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Abstract

Option prices observed in the market do not agree with the assumptions of the Black and Scholes model. In particular, the volatility, which is assumed constant in the model, takes different values for options with different maturities or strike prices. Dupire showed that, if we relax the hypotheses of the Black and Scholes model, and we assume the stock prices are driven by a general diffusion, it is always possible to recover such a process that mimics a given price system (provided that the price system is sufficiently regular). The aim of this paper is to investigate the behaviour of the system of option prices whose underlying is driven by a discontinuous process (more precisely, by a lognormal jump process): we want to check whether this prices system can be recovered by a diffusion or not, and we want to analyse its local volatility function.

1 Introduction

The problem of option pricing is to specify a model so that it is possible to compute the option prices as a function of the parameters of the model. The Black and Scholes model is at the same time a typical example and a cornerstone in option pricing. It assumes that the stock price is driven by a geometric Brownian motion: the solution of the model depends therefore on

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two features of the process that drives the underlying asset price, *continuity* and the *Markov property*. It is well known that many of the hypotheses underlying the Black and Scholes model are quite implausible. In particular, the volatility of the underlying asset is not constant. In fact, we can observe option prices in the market, and we can compute the value of the volatility that fits exactly the price of the option in the Black and Scholes model – i.e., we can compute the *implied volatility* of the option. Then, the values of the implied volatility for different options show a strong dependence both on the strike price and on the maturity of the option.

Following Dupire [2], volatility can be allowed to be a function of time and of the current price of the stock. The underlying asset price is then driven by a diffusion. On the other hand, there is a price that we have to pay: the system of option prices must be sufficiently smooth – i.e., it must fulfill the regularity conditions shown below. Provided that the option prices satisfy these conditions, it is always possible to recover a diffusion that mimics these prices.

Instead, we can choose a different model, keeping the Markov property of the process that drives the stock price, but relaxing the continuity. In this model the underlying asset price is driven by a jump process (more precisely by a *lognormal jump process*): option pricing is still possible, but some additional assumptions on the structure of the market is needed, as Merton [6] showed.

In this paper we want to investigate what happens if we apply the Dupire methodology to the system of option prices generated by the Merton jump model. In section 2 the Merton jump model is briefly explained, in section 3 it is proved that the prices system generated by a lognormal jump process cannot be recovered by a diffusion, since only one of the two regularity condition is fulfilled. In section 4 it is discussed the behaviour of the local volatility function corresponding to the prices system generated by a lognormal jump process. Section 5 concludes.

2 The Merton Jump Model

Merton assumes that the stock price is the composition of two types of changes: a normal change, driven by a Brownian motion, and an abnormal change, driven by a Poisson process. Such a process is called a *jump process*. When a Poisson event occurs, moreover, there is a drawing from a distribution to determine the impact of the event on the stock price: if S_t is the stock price at time t and Y is the random variable describing the drawing, then

$$S_t = Y S_{t^-}$$

where S_{t^-} is the left limit of S_t, i.e. $S_{t^-} = \lim_{s \to t^-} S_s$. The stock price process can be described by a stochastic differential equation

$$dS_t = (\mu - \lambda k) S_{t^-}\, dt + \sigma S_{t^-}\, dB + (Y - 1)\, S_{t^-}\, dq \tag{1}$$

where μ is the expected return on the stock, σ^2 is the variance of the return (given that no Poisson event occurs), dB is a standard Brownian motion, dq is a Poisson process with rate λ and k is defined as follows:

$$k = \mathbb{E}\,[Y - 1]$$

μ, σ^2, λ and k are real, positive constants;The stochastic differential equation (1) has an explicit solution

$$S_t = S_0 \exp\left[\left(\mu - \lambda k - \frac{\sigma^2}{2}\right) t + \sigma B_t\right] Y^{(n)}$$

where B_t is a Gaussian random variable with mean zero and variance t; $Y^{(0)} = 1$ and $Y^{(n)} = \prod_{j=1}^{n} Y_j$ for $n \geq 1$, where $\{Y_j\}$ are all independent and identically distributed to Y; n is Poisson distributed with mean λt. In particular, if Y is itself lognormally distributed, i.e. $\log Y$ is normally distributed with mean $\log(k+1) - \delta^2/2$ and variance δ^2,then the distribution of S_t is itself lognormal. Such a process is called a *lognormal jump process*[1].

It is possible to derive the partial differential equation for the price $\Lambda\,(S,t)$ of a contingent claim on a stock whose prices follows the process (1) expiring at time $T > t$, if we make the further assumption that the jump component is uncorrelated with the market:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Lambda}{\partial S^2} + rS\frac{\partial \Lambda}{\partial S} + \frac{\partial \Lambda}{\partial t} + \lambda \mathbb{E}\left[\Lambda\,(SY, t) - \Lambda\,(S, t)\right] = r\Lambda \tag{2}$$

subject to the appropriate boundary conditions. The solution of the equation can be computed explicitly, since we know the distribution of Y.

The price $C_{S,t}(X,T)$ of a standard European call option expiring at time T with strike price X (given that the of the underlying at time t is S) satisfies the boundary conditions

$$C_{0,t}(X,T) = 0$$

$$C_{S,T}(X,T) = \max(0, S - X)$$

[1]It is not a standard definition.

It is possible to compute the explicit solution of the equation (2) with the above boundary conditions for a lognormal jump process. The price of an European call option whose underlying asset is driven by such a process – as Merton showed [6] – is then given by:

$$\begin{aligned} C_{S,t}(X,T) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} W(X,\tau,\sigma_n,r_n) \qquad (3) \\ &= \sum_{n=0}^{\infty} p'_n W_n \end{aligned}$$

where r is the risk free interest rate,

$$\tau = T - t$$

$$\lambda' = \lambda(1+k)$$

and for $n = 0, 1, 2, \ldots$:

$$\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}$$

$$r_n = r - \lambda k + \frac{n \log(1+k)}{\tau}$$

$$p'_n = \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}$$

$W(X,\tau,\sigma,r)$ is defined by the Black and Scholes formula, i.e.

$$W(X,\tau,\sigma,r) = SN(d_1) - Xe^{-r\tau}N(d_2) \qquad (4)$$

$$d_1 = \frac{\log S - \log X + r\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

where $N(x)$ is the standard cumulative normal distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

For $n = 0, 1, \ldots$, W_n is then defined as follows:

$$W_n = N(d_1^{(n)}) - Xe^{-r_n\tau}N(d_2^{(n)})$$

$$
\begin{aligned}
d_1^{(n)} &= \frac{\log S - \log X + r_n \tau}{\sigma_n \sqrt{\tau}} + \frac{1}{2}\sigma_n \sqrt{\tau} \\
d_2^{(n)} &= d_1^{(n)} - \sigma_n \sqrt{\tau}
\end{aligned}
$$

For the sake of simplicity, in the following we assume that jumps up and jumps down are equally likely. It implies that

$$k \equiv \mathbb{E}\left[Y - 1\right] = 0$$

Moreover, we use a saving account which grows at the risk free rate as numeraire, and hence

$$r = 0$$

Then, the above formulas are further simplified:

$$\lambda' = \lambda$$

and for $n = 0, 1, 2, \ldots$:

$$
\begin{aligned}
r_n &= r = 0 \\
p'_n &= p_n = \frac{e^{-\lambda\tau} (\lambda\tau)^n}{n!} \\
d_1^{(n)} &= \frac{\log S - \log X}{\sigma_n \sqrt{\tau}} + \frac{1}{2}\sigma_n \sqrt{\tau} \\
d_2^{(n)} &= d_1^{(n)} - \sigma_n \sqrt{\tau}
\end{aligned}
$$

3 Diffusion versus Jump Processes

Given a system of option prices $C_{S,t}(X, T)$, i.e. the set of option prices for all the possible values of the strike price and of the maturity[2], Dupire showed that it is possible to recover for the underlying price a diffusion process under the risk neutral probability measure which mimics the prices system. The process is driven by the following stochastic differential equation,

$$dS = s(S, t)\, dB \tag{5}$$

[2]In this idealized model, the strike price X can take any value in the interval $(0, +\infty)$ and the maturity T varies in the interval $(t, \overline{T})$, where $\overline{T}$ is the time horizon.

It can be shown that the local volatility s is then given by a function of the derivatives of $C_{S,t}(X,T)$

$$s(X,T) = \frac{1}{X}\sqrt{2\frac{\partial C_{S,t}(X,T)}{\partial T} \Big/ \frac{\partial^2 C_{S,t}(X,T)}{\partial X^2}} \tag{6}$$

This reconstruction is always possible, provided that the following conditions are fulfilled by the prices system $C_{S,t}(X,T)$:

1. *Slow growth condition*

$$\frac{\partial C_{S,t}(X,T)}{\partial T} \leq \gamma X^2 \frac{\partial^2 C_{S,t}(X,T)}{\partial X^2} \tag{7}$$

   for a certain positive real number γ and for large values of X; this condition is necessary in order to ensure the existence and uniqueness of the solution of the stochastic differential equation (5) and it is also required for (6) to remain bounded;

2. *Dupire condition*

$$\lim_{X \to +\infty} \frac{\partial C_{S,t}(X,T)}{\partial T} = 0 \tag{8}$$

   this condition is necessary in order to derive relation (6).

In the following, we will show that the prices system (3) generated by a lognormal jump process fulfills the second condition, but not the first, and therefore cannot be recovered by a diffusion process (5).

3.1 Slow Growth Condition

It can be checked directly that condition (7) does not hold. The derivative of $C_{S,t}(X,T)$ with respect to T is given by:

$$\frac{\partial C_{S,t}(X,T)}{\partial T} = \frac{\partial C_{S,t}(X,T)}{\partial \tau} = \sum_{n=0}^{\infty} \left[p_n \left(\frac{n}{\tau} - \lambda \right) W_n + p_n \frac{\partial W_n}{\partial \tau} \right] \tag{9}$$

and the second derivative of $C_{S,t}(X,T)$ with respect to X is given by:

$$\frac{\partial^2 C_{S,t}(X,T)}{\partial X^2} = \sum_{n=0}^{\infty} p_n \frac{\partial^2 W_n}{\partial X^2}$$

Recalling the following relations:

$$\frac{\partial W}{\partial \tau} = \frac{X\sigma e^{-r\tau}}{2\sqrt{\tau}}n(d_2) + rXe^{-r\tau}N(d_2) \tag{10}$$

$$\frac{\partial^2 W}{\partial X^2} = \frac{e^{-r\tau}n(d_2)}{X\sigma\sqrt{\tau}} \tag{11}$$

where

$$n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

is the standard normal density function, and that the risk free asset is used as numeraire, we get:

$$\frac{\partial C_{S,t}(X,T)}{\partial T} = \sum_{n=0}^{\infty}\left[p_n\left(\frac{n}{\tau}-\lambda\right)W_n + p_n\frac{X\sigma_n}{2\sqrt{\tau}}n(d_2^{(n)})\right]$$

where the first term in the summation arises from differentiating p_n with respect to T, and :

$$\frac{\partial^2 C_{S,t}(X,T)}{\partial X^2} = \sum_{n=0}^{\infty} p_n \frac{n(d_2^{(n)})}{X\sigma_n\sqrt{\tau}}$$

Both $\partial C_{S,t}(X,T)/\partial T$ and $\partial^2 C_{S,t}(X,T)/\partial X^2$ are positive quantities since all the terms of the two series, but $(n-\lambda\tau)/\tau$, are positive and by the Harris inequality[3]

$$\sum_{n=0}^{\infty} p_n\left(\frac{n}{\tau}-\lambda\right)W_n \geq 0$$

[3]The Harris inequality states that if f and g are both increasing (or both decreasing) functions, then

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\,\mathbb{E}[g(X)]$$

Since n is a random variable Poisson distributed with parameter λ, and $W_n \leq W_{n+1}$ (in fact, $\partial W/\partial\sigma > 0$ and $\sigma_n \leq \sigma_{n+1}$), we have:

$$\begin{aligned}\sum_{n=0}^{+\infty} p_n n W_n &= \mathbb{E}_n[nW_n] \\ &\geq \mathbb{E}_n[n]\,\mathbb{E}_n[W_n] = \lambda\tau\mathbb{E}_n[W_n] \\ &= \lambda\tau\sum_{n=0}^{+\infty} p_n W_n\end{aligned}$$

and therefore

$$\sum_{n=0}^{+\infty} p_n(n-\lambda\tau)W_n \geq 0$$

A complete proof of the Harris inequality can be found in Durrett [3].

Hence:

$$\frac{\partial C_{S,t}(X,T)}{\partial T} \Big/ X^2 \frac{\partial^2 C_{S,t}(X,T)}{\partial X^2} \geq \frac{\sum_{n=0}^{\infty} p_n \sigma_n^2 g_n(X)}{\sum_{n=0}^{\infty} p_n g_n(X)}$$

where

$$g_n(X) = \frac{X n(d_2^{(n)})}{\sigma_n \sqrt{T}}$$

Since:

$$\begin{aligned}
\lim_{X \to +\infty} \frac{g_j(X)}{g_k(X)} &= \lim_{X \to +\infty} \frac{\sigma_k\, n(d_2^{(j)})}{\sigma_j\, n(d_2^{(k)})} \\
&= \lim_{X \to +\infty} \frac{\sigma_k}{\sigma_j} \exp\left\{ [\log S - \log X]^2 \frac{(j-k)\,\delta^2}{\sigma_j^2 \sigma_k^2 T^2} + \frac{1}{4}(k-j)\,\delta^2 \right\} \\
&= \begin{cases} +\infty & \text{if } j > k \\ 1 & \text{if } j = k \\ 0 & \text{if } j < k \end{cases}
\end{aligned}$$

If we choose X large enough, for $j = 0, 1, \ldots, k$ we have:

$$p_j g_j(X) \leq \frac{1}{k} p_k g_k(X)$$

Hence:

$$\begin{aligned}
\frac{\sum_{n=0}^{\infty} p_n \sigma_n^2 g_n(X)}{\sum_{n=0}^{\infty} p_n g_n(X)} &\geq \sigma_k^2 \frac{\sum_{n=k}^{\infty} p_n g_n(X)}{\sum_{n=0}^{\infty} p_n g_n(X)} \\
&\geq \sigma_k^2 \frac{\sum_{n=k}^{\infty} p_n g_n(X)}{\underbrace{\frac{1}{k} p_k g_k(X) + \ldots + \frac{1}{k} p_k g_k(X)}_{k} + \sum_{n=k}^{\infty} p_n g_n(X)} \\
&= \sigma_k^2 \frac{\sum_{n=k}^{\infty} p_n g_n(X)}{p_k g_k(X) + \sum_{n=k}^{\infty} p_n g_n(X)} \\
&\geq \sigma_k^2 \frac{\sum_{n=k}^{\infty} p_n g_n(X)}{2 \sum_{n=k}^{\infty} p_n g_n(X)} = \frac{\sigma_k^2}{2}
\end{aligned}$$

Since

$$\lim_{k \to +\infty} \sigma_k^2 = \lim_{k \to +\infty} \sigma^2 + k \frac{\delta^2}{T} = +\infty$$

it follows

$$\lim_{X \to +\infty} \frac{\partial C_{S,t}(X,T)}{\partial T} \Big/ X^2 \frac{\partial^2 C_{S,t}(X,T)}{\partial X^2} = +\infty$$

and therefore condition (7) does not hold.

3.2 Dupire Condition

Since the series (9) converges uniformly, the Dupire condition (8) is fulfilled. We can check the uniform convergence of the series (9). The function $f_n(X) = Xn(d_2^{(n)})$ vanishes as X goes to infinity, $f_n(0) = 0$ and has its absolute maximum in $X_n^{\max} = S \exp\left[\sigma_n\sqrt{\tau} - \sigma_n^2\tau/2\right]$. The maximum value of f_n is then

$$f_n(X_n^{\max}) = S\frac{e^{\sigma_n\sqrt{\tau}-\sigma_n^2\tau/2-1/2}}{\sqrt{2\pi}} = F_n$$

Then, let F be

$$F = \max_{n=0,1,\dots} F_n$$

since $\lim_{n\to+\infty} F_n = 0$, F is well defined. Therefore:

$$\frac{\partial W_n}{\partial\tau} = \frac{\sigma_n}{2\sqrt{\tau}} f_n(X) \le \frac{\sigma_n}{2\sqrt{\tau}} F = \kappa_n$$

If X is large enough, then

$$\begin{aligned} \frac{\partial C_{S,t}(X,T)}{\partial T} &= \sum_{n=0}^{\infty}\left[p_n\left(\frac{n}{\tau}-\lambda\right)W_n + p_n\frac{\partial W_n}{\partial\tau}\right] \\ &\le \sum_{n=0}^{\infty}\left[p_n\left|\frac{n}{\tau}-\lambda\right| + p_n\kappa_n\right] \end{aligned}$$

Hence, for any given $\varepsilon > 0$, provided that n_0 is sufficiently large,

$$\sum_{n=n_0}^{\infty}\left[p_n\left(\frac{n}{\tau}-\lambda\right)W_n + p_n\frac{\partial W_n}{\partial\tau}\right] \le \sum_{n=n_0}^{\infty} p_n\left|\frac{n}{\tau}-\lambda\right| S + p_n\kappa_n < \varepsilon$$

since both $\sum_{n=n_0}^{\infty} p_n\left(n/\tau-\lambda\right)$ and $\sum_{n=0}^{+\infty} p_n\kappa_n$ converge absolutely, because $\kappa_n = O(\sqrt{n})$.

By the fact

$$\lim_{X\to+\infty} d_1^{(n)} = \lim_{X\to+\infty} d_2^{(n)} = -\infty$$

it follows that

$$\lim_{X\to+\infty} W_n = 0$$

and from (10) we have that

$$\lim_{X\to+\infty}\frac{\partial W_n}{\partial\tau} = 0$$

Therefore, since the series (9) converges uniformly as function of X we have

$$\begin{aligned}
\lim_{X\to+\infty} \frac{\partial C_{S,t}(X,T)}{\partial T} &= \lim_{X\to+\infty} \sum_{n=0}^{\infty} \left[p_n \left(\frac{n}{\tau} - \lambda \right) W_n + p_n \frac{\partial W_n}{\partial \tau} \right] \\
&= \sum_{n=0}^{\infty} \lim_{X\to+\infty} \left[p_n \left(\frac{n}{\tau} - \lambda \right) W_n + p_n \frac{\partial W_n}{\partial \tau} \right] \\
&= 0
\end{aligned}$$

and the condition (8) holds.

4 The Local Volatility Function

Since the slow growth condition (7) does not hold, it is not possible to recover a diffusion that mimics the price system generated by a lognormal jump process. Nevertheless, we can use formula (6) to compute the local volatility[4] $s(X,T)$. $s(X,T)$ is then the local value of the volatility of the diffusion that mimics the price system C, if the behaviour for large X of C were different and the conditions (7) and (8) were fulfilled. The meaning of local is here twofold: s gives a local value of the volatility and the diffusion can recover the price system only locally (since the conditions (7) and (8) are fulfilled only locally).

The first feature of the function $s(X,T)$ is that it has a singularity for T approaching t, i.e. for τ vanishing. In fact:

$$\lim_{T\to t} s(X,T) = \begin{cases} +\infty & \text{if } X \neq S \\ 0 & \text{if } X = S \end{cases} \tag{12}$$

This fact can be proved following the same steps shown in section 3.1. In fact:

$$\begin{aligned}
s^2(X,T) &= 2\frac{\partial C_{S,t}(X,T)}{\partial T} \Big/ X^2 \frac{\partial^2 C_{S,t}(X,T)}{\partial X^2} \\
&\geq \frac{\sum_{n=0}^{\infty} p_n \sigma_n^2 g_n(\tau)}{\sum_{n=0}^{\infty} p_n g_n(\tau)}
\end{aligned} \tag{13}$$

where

$$g_n(\tau) = \frac{X n(d_2^{(n)})}{\sigma_n \sqrt{\tau}}$$

[4]The variables of the function s are called X and T to stress the close relation with the strike price and the matutity of the options that form the price system, as it is clear from (6).

is defined as in section 3.1, but it is considered here as function of τ. If $S \neq X$, then

$$\lim_{\tau\to 0}\frac{g_j(\tau)}{g_k(\tau)} = \lim_{\tau\to 0}\frac{\sigma_k\, n(d_2^{(j)})}{\sigma_j\, n(d_2^{(k)})} = \begin{cases} +\infty & \text{if } j > k \\ 1 & \text{if } j = k \\ 0 & \text{if } j < k \end{cases}$$

and hence, for each chosen k, if τ is small enough

$$\frac{\sum_{n=0}^{\infty} p_n \sigma_n^2 g_n(\tau)}{\sum_{n=0}^{\infty} p_n g_n(\tau)} \geq \frac{\sigma_k^2}{2}$$

It implies that

$$\lim_{T\to t} s(X,T) = +\infty$$

Instead, if $S = X$

$$\lim_{\tau\to 0}\frac{g_j(\tau)}{g_k(\tau)} = \frac{\sigma_k}{\sigma_j}\exp\left[\frac{1}{4}(k-j)\,\delta^2\right]$$

and then the inequality (13) is not useful to compute the limit. In this case we can compute which is the infinitesimal of higher order in both the numerator and the denominator. We have

$$s^2(X,T) = \frac{\sum_{n=0}^{+\infty} p_n\left[2\,(n-\lambda\tau)\,W_n + X\sigma_n\sqrt{\tau}\,n(d_2^{(n)})\right]}{\sum_{n=0}^{+\infty} p_n\sqrt{\tau}\sigma_n^{-1}n(d_2^{(n)})}$$

Since, for τ vanishing

$$d_1^{(n)} = \sqrt{\tau}\frac{\sigma_n}{2}$$

$$d_2^{(n)} = -\sqrt{\tau}\frac{\sigma_n}{2}$$

$$N(d_1^{(n)}), N(d_2^{(n)}) \underset{\tau\to 0}{\longrightarrow} \frac{1}{2}$$

$$n(d_1^{(n)}), n(d_2^{(n)}) \underset{\tau\to 0}{\longrightarrow} \frac{1}{\sqrt{2\pi}}$$

$$p_n = O(\tau^n)$$

$$W_n = Sn(d_2^{(n)})\sigma_n\sqrt{\tau} + o(\sqrt{\tau})$$

for small values of τ we have

$$s^2(X,T) = \frac{p_0 S \sigma_0 \sqrt{\tau}\, n(d_2^{(0)}) + o(\sqrt{\tau})}{p_0 S \sigma_0^{-1} \sqrt{\tau}\, n(d_2^{(0)}) + o(\sqrt{\tau})} \underset{T \to t}{\longrightarrow} \sigma_0^2 = \sigma^2$$

Therefore, formula (12) is correct.

The function $s(X,T)$ can be studied numerically, for fixed values of the parameters and for certain range of X and T. With the help of a FORTRAN code, we have plotted the graph of the local volatility function for this choice of the parameter

$$\lambda = 0.1 \qquad\qquad \sigma = 0.2 \qquad\qquad \delta = 0.1$$

Since the risk free asset is used as numeraire, $r = 0$. We have set the current asset price equal to one

$$S = 1$$

and $t = 0$ – and hence $T = \tau$.The range of X is the interval $[0.5, 2]$ and the range of T is $[0.001, 2]$. We can think that T is measured in years, so that the values of the parameter are realistic.

The numerical results agree with the theoretical behaviour of the local volatility for small values of T. The price of an option close to maturity exactly at the money behaves as if it is affected only by the Brownian motion. If a jump happens, the option would become either out or in the money with virtually no chance to change its character, since it is very close to expiration. Because jumps up and jumps down have the same probability ($k = \mathbb{E}[Y - 1] = 0$), these two effects cancel out and the price varies as if it feels only the Brownian motion. On the other hand, if $S \neq X$ a jump with suitable amplitude could change the character of the option in such a way that it is very unlikely to change it again, since the option is very close to expiration. In this case the chance are not even, and it causes a great uncertainty, which is reflected by very high values of the local volatility.

For large values of T, instead, the option price becomes less and less sensitive to the strike price and the local volatility approaches (indepedently from X) the value of the unconditional standard deviation per unit time, which is $\sqrt{\sigma^2 + \lambda\delta^2}$.

This behaviour is clear from the three dimensional graph of s. For large values of T, s is flat – but as T approaches 0 the surface ripples and forms a valley corresponding to $X = S$. This valley becomes more and more narrow and deep as T gets smaller. For T vanishing, we observe the expected singularity.

5 Conclusions

As Merton showed, option pricing is still possible if prices are driven by a log-normal jump process. Nevertheless, the set of prices generated by this model is not compatible with a diffusion, even if we consider a general volatility which is both time and price dependant. In fact, it is not possible to find a diffusion that genarates the same prices system, since condition (7) is not fulfilled.

The local volatility function is still well defined, as it can be computed by (6).In this case, the local volatility gives the parameter of a diffusion which can recover the prices system only locally (since the regularity conditions are fulfilled only locally). If we plot the local volatility as function of X and T, we observe strange behaviour. For large T, the volatility approaches the unconditional standard deviation (per unit time). For T vanishing, instead, we can observe a singularity: the volatility diverges to infinity if $X \neq S$, while for $X = S$ approaches the volatility of the continuous part of the process. For small values of T this effect is clearly recognizable: the volatility surface ripples and forms a deep and narrow valley around $X = S$.

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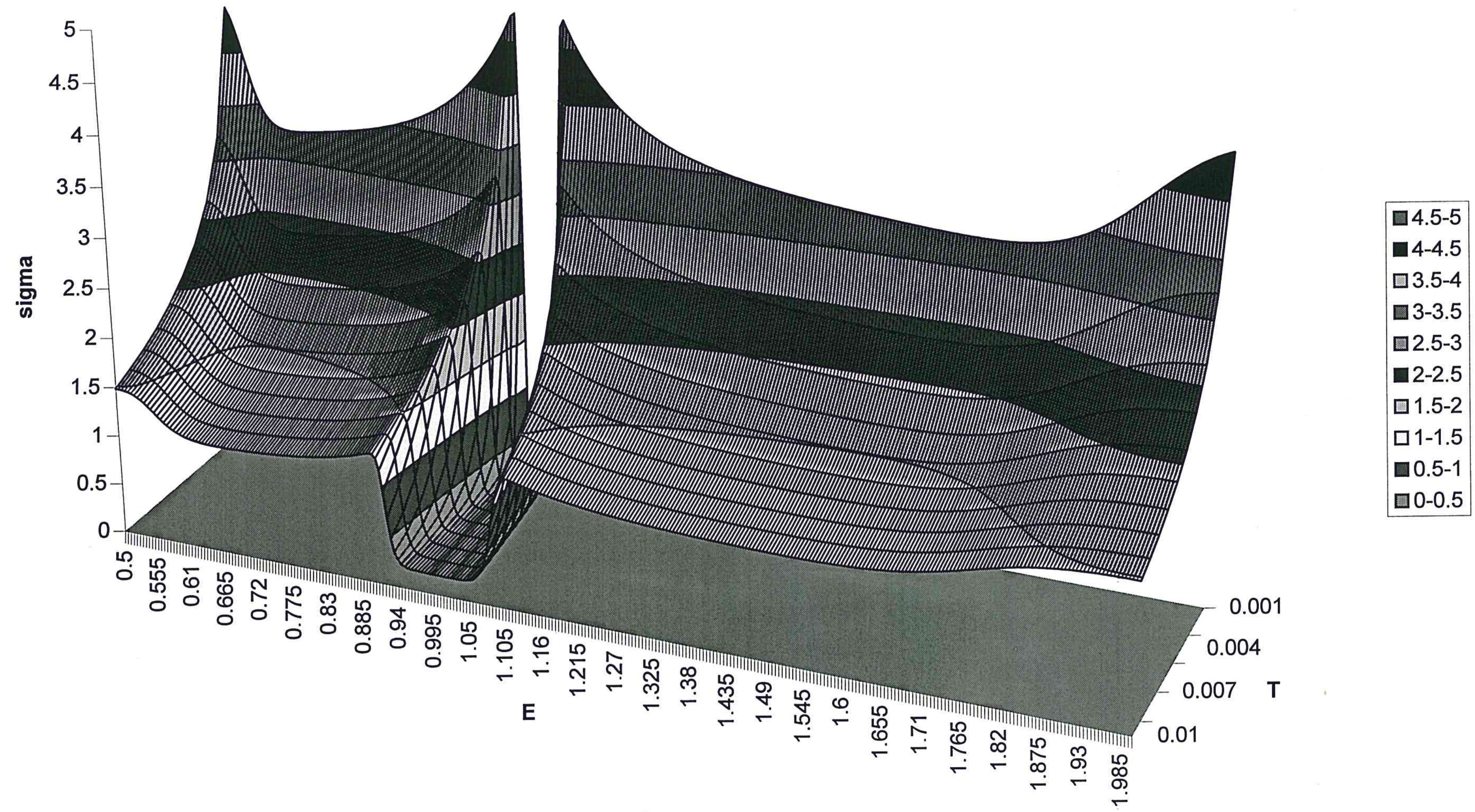
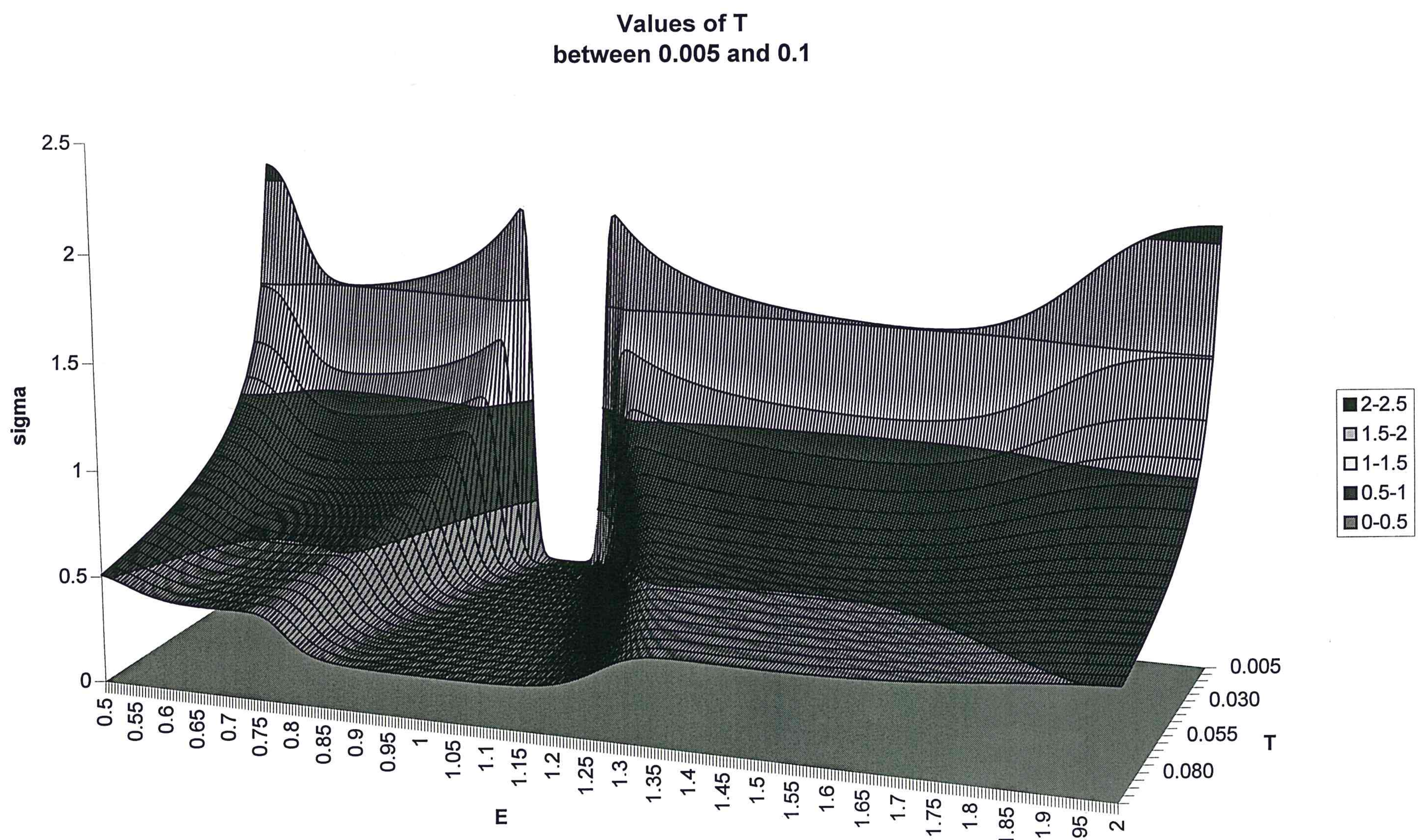
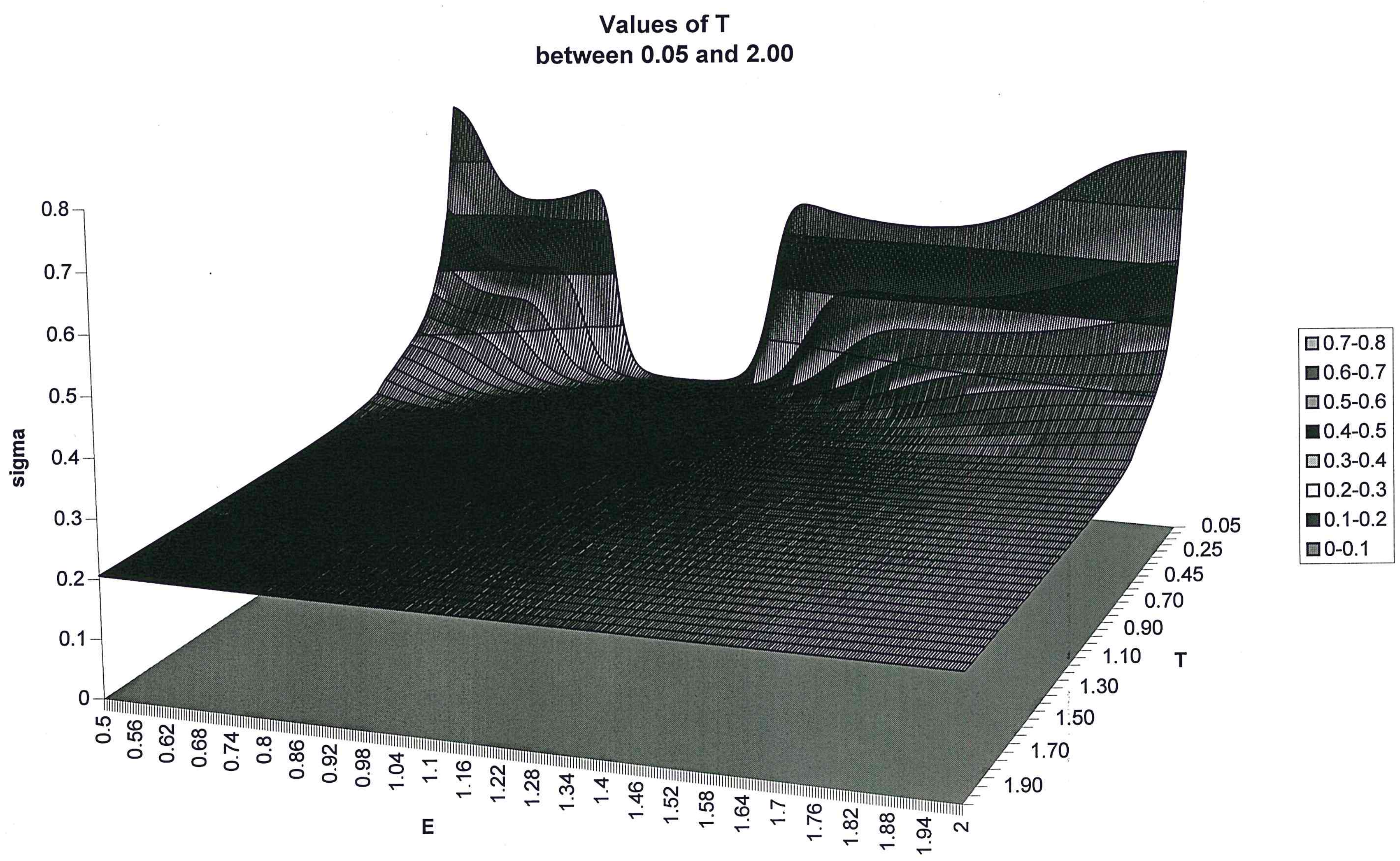
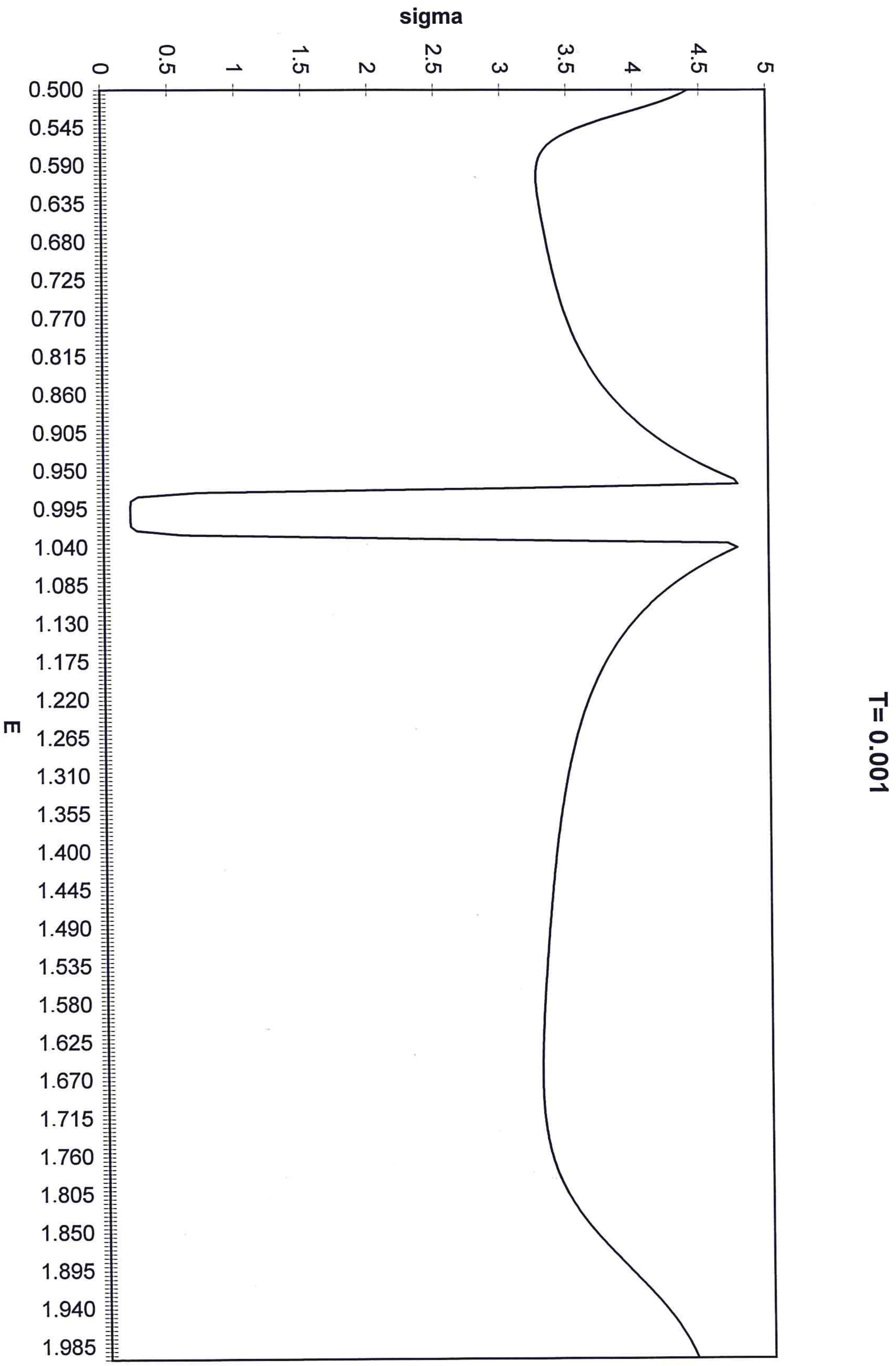
Values of T between 0.001 and 0.01



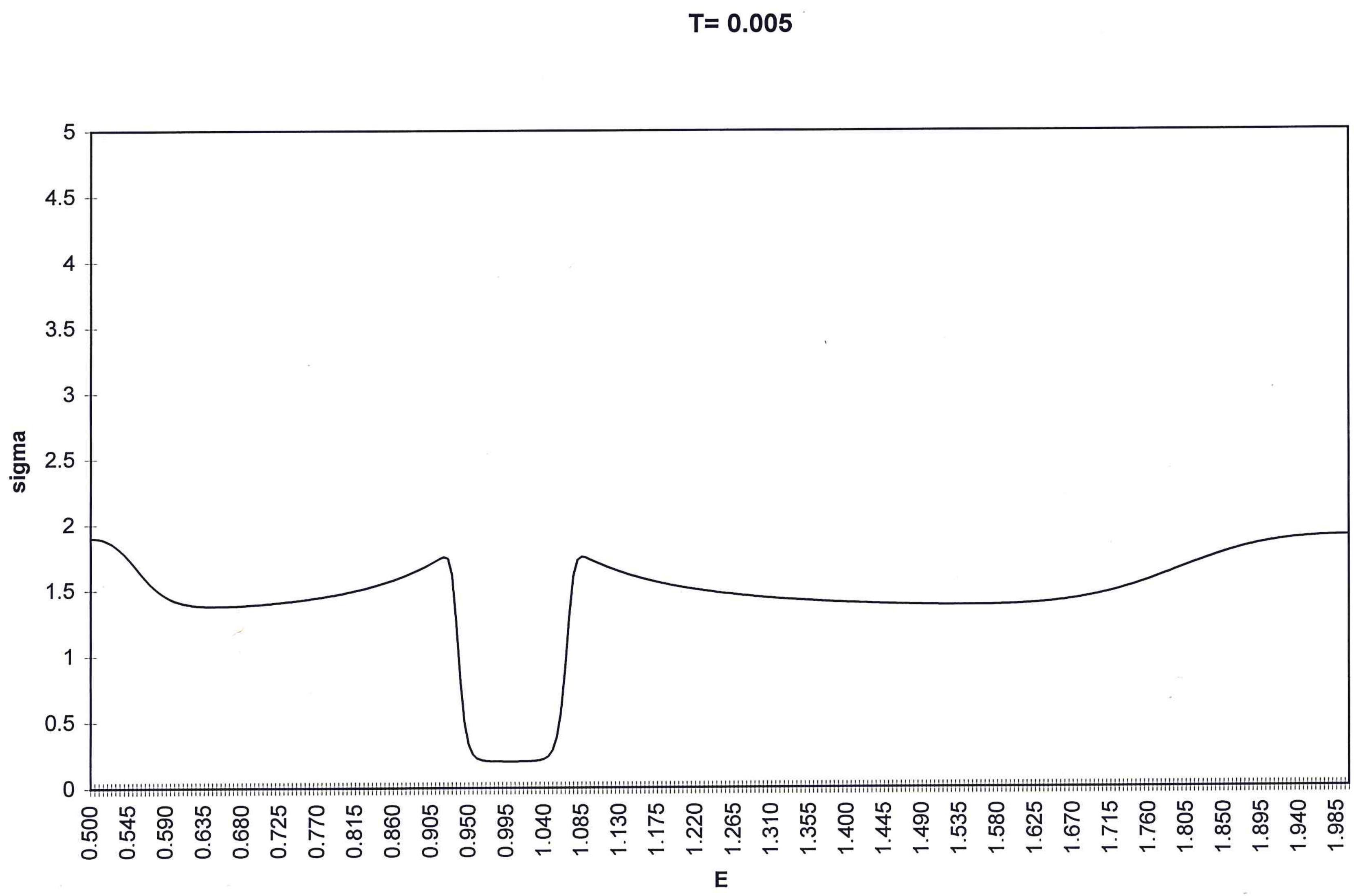

Values of T
between 0.005 and 0.1
sigma
2.5
2
1.5
1
0.5
0
E
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25
1.3
1.35
1.4
1.45
1.5
1.55
1.6
1.65
1.7
1.75
1.8
1.85
1.9
1.95
2
T
0.005
0.030
0.055
0.080
2-2.5
1.5-2
1-1.5
0.5-1
0-0.5



Values of T
between 0.05 and 2.00
sigma
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
E
0.5
0.56
0.62
0.68
0.74
0.8
0.86
0.92
0.98
1.04
1.1
1.16
1.22
1.28
1.34
1.4
1.46
1.52
1.58
1.64
1.7
1.76
1.82
1.88
1.94
2
T
0.05
0.25
0.45
0.70
0.90
1.10
1.30
1.50
1.70
1.90
0.7-0.8
0.6-0.7
0.5-0.6
0.4-0.5
0.3-0.4
0.2-0.3
0.1-0.2
0-0.1



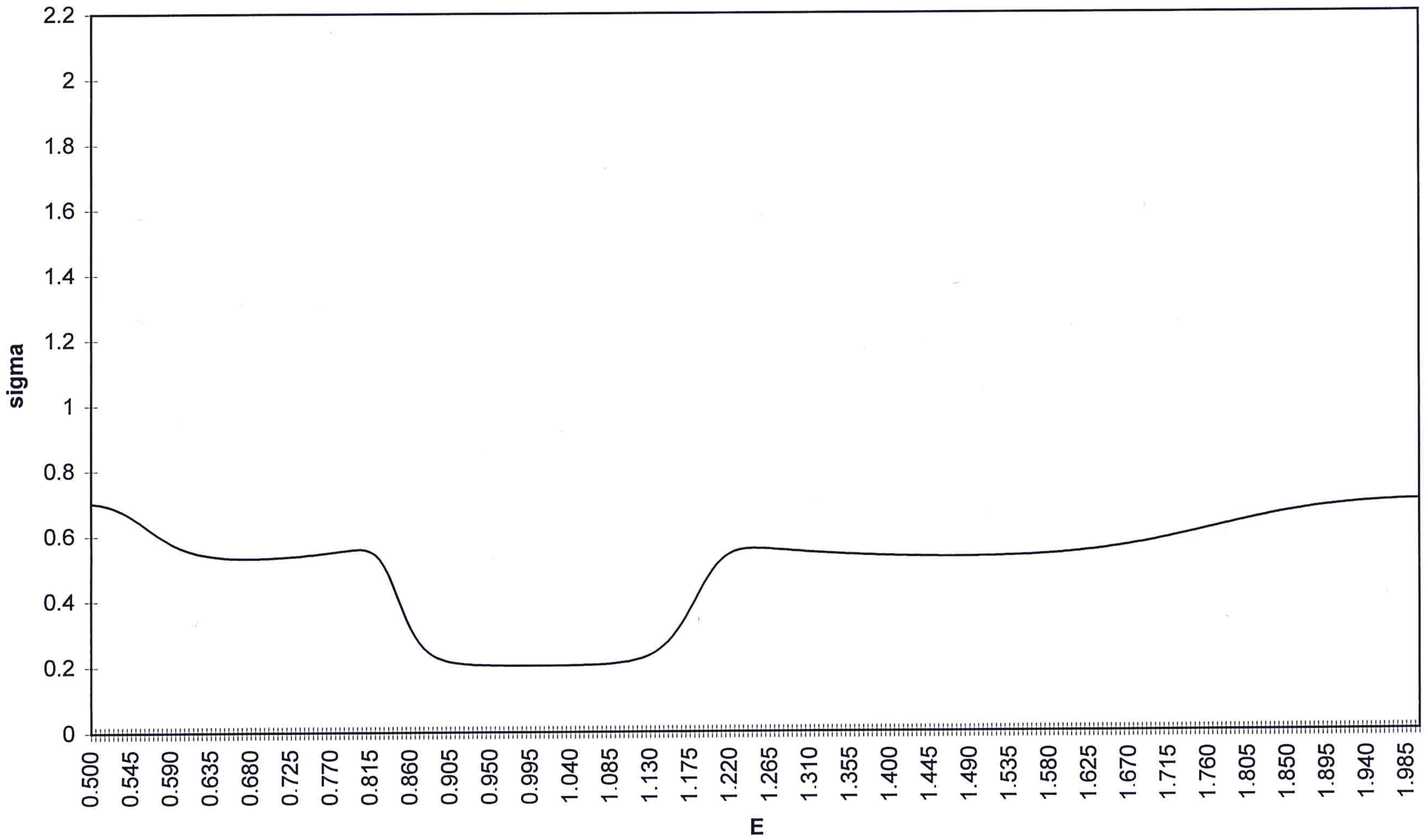
T= 0.001
sigma
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
E
0.500
0.545
0.590
0.635
0.680
0.725
0.770
0.815
0.860
0.905
0.950
0.995
1.040
1.085
1.130
1.175
1.220
1.265
1.310
1.355
1.400
1.445
1.490
1.535
1.580
1.625
1.670
1.715
1.760
1.805
1.850
1.895
1.940
1.985



T= 0.005
sigma
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0.500
0.545
0.590
0.635
0.680
0.725
0.770
0.815
0.860
0.905
0.950
0.995
1.040
1.085
1.130
1.175
1.220
1.265
1.310
1.355
1.400
1.445
1.490
1.535
1.580
1.625
1.670
1.715
1.760
1.805
1.850
1.895
1.940
1.985
E

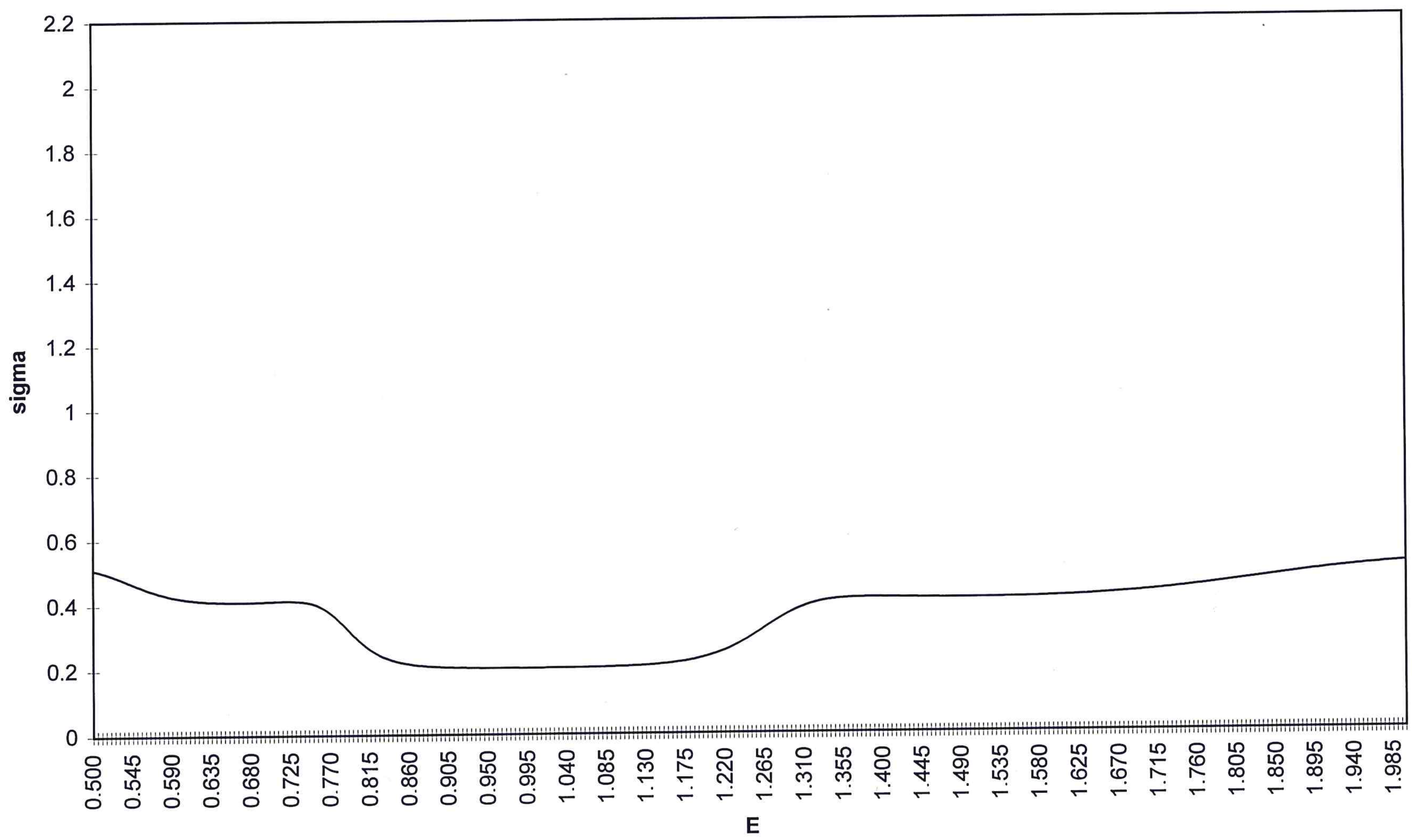

T= 0.01
sigma
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
E
0.500
0.545
0.590
0.635
0.680
0.725
0.770
0.815
0.860
0.905
0.950
0.995
1.040
1.085
1.130
1.175
1.220
1.265
1.310
1.355
1.400
1.445
1.490
1.535
1.580
1.625
1.670
1.715
1.760
1.805
1.850
1.895
1.940
1.985

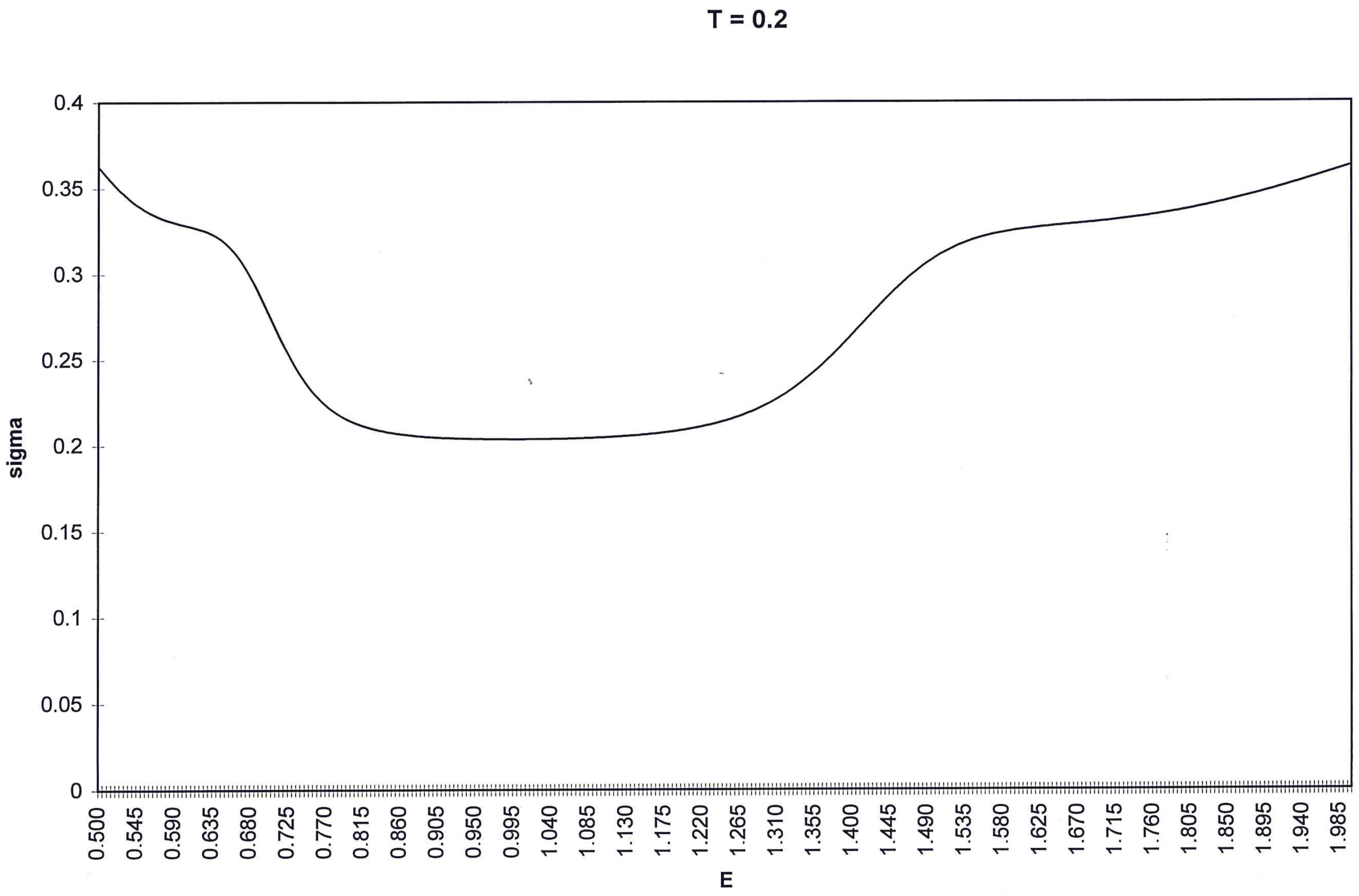
T= 0.05



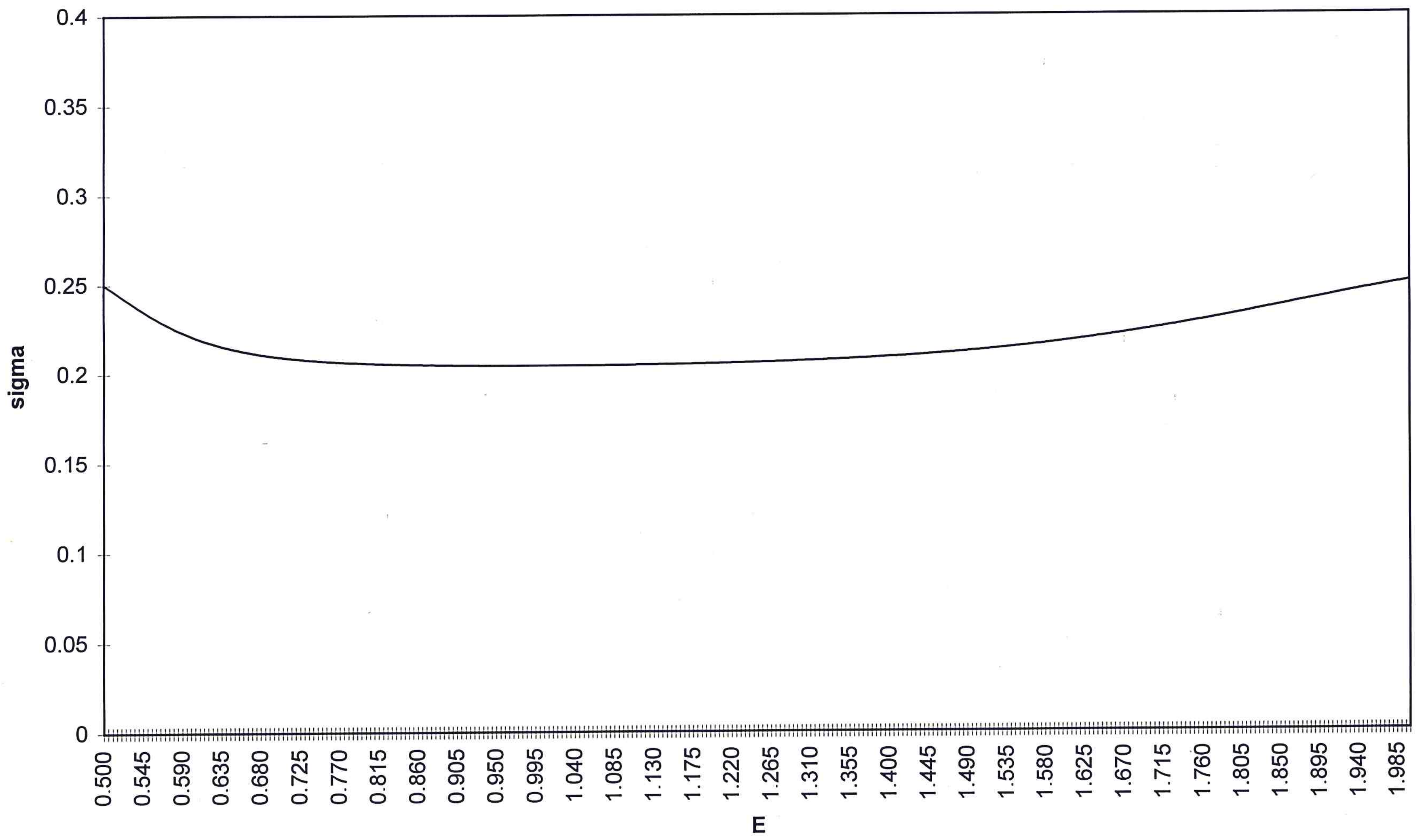
sigma
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.500
0.545
0.590
0.635
0.680
0.725
0.770
0.815
0.860
0.905
0.950
0.995
1.040
1.085
1.130
1.175
1.220
1.265
1.310
1.355
1.400
1.445
1.490
1.535
1.580
1.625
1.670
1.715
1.760
1.805
1.850
1.895
1.940
1.985
E

T = 0.1

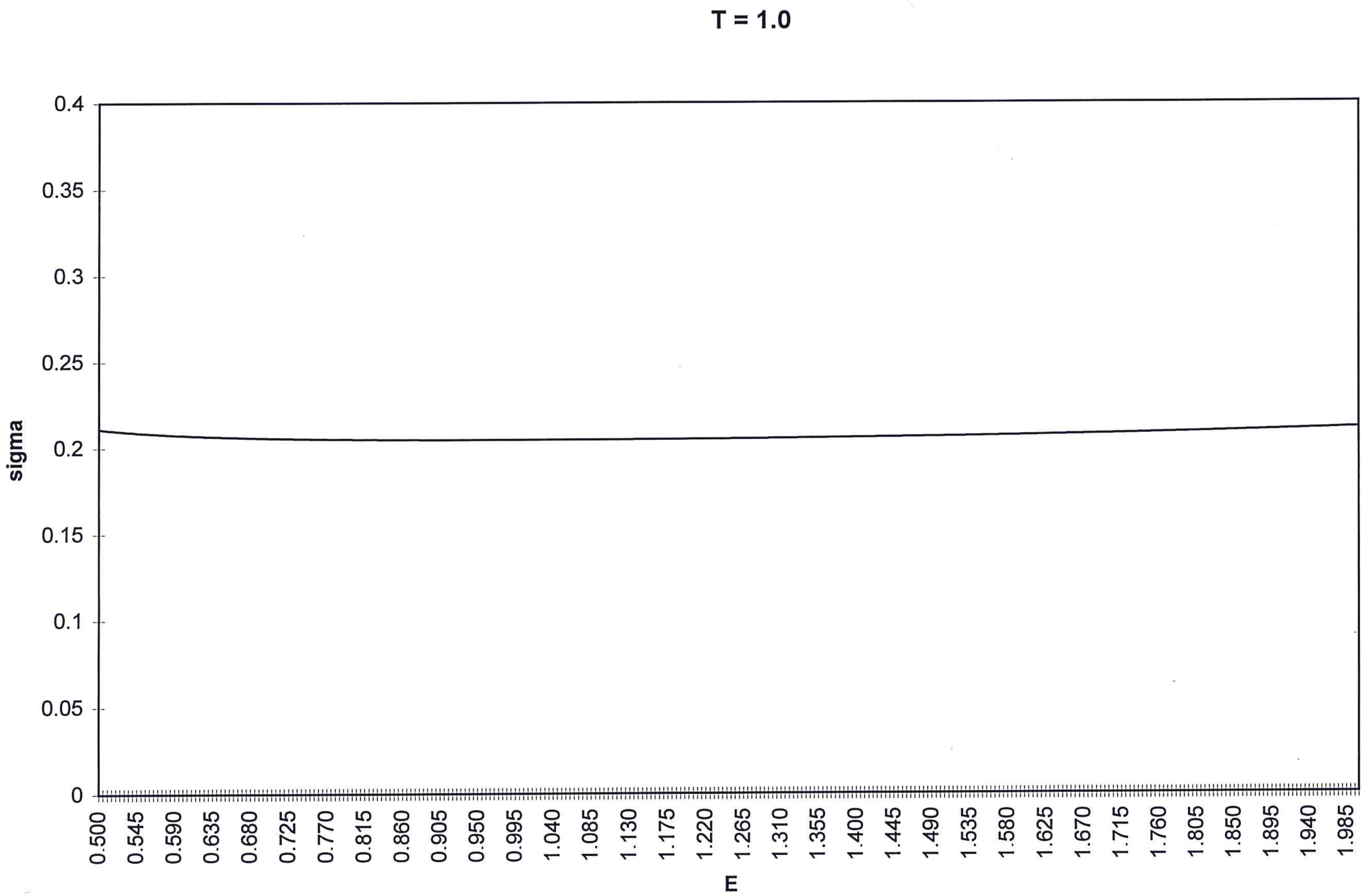

sigma
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.500 0.545 0.590 0.635 0.680 0.725 0.770 0.815 0.860 0.905 0.950 0.995 1.040 1.085 1.130 1.175 1.220 1.265 1.310 1.355 1.400 1.445 1.490 1.535 1.580 1.625 1.670 1.715 1.760 1.805 1.850 1.895 1.940 1.985
E


T = 0.2
sigma
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.500
0.545
0.590
0.635
0.680
0.725
0.770
0.815
0.860
0.905
0.950
0.995
1.040
1.085
1.130
1.175
1.220
1.265
1.310
1.355
1.400
1.445
1.490
1.535
1.580
1.625
1.670
1.715
1.760
1.805
1.850
1.895
1.940
1.985
E

T= 0.5



sigma
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.500
0.545
0.590
0.635
0.680
0.725
0.770
0.815
0.860
0.905
0.950
0.995
1.040
1.085
1.130
1.175
1.220
1.265
1.310
1.355
1.400
1.445
1.490
1.535
1.580
1.625
1.670
1.715
1.760
1.805
1.850
1.895
1.940
1.985
E


T = 1.0
sigma
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.500
0.545
0.590
0.635
0.680
0.725
0.770
0.815
0.860
0.905
0.950
0.995
1.040
1.085
1.130
1.175
1.220
1.265
1.310
1.355
1.400
1.445
1.490
1.535
1.580
1.625
1.670
1.715
1.760
1.805
1.850
1.895
1.940
1.985
E