0525** | European | receiver | 0.0116 | 0.0129 | 0.0139 |
| | European | payer | 0.0067 | 0.0082 | 0.0092 |
| | American | receiver | 0.0120 | 0.0137 | 0.0151 |
| | American | payer | 0.0069 | 0.0087 | 0.0101 |

### 3.12 American Payer Swaption Into Existing Swap

For this type of swaption the cashflows are fixed relative to time $t$ and not the exercise date, $T'$. Suppose that the cashflows to the swap occur at dates $s_1, s_2, \ldots, s_n$ where $t < s_1 < s_2 < \ldots < s_n$, $s_k = t + k\Delta\tau$, $k = 1, \ldots n$, $s_x \equiv T$, $1 \le x \le n$, $\Delta\tau$ is the reset period of the swap, $s_n$ is the maturity date of the swap, $s_n - s_x$ is the life of the swap at the exercise date $s_x$. When the option holder exercises the option they are entitled to the remaining cashflows after the date of exercise. We are assuming here that the terms of the contract only allow the holder to exercise the option on a swap reset date (our result easily generalises to the case where the holder can exercise the option at any time).

If the option is held until expiry the payoff is equal to;

$$C(T) = L \max\left\{0, 1 - \left[\sum_{k=x+1}^{n} \frac{K}{2} P(T, s_k) + P(T, s_n)\right]\right\} \qquad (10)$$

In terms of the tree the equivalent payoff is given by;

$$C(N,j) = L\max\left\{0, 1 - \left[\sum_{k=x+1}^{n} \frac{K}{2} P(N,j,s_k) + P(N,j,s_n)\right]\right\} \qquad (11)$$

Next, take discounted expectations, defined by *DE*, back to time $s_{x-1}$ and apply the early exercise condition;

$$C(i,j) = L\max\left\{DE, 1 - \left[\sum_{k=x}^{n} \frac{K}{2} P(i,j,s_k) + P(i,j,s_n)\right]\right\} \qquad (12)$$

The process is repeated at each of the permissible exercise dates back to the origin of the tree. Table 3 shows values for these American swaption values under the same assumptions as Table 2.

**Table 3 Swaption Into Existing Swap Prices**

| Strike Rate | Swaption Type | Swap Tenor | | |
|---|---|---|---|---|
| | | 3.0 | 4.0 | 5.0 |
| **0.0475** | American receiver | 0.00556 | 0.00578 | 0.00594 |
| | American payer | 0.00823 | 0.00826 | 0.00813 |
| **0.0500** | American receiver | 0.00771 | 0.0079 | 0.00805 |
| | American payer | 0.00969 | 0.00973 | 0.00963 |
| **0.0525** | American receiver | 0.01042 | 0.0106 | 0.01072 |
| | American payer | 0.01187 | 0.01192 | 0.01183 |

## 3.2 Path Dependent Interest Rate Exotics

So far we have concentrated on pricing interest rate derivatives where the payoff to the derivative only depends on the level of the short rate and not on the path that the short rate took to achieve that level. We now show how a range of path-dependent exotics can be valued within the short rate tree framework.

To value path-dependent options we need to value the option at each node of the tree for all of the values of the path function (e.g. maximum, minimum, average, etc.) that could occur. However, the number of alternative function values can grow very quickly, in the worse case exponentially i.e. equal to the number of paths reaching a particular node. In these cases Hull and White [1993], using a binomial tree for an asset that follows a lognormal diffusion, show how to keep track of a representative subset of all possible values of the function. The value of the option for other values of the function is computed via interpolation. In the following sections we extend the techniques developed in the Hull and White paper to pricing a range of interest rate exotics using short rate trees.

We define $f(t,r)$ to be the value of the path dependent function at time $t$ when the level of the short rate is $r$ for a particular path. To value a path dependent derivative ideally the derivative should be evaluated at all the possible values of the function that can occur. For this to be computationally tractable we need to be able to evaluate $f(t+\Delta t,r)$ from $f(t,r)$ and the level of the short rate at $t+\Delta t$. For many derivatives the number of possible function paths increases rapidly as the number of time steps is increased and so evaluating all the possible values that can occur becomes impractical. Instead of keeping track of all the possible alternatives we compute the value of the derivative at each node for only certain values of the function. Specifically, we evaluate the maximum and minimum values of $f$ at each node and then approximate the set of all possible values of $f$ with $M$ equally spaced values.

Let $f(i,j,l)$ represent the $l$'th value ($l=0,1,\ldots,M$) of the path function in the tree at the node $(i,j)$ and $C(i,j,l)$ now represent the value of the derivative when $f$ has this value. To calculate the function values we apply forward induction. The general procedure involves stepping forward through the tree computing only the maximum and minimum values of the path dependent function at each node i.e. $f(i,j,0)$ and $f(i,j,M)$ at node (*i*,*j*) respectively. We then choose a functional approximation to represent the path dependent function and/or the discrete set of values for which the option will be valued. Finally, we step back through the tree computing the option values in the usual way but for each of the set of values of *f* at each node.

Once the forward induction step is completed the value of the derivative security is known at maturity, $C(N,j,l)$ and is given by the maturity condition. At previous time steps the derivative value is found analogously to equation (3) by implementing backwards induction;

$$C(i,j,l) = \\ e^{-r(i,j)\Delta t}\left[p_u C(i+1,j_u,l_u) + p_m C(i+1,j_m,l_m) + p_d C(i+1,j_d,l_d)\right] \quad (13)$$

where $C(i+1,j_u,l_u)$ is the price of the derivative if the function moves from $f(i,j,l)$ to $f(i+1,j_u,l_u)$ in the upstate. The others terms are defined similarly. Due to the nature of the forwards induction (we only hold *M* values of the function at each node) $C(i+1,j_u,l_u)$, $C(i+1,j_m,l_m)$, and $C(i+1,j_d,l_d)$ might not be explicitly available at time step $i+1$, and so we must interpolate from the values which are available. For example, we interpolate $C(i+1,j_u,l_u)$ from $C(i+1,j_u,l_1)$ and $C(i+1,j_u,l_2)$ where $l_1$ and $l_2$ are the closest values of $f$ to $f(i+1,j_u,l_u)$ such that $f(i+1,j_u,l_1) \leq f(i+1,j_u,l_u) \leq f(i+1,j_u,l_2)$. This procedure is repeated for all *l* at node (*i*,*j*). We determine $C(i+1,j_m,l_m)$, and $C(i+1,j_d,l_d)$ similarly.

In the following many of our examples are based on interest rate cap agreements and so here we outline our notation for a standard cap. Consider a *n*-year cap with semi-annual reset dates, a cap rate of $r_C$ on an underlying principal of *L*, where the capped rate is, say, 6 months LIBOR. Let $\Delta\tau$ be the reset frequency of the cap (i.e. $\Delta\tau = 0.5$). The cap can be interpreted as a series of 'caplets', covering each period $\Delta\tau$. If $R((k+1)\Delta\tau)$ [1] is the LIBOR level at time $(k+1)\Delta\tau$ then the payoff to the $k$'th caplet for the period between $k\Delta\tau$ and $(k+1)\Delta\tau$ at time $(k+1)\Delta\tau$, for some integer $k$ is given by[2]

$$\Delta\tau L \max(R((k+1)\Delta\tau) - r_c, 0) \tag{14}$$

In order to be able to use the tree to price the caplet we need to be able to recover the level of LIBOR at each node of the tree. Because of the discrete nature of the compounding of LIBOR we can write $\Delta\tau$ -LIBOR at time $t$, $R(t,\tau)$, in terms of the discount function at time $t$ [3];

$$R(t,\tau) = \frac{1}{\Delta\tau}\left[\frac{1}{P(t,t+\Delta\tau)} - 1\right] \tag{15}$$

Therefore can determine $\Delta\tau$ -LIBOR at node $(i,j)$ in the tree, $R(i,j,\Delta\tau)$, according to the following;

$$R(i,j,\Delta\tau) = \frac{1}{\Delta\tau}\left[\frac{1}{P(i,j,i\Delta t+\Delta\tau)} - 1\right] \tag{16}$$

The tree equivalent payoff for each node at maturity therefore becomes;

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1 Where $R(k\Delta\tau)$ should be read as $R(k\Delta\tau, k\Delta\tau + \Delta\tau)$.

2 Note that this is equivalent to a LIBOR-*in-arrears* caplet. Normally the realised LIBOR at $k\Delta\tau$ determines the payoff at $(k+1)\Delta\tau$ as $\Delta\tau L \max(R(k\Delta\tau) - r_c, 0)$. We write the payoff as equation (14) as it fits in with the definition of the knockout, lookback, and barriers that we go on to study.

3 We are assuming here that the LIBOR maturity is the same as the reset frequency. It is straightforward to relax this assumption.

$$C(N,j) = \Delta\tau \max\{R(N,j,\Delta\tau) - r_c, 0\} \tag{17}$$

for all $j$ at time step $N$ where $N\Delta t = (k+1)\Delta\tau$.

We now turn our attention to exotic versions of this basic instrument.

### 3.2.1 Down and Out Barrier Caps

Down and out barrier caps are interest rate caps that extinguish worthless if the index rate goes below a certain level $H$ (the barrier level). For example the $k$'th individual caplet expires worthless if the reference LIBOR rate crosses the barrier during the caplet period $k\Delta\tau$ to $(k+1)\Delta\tau$. If the barrier is not crossed the reference LIBOR rate determining the payoff at $(k+1)\Delta\tau$ is $R((k+1)\Delta\tau)$. The payoff can therefore be represented by;

$$\{\Delta\tau L \max(R((k+1)\Delta\tau) - r_c, 0)\} \mathbf{1}_{R(t;k\Delta\tau<t\le(k+1)\Delta\tau)>H} \tag{18}$$

where $\mathbf{1}_{R(t;k\Delta\tau<t\le(k+1)\Delta\tau)>H}$ is the indicator function;

$$\mathbf{1}_{R(t;k\Delta\tau<t\le(k+1)\Delta\tau)>H} = \begin{cases} 1 & R(t) > H \qquad k\Delta\tau < t \le (k+1)\Delta\tau \\ 0 & R(t) \le H \qquad otherwise \end{cases} \tag{19}$$

In order to incorporate this extra boundary condition in the tree we set the value of the option to be zero at every node below the barrier after performing discounted expectations at each time step. For example, if $n\Delta t = k\Delta\tau$ then we apply;

$$\begin{aligned} &if \quad R(i,j,\Delta\tau) < H \\ &\quad then \;\; C(i,j) = 0 \qquad \forall j \\ &\quad else \;\; C(i,j) = e^{-r(i,j)\Delta t}\left[p_u C(i+1,j_u) + p_m C(i+1,j_m) + p_d C(i+1,j_d)\right] \end{aligned} \tag{20}$$

**Table 4: Barrier Cap Prices**

| Maturity | Cap Rate | | | |
|---|---|---|---|---|
| (Years) | 0.053 | 0.054 | 0.055 | 0.056 |
| | | | | |
| 3 | 0.0101 | 0.0091 | 0.0083 | 0.0074 |
| 4 | 0.0155 | 0.0142 | 0.0130 | 0.0118 |
| 5 | 0.0214 | 0.0197 | 0.0182 | 0.0167 |

at time step $i$, $i = n$ to $N$. Table 4 shows the relationship between the relative cap rate and barrier level and the price of a down and out barrier cap for different maturities. The table is constructed under the assumptions of a flat term structure of 5% continuously compounded, $\alpha = 0.02$, $\sigma = 0.01$, and with 52 time steps per year. The cap is assumed to reset semi-annually based on the level of 6-month LIBOR with the barrier level fixed at 3.0% and barrier fixing weekly.

### 3.2.2 Lookback Caps

For lookback caps the cap rate for each caplet is determined by the maximum level of LIBOR over the life of the capping period, i.e. the reference rate for the caplet period from $k\Delta\tau$ to $(k+1)\Delta\tau$ is set by the maximum level of LIBOR during that period. The payoff at time $(k+1)\Delta\tau$ is therefore;

$$\Delta\tau L \max(M((k+1)\Delta\tau) - r_c, 0) \tag{21}$$

where $M((k+1)\Delta\tau) = \max(R(t); k\Delta\tau \le t \le (k+1)\Delta\tau)$ is the maximum level of LIBOR achieved over the period $k\Delta\tau$ and $(k+1)\Delta\tau$.

In order to value lookback caplets within the tree we apply the principles outlined at the beginning of this section. At $k\Delta\tau$ set $f(n,j,0) = f(n,j,M) = R(n,j,\Delta\tau)$, where $n\Delta t = k\Delta\tau$, for all $j$ at time step $n$. Sweep through the tree until $(k+1)\Delta\tau$ updating the maximum and minimum of the maximum level of LIBOR attained in the tree. e.g. for the maximum ($l = M$);

$$f(i+1, j_u, M) := \max\{f(i,j,M), R(i+1, j_u, \Delta\tau), f(i+1, j_u, M)\} \qquad (22)$$

At maturity time $N\Delta t$ $(= (k+1)\Delta\tau)$ the maturity condition is evaluated as;

$$C(N,j,l) = \Delta\tau L \max\{F(N,j,l) - r_c, 0\} \qquad l = 0, \ldots M$$

where the values $l = 1$ to $M$ of $f(N,j,l)$ are interpolated from $f(N,j,0)$ and $f(N,j,M)$. Backwards induction is then implemented until time step $n$ when a version of equation (5) can be applied.

Table 5 shows the relationship between the cap rate and the price of a lookback cap for varying maturities. The table is constructed under the same assumptions of table 4. The cap is assumed to reset semi-annually based on the level of 6-month LIBOR with the lookback fixing weekly.

**Table 5: Prices of Lookback Caps**

| **Maturity** | **Cap Rate** | | | |
|---|---|---|---|---|
| | **0.053** | **0.054** | **0.055** | **0.056** |
| **3** | 0.0166 | 0.0153 | 0.0141 | 0.0129 |
| **4** | 0.0245 | 0.0227 | 0.0210 | 0.0194 |
| **5** | 0.0326 | 0.0304 | 0.0283 | 0.0263 |

### 3.2.3 Average Rate Caps

For average rate caps the payoff for each caplet is determined by the average level of LIBOR over the life of the capping period, i.e. the payoff for the caplet period from $k\Delta t$ to $(k+1)\Delta t$ is set by the average level of LIBOR between $k\Delta t$ and $(k+1)\Delta t$. The payoff at time $(k+1)\Delta t$ is therefore;

$$\Delta\tau L \max(A((k+1)\Delta\tau) - r_c, 0) \tag{23}$$

where $A((k+1)\Delta\tau)$ is the average level of LIBOR realised between $k\Delta\tau$ and $(k+1)\Delta\tau$.

At $k\Delta t$ set $f(n,j,0) = f(n,j,M) = R(n,j,\Delta\tau)$ for all *j* at time step *n*. Sweep through the tree until $(k+1)\Delta\tau$ updating the maximum and minimum of the average of LIBOR attained in the tree. Suppose that there have been *m* fixings after *i* time steps, therefore $m=1$ when $i=n$, $m=2$ when $i=n+1$, etc. For the maximum average value at $(i+1)\Delta t$ we have;

$$f(i+1, j_u, M) := \max\left\{\frac{mf(i,j,M) + R(i+1, j_u, \Delta\tau)}{m+1}, f(i+1, j_u, M)\right\} \tag{24}$$

At maturity $N\Delta t$ the maturity condition is evaluated as;

$$C(N,j,l) = \Delta\tau L \max\{f(N,j,l) - r_c, 0\} \qquad l = 0, \ldots M \tag{25}$$

Table 6 shows the relationship between the cap rate and the price of an average rate cap for varying maturities. The table is constructed under the same assumptions of table 4. The cap is assumed to reset semi-annually based on the level of 6-month LIBOR with the averaging fixed weekly.

**Table 6: Prices of Average Rate Caps**

| Maturity | Cap Rate | | | |
|---|---|---|---|---|
| | **0.053** | **0.054** | **0.055** | **0.056** |
| 3 | 0.0102 | 0.0092 | 0.0084 | 0.0075 |
| 4 | 0.0156 | 0.0143 | 0.0131 | 0.0119 |
| 5 | 0.0215 | 0.0198 | 0.0183 | 0.0168 |

### 3.2.4 Index Amortising Rate Swaps[4]

Index amortising rate (IAR) swaps are agreements to exchange fixed for floating rate payments on pre-specified dates on a principal amount that may decline through time; the reduction in principal depending on the level of interest rates. A typical principal reduction schedule is illustrated in table 7. The schedule determines how the principal underlying the swap will be reduced according to some index. Popular choices for the index include LIBOR of different tenors and Constant Maturity Treasury rates (CMT).

**Table 7: Principal Reduction Schedule for an Index Amortising Swap**

| **Index Level (offset in basis points)** | **Principal Reduction** |
|---|---|
| $X$ - 200 or lower | 100 |
| $X$ - 100 | 60 |
| $X$ | 40 |
| $X + 100$ | 20 |
| $X + 200$ | 10 |
| $X + 300$ or higher | 5 |

Consider a 3 year IAR swap with a starting principal of $100m where the principal repayments are determined according to the above schedule. The base rate, $X$, is

4 This section is based on Clewlow, et al. [1996].

determined at the outset of the swap at, say 5%. Payments are only made on the semi-annual interest payment dates and the principal reduction may not exceed the outstanding principal. Suppose that 3-month LIBOR is currently 5%. The annual amortisation rate is therefore 40%. A LIBOR level of 6.5% would lead to an amortisation rate of 15% per annum[5]. The terms of the swap sometimes preclude any reduction in the principal until after a given time called the *lockout period.*

Like a plain vanilla swap, the value of an index IAR swap is the difference between the value of the two bonds, which we denote by $B_{fix}$ for the fixed side, and $B_{fl}$ for the floating side, underlying the swap. The floating side equals the swap principal immediately after a reset date, whilst the fixed side must be calculated from the short rate tree. The value of the fixed rate bond depends on the level of interest rates and the outstanding principal. The principal satisfies the condition that its level at time $t+\Delta t$ can be calculated from the level at time $t$ and the interest rate at time $t+\Delta t$.

Depending on the path of the short rate through the tree, the number of alternative principal amounts that can be realised at any node grows quickly with the number of time steps and so we apply our techniques for valuing path dependent derivatives with $f(i,j,l)$ denoting the outstanding principal at node $(i,j)$ and $C(i,j,l)$ for the value of $B_{fix}$ at $(i,j)$ when the principal has value $f(i,j,l)$. Also let $pr(i,j)$ denote the principal reduction at node $(i,j)$, that is the amount deducted from the current principal, which is determined according to the scheme in table 7 dependent on $R(i,j,\Delta\tau)$.

If we are moving from node $(i,j)$ to node $(i+1,j_u,l)$ and if the amortisation rate applies to the outstanding principal, then the principal reduction is given by

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5 We are assuming that the terms of the deal specify that linear interpolation applies to the reduction schedule.

$pr(i+1, j_u) f(i,j,l)$ which is used to determine the maximum and minimum principals at the node $(i+1, j_u)$. If the rate applies to the original principal, $f(0,0,k)$, then, in order that the principal cannot become negative, the reduction is given by $\min(pr(i+1, j_u) f(0,0,l), f(i,j,l))$. This process continues until the end of the life of the swap.

Once the forward induction step is completed, the value of the bond at maturity, $C(n,j,l)$, can be calculated for all *j* and all *l*[6]:

$$C(n,j,l) = f(n,j,l) \tag{26}$$

To calculate the value of $B_{fix}$ at node (*i,j*) for $i < n$ we use backwards induction. For a European derivative this implies that

$$C(i,j,l) = e^{-r(i,j)\Delta t}\left[p_u C(i+1,u,l_u) + p_m C(i+1,m,l_m) + p_d C(i+1,d,l_d) + c(i,j)\right] \tag{27}$$

where $c(i,j)$ is the cashflow (interest plus principal reduction) during the period $i\Delta t$ and $(i+1)\Delta t$.

The value of the swap for the receiver of fixed payments is then the difference between $B_{fix}$ and the principal of the swap.

Table 8 presents fair value swap rates for index amortising swaps of various maturities. The table is constructed under the same modeling assumptions of table 4. In addition the principal reduction schedule is given by table 7 with linear

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6 For $k \neq 0$ and $k \neq M$ we interpolate *f*.

interpolation and an initial lockout period of 2 years. The swap is assumed to reset semi-annually based on the level of 3-month LIBOR.

**Table 8: Index Amortising Swap Rates**

| Maturity | Swap Rate |
|---|---|
| 3 | 0.0515 |
| 4 | 0.0525 |
| 5 | 0.0535 |

## 4. Conclusions

In this paper we have described numerical procedures for pricing a wide range of interest rate exotics working within the short rate tree framework favoured by Hull-White, Black-Derman-Toy, and Black-Karasinski amongst others. We have shown that both American featured and path dependent interest rate options can be priced efficiently given that the tree has already been constructed. In a forthcoming paper we show how these procedures can be further improved by using more appropriate discretisation and interpolation methods and we analyse the trade-off between the approximation errors and the computational speed.

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