FINANCIAL OPTIONS RESEARCH CENTRE

University of Warwick

Kalman Filtering of Generalized Vasicek Term Structure Models

Simon H Babbs

and

K Ben Nowman

June 1997

Financial Options Research Centre
Warwick Business School
University of Warwick
Coventry CV4 7AL
Phone: (01203) 524118

FORC Preprint: 97/80

KALMAN FILTERING OF GENERALIZED VASICEK TERM STRUCTURE MODELS

SIMON H. BABBS

First National Bank of Chicago*, London and University of Warwick

K. BEN NOWMAN

First National Bank of Chicago*, London

Current Version: June 1997

ABSTRACT

We present a subclass of Langetieg's (1980) linear Gaussian models of the term structure. The bond price is derived in terms of a finite set of state variables with correlated innovations. The subclass contains a reformulation of the double decay model of Beaglehole and Tenney (1991), enabling us to clarify interpretation of their parameters. We apply Kalman filtering to a state space formulation of the model, allowing measurement errors in the data. One and two factor models are estimated on US data over 1987 - 1996. Results indicate our subclass of models can explain movements in the US term structure.

*Correspondence:

First National Bank of Chicago,
First Chicago House,
90 Long Acre,
London WC2E 9RB,
United Kingdom.

It is a pleasure to thank Jesper Lund for his advice, comments and encouragement in this research. We thank also Rex Bergstrom, Marcus Chambers, Andrew Johnson, Joao Neves, Alois Geyer, Stefan Pichler and Paul Merriman for advice and comments and to seminar participants at First National Bank of Chicago, London and the University of Essex. We are grateful to Tim Bissell and Benoit-Gilles Cambier for their work on the construction of the data. The views expressed are those of the authors and not necessarily those of First National Bank of Chicago, London.

1. INTRODUCTION

Since the mid-1980's, there has been a plethora of "arbitrage-based" models of the term structure of interest rates. Such models take the observed current term structure as given, and seek to price interest rate derivative securities by arbitrage alone, based on assumptions concerning the future dynamics of the term structure. Arbitrage-based models are able to leave arbitrary the market prices of risk while taking any initial term structure as given. The drifts of the state variables, under the objective probabilities, depend on the market prices of risk, and are therefore left totally unspecified. Clearly these features make econometric investigation of arbitrage-based models problematic, whether to estimate their parameters and even to examine their plausibility. Moreover, in seeking to mount an econometric investigation of the realism of a model, it would seem natural to enquire whether the initial term structure might plausibly be explained by the model rather than treating the term structure at an arbitrarily chosen start date as a pre-specified functional parameter.

One obvious way to address this situation is econometric analysis of equilibrium-based analogues of arbitrage-based models. We use the term "equilibrium-based", to identify versions of the models, in which market prices of risk are (directly or indirectly) given, and the instantaneous spot interest rate depends explicitly upon the state variables, thus determining the initial term structure endogenously within the model. We define the term "analogues" to mean: that the equilibrium-based version can be described in terms of a set of state variables whose risk-adjusted probability law is identical with those of the state variables of the arbitrage-based version. In addition, the two versions possess an identical equation to link the initial term structure to the term structure at a subsequent date via the evolution of the state variables over the intervening period.

In this paper we consider a subclass of the general linear Gaussian model of the term structure, which goes back to Langetieg (1980). Models in this class have been investigated by many researchers, for example, Langetieg (1980), Hull and White (1990), Beaglehole and Tenney (1991) and Babbs (1990, 1993). The subclass we consider here is distinguished by the characteristic that the drift of each state variable depends on no other state variable.[1] As detailed consideration of Langetieg (1980) readily makes clear, this drift restriction makes the model particularly analytically tractable[2], even if the coefficients are time-varying and innovations in the state variables are correlated[3], in that pure discount bond prices can be related to the state variables by a closed-form formula. Such a formula is a considerable computational benefit possessed by our subclass of models. We confirm that the arbitrage-based version of models in this subclass are precisely the multifactor "Generalized Vasicek" family discussed by Babbs (1993).

Superficially, it would appear that the "Generalized Vasicek" subclass involves a further substantive restriction, namely that each state variable is mean-reverting to zero rather than to a arbitrary function of time, and that the weight of each state variable in determining *r* is minus unity rather than an arbitrary function of time. We proceed to show that our formulation holds without loss of generality so long as the state variables are unobservable (rather than, say, identified exactly[4] with spot rates of various maturities as in Duffie and Kan (1996)). This has the advantage of reducing the number of coefficient functions in the model by twice the number of state variables. Another advantage of our formulation is linked to the "double decay" model of Beaglehole and Tenney (1991)(and an equivalent model of Hull and White (1994)).

We show, in the case of a two state variable model driven by two Brownian motions that though it would appear that our formulation excludes the model of Beaglehole and Tenney (1991) in which the short rate reverts towards a level which itself follows a mean-reverting random walk about a constant long-run average we show that, excluding some exceptional cases, those models can be re-expressed as special cases of our model. Indeed, we show that our formulation may be preferable in that the common intuitive interpretation of the mean-reversion speeds is shown to be, in fact, illusory!

Recent empirical testing of term structure models have concentrated on the dynamic implications of the models using time series data. Recent examples, include, Chan, Karolyi, Longstaff and Sanders (1992), Broz, Scaillet and Zakoian (1995), Brenner, Harjes and Kroner (1996), Dahlquist (1996), Nowman (1997a,b)[5] and Andersen and Lund (1997).

An alternative approach has concentrated on the cross-sectional implications of term structure models; examples include: Brown and Dybvig (1986), Barone, Cuoco and Zautzik (1991), Brown and Schaefer (1994), Rogers and Stummer (1994) and De Munnik and Schotman (1994). Both the above approaches suffer from the disadvantage that they do not use the full information available from the yield curve obtained over time and across maturities in the estimation procedure. Recently, approaches providing a solution to this have been put forward by Gibbons and Ramaswamy (1993), Chen and Scott (1993), and Pearson and Sun (1994). A drawback of Pearson and Sun (1994) is that they assumed the two data points on the yield curve are measured without error and in Chen and Scott (1993) they assumed one bond price is observed without error (though allowing for measurement errors in others).

The application of Kalman filtering methods in the estimation of term structure models using cross-sectional/time series data, has been investigated by Pennacchi (1991), Lund (1994, 1997), Chen and Scott (1995), Duan and Simonato (1995), Geyer and Pichler (1996), Ball and Torous (1996), and Jegadeesh and Pennacchi (1996) (see also Harvey (1989) for an extensive treatment of Kalman filtering in econometrics). The use of the state space model formulation of term structure models and the application of the Kalman filter has the advantage that it allows the underlying state variables to be handled correctly as unobservable variables compared to using a short term rate as a proxy (e.g. Chan et al (1992), Nowman (1997a,b). In addition, estimated time series of the unobserved state variables are obtained and measurement errors in the data (bond prices or interest rates) are handled explicitly (see Geyer and Pichler (1996) for further advantages). In this paper we consider the application of Kalman filtering methods to one and two factor versions of our subclass of models using US data.

The paper is organised as follows. In Section 2 the subclass of models are outlined and the general formula for the bond price derived. Section 2.1 establishes the analogue relationship between the equilibrium based model and its arbitrage based version and Section 3 discusses the relationship to the models of Langetieg (1980) and Beaglehole and Tenney (1991). The bond price for the general constant parameter case is then given in Section 4. Section 5 discusses the state space formulation of the model and the estimation of the parameters. The data and empirical results are presented in Section 6 which also includes the factor loadings implied by the model. Some conclusions are offered in Section 7. Key derivations are contained in the Appendix.

2. SUBCLASS OF MODELS

A possible description of the instantaneous spot interest rate, r, is:

$$r(t) = \mu(t) - \sum_{j=1}^{J} X_j(t) \tag{1}$$

where μ is some deterministic process (i.e. most a function of time), and $X_1(t), \ldots, X_J(t)$ represent the current effect of J streams of economic "news". The economic "news" in a two factor model can be interpreted as short term economic "news" (for example, "rumours" of interest rate decisions from the *Federal Open Market Committee*) and long term economic "news" (for example, monthly and quarterly economic statistics). The arrival of each type of "news" is modelled by the innovations of Brownian motions, which may be correlated, while the impact of a piece of news dies away exponentially as the time since it was received increases. In equation form

$$dX_j = -\xi_j X_j dt + c_j dW_j \tag{2}$$

where each ξ_j and c_j are deterministic processes of mean reversion and diffusion coefficients, and $W_1, \ldots, W_J$ are standard Brownian motions with deterministic instantaneous correlation processes, $\rho_{jk} : j, k = 1, \ldots, J$. Equation (2) can equivalently be expressed as

$$dX_j = -\xi_j X_j dt + \sum_{q=1}^{Q} \kappa_{jq} dZ_q \qquad (Q \leq J)^6 \tag{3}$$

where $Z_1, \ldots, Z_Q$ are independent standard Brownian motions and

$$\sum_{q=1}^{Q} \kappa_{jq}\kappa_{kq} = \rho_{jk}\, c_j c_k . \tag{4}$$

In the Appendix we apply the results of Babbs and Selby (1993) to show that, with deterministic market prices of risk attaching to the orthogonal sources of risk, $Z_1, \ldots, Z_Q$, the model can be supported in an incomplete markets general equilibrium, and that interest rate contingent claims including, of course, pure discount bonds are uniquely priced, notwithstanding that incompleteness. The key to this uniqueness is that under any risk-adjusted probability measure, P^*, corresponding to using the continuously compounding money-market account process[7] as numeraire, the processes

$$Z_q^* \equiv Z_q + \int_0^{\cdot} \theta_q(u)\,du; \qquad q = 1, \ldots, Q \tag{5}$$

are independent standard Brownian motions. This fact uniquely determines the risk-adjusted probability law of the state variables, $X_1, \ldots, X_J$, and hence of the short rate, r, and of all payoffs contingent on the state variables.

The resulting[8] formula for the price $B(M,t)$ at time t of unit nominal of a pure discount bond maturing at time M is

$$B(M,t) = \exp\left\{ -\int_t^M \mu(u)\,du - \sum_{q=1}^{Q} \int_t^M \theta_q(u)\sigma_q(M,u) - \frac{1}{2}\sigma_q^2(M,u)\,du + \sum_{j=1}^{J} \frac{G_j(M) - G_j(t)}{G_j'(t)} X_j(t) \right\} \tag{6}$$

where

$$G_j(t) = \int_0^t \exp\left\{ -\int_0^u \xi_j(s)\,ds \right\} du \tag{7a}$$

and where

$$\sigma_q(M,t) = \sum_{j=1}^{J} \frac{G_j(M) - G_j(t)}{G_j'(t)} \kappa_{jq}(t) \tag{7b}$$

represents the component of the (proportional) volatility of $B(M,t)$ attributable to Z_q.

2.1. THE ANALOGUE RELATIONSHIP

We now establish the desired analogue relationship between the equilibrium-based model set out above and its arbitrage-based version discussed in Babbs (1993). Dividing (6) through by the expressions that equation gives for $B(M,0)$ and $B(t,0)$, and undertaking some re-arrangement by means of (7b), we obtain

$$B(M,t) = \frac{B(M,0)}{B(t,0)} \exp\left\{-\frac{1}{2}\sum_{q=1}^{Q}\int_0^t \sigma_q^2(M,u) - \sigma_q^2(t,u)du + \sum_{j=1}^{J}\{G_j(M)-G_j(t)\}Y_j(t)\right\} \quad (8)$$

where

$$Y_j(t) = \frac{X_j(t)}{G_j'(t)} - \frac{X_j(0)}{G_j'(0)} + \sum_{q=1}^{Q}\int_0^t \frac{\theta_q(u)\kappa_{jq}(u)}{G_j'(u)}du \quad (9)$$

On the RHS of (9) all terms except $X_j(t)$ are deterministic. It follows that $Y_1,\ldots,Y_J$ are an equivalent set of state variables to $X_1,\ldots,X_J$. Moreover, if we substitute the solution to (3), and also (5), into (9), we can rewrite (9) as

$$Y_j(t) = \sum_{q=1}^{Q}\int_0^t \frac{\kappa_{jq}(u)}{G_j'(u)}dZ_q^* \quad (10)$$

Equations (9) and (10) hold also under the arbitrage-based version of the "Generalized Vasicek" family of models, as discussed in Babbs (1993) being precisely equivalent to his equations (9)[9] and (7) respectively. This establishes the desired analogue relationship between the equilibrium-based model in this paper and the arbitrage-based version. The simple form of the pure discount bond price formula given by (6) confirms the tractability of the "Generalized Vasicek" family of Linear Gaussian models. In particular, it can be obtained by specialising appropriately the general linear Gaussian model analysed by Langetieg (1980). It is the relation of the "Generalized Vasicek" family to that general class, that we will now explore in more detail.

3. RELATION TO GENERAL LINEAR GAUSSIAN MODELS

In the general model described by Langetieg (1980) (see Langetieg's equations (5) and (6)), the dynamics of the state variables can be expressed as

$$dX_j = \left(a_j + \sum_{k=1}^{J} B_{jk} X_k\right) dt + \sum_{q=1}^{Q} \kappa_{jq} \, dZ_q \tag{11}$$

and the short rate is a general linear combination of the state variables

$$r = w_0 + \sum_{j=1}^{J} w_j X_j . \tag{12}$$

Each coefficient $\left(a_j, B_{jk}, \kappa_{jq}, w_0, w_j\right)$ is allowed to be deterministically time-varying. Our subclass appears to involve three substantive restrictive measures:

I. the off-diagonal terms in the matrix of mean-reversion coefficient functions, B_{jk}, have been set identically to zero;

II. the levels coefficient functions, a_j, have been set identically to zero;

III. the weighting functions, w_j, with which the state variables appear in the short rate expression (12) have been set identically to minus unity.

The first of these restrictions is indeed substantive. As discussed earlier, it is motivated by the desire to restrict attention to the subclass of fully analytically tractable models. Having restricted the off-diagonal B_{jk} to be zero, we can now construct an invertible transformation whereby, far from being substantive restrictions, the second and third of the measures above simply remove redundant functional parameters from the model. In econometric analysis, it is preferable to treat $X_1, \ldots, X_J$ as unobservable state variables, as opposed to identifying them with particular data. We are therefore at liberty to apply any convenient invertible transformation to them. Consider the processes $\tilde{X}_1, \ldots, \tilde{X}_J$ defined by

$$\tilde{X}_j(t) \equiv -w_j(t)\left[X_j(t) - \int_0^t a_j(u) \exp\left\{\int_u^t B_{jj}(s)\,ds\right\} du\right]. \tag{13}$$

Since the terms on the RHS of (13), other than $X_j(t)$ are deterministic, $\tilde{X}_1,\ldots,\tilde{X}_J$ are an equivalent set of state variables. Moreover, expressing (13) in differential form, and using (11) and (13), we find that the dynamics of these new state variables are of the form given by (2)

$$d\tilde{X}_j = -\xi_j \tilde{X}_j \, dt + \sum_{q=1}^{Q} \tilde{\kappa}_{jq} \, dZ_q \tag{14}$$

where[10]

$$\xi_j(t) = -B_{jj}(t) - \frac{w_j'(t)}{w_j(t)} \tag{15a}$$

$$\tilde{\kappa}_{jq}(t) = -w_j(t)\,\kappa_{jq}(t). \tag{15b}$$

Finally, substituting (13) into (12) and rearranging, we obtain an expression of the form of (1)

$$r(t) = \mu(t) - \sum_{j=1}^{J} \tilde{X}_j(t) \tag{16}$$

where

$$\mu(t) = w_0(t) + \sum_{j=1}^{J} w_j(t) \int_0^t a_j(u) \exp\left\{ \int_u^t B_{jj}(s)\,ds \right\} du. \tag{17}$$

Equations (14), (16) and (17) show, that, having opted to exclude off-diagonal entries from the matrix B, the set of remaining parameter functions can be reduced from the $3J+1$ functions (excluding the diffusion coefficients) $w_0, w_1, \ldots, w_J$, $a_1, \ldots, a_J$, $B_{11}, \ldots, B_{JJ}$ to the set of $J+1$ functions μ, $\xi_1, \ldots, \xi_J$.

At first glance, our restriction that off-diagonal B_{jk} be zero would appear to exclude the *double decay* model of Beaglehole and Tenney (1991)

$$dr = \xi_1(y-r)dt + \kappa_{11} dZ_1 \tag{18a}$$

$$dy = \xi_2(m-y)dt + \kappa_{21} dZ_1 + \kappa_{22} dZ_2. \tag{18b}$$

Since the obvious representation in terms of (11)-(12) is to identify X_1 directly with r and X_2 with y we then have

$$dX_1 = (-\xi_1 X_1 + \xi_1 X_2)dt + \kappa_{11} dZ_1 \quad (19a)$$

$$dX_2 = (\xi_2 m - \xi_2 X_2)dt + \kappa_{21} dZ_1 + \kappa_{22} dZ_2 \quad (19b)$$

with off-diagonal coefficients in (19a). Fortunately, as we show in the Appendix, Beaglehole and Tenney's model can be written in our form[11] i.e. in terms of (1) and (3). Moreover, given constant mean-reversion speeds, if m is a constant, then μ is also constant at precisely the same level. We note in passing that we prefer our formulation of Beaglehole and Tenney's model to theirs for the following reason. We were initially tempted by (18a)-(18b) into interpreting ξ_1 as the speed of mean reversion of the short rate, r, and ξ_2 as the speed of mean reversion of the stochastic level, y towards which r is tending. Unfortunately such an interpretation is built on sand since, assuming that y is unobservable, we could just as well define the new unobservable level[12]

$$y^* = \left(1 - \frac{\xi_1}{\xi_2}\right) r + \frac{\xi_1}{\xi_2} y \quad (20)$$

and re-write the model in the form[13]

$$dr = \xi_2 (y^* - r)dt + \kappa_{11} dZ_1 \quad (21a)$$

$$dy^* = \xi_1 (m - y^*)dt + \kappa^*_{21} dZ_1 + \kappa^*_{22} dZ_2 \quad (21b)$$

in which the roles of ξ_1 and ξ_2 are interchanged !.

4. THE CONSTANT PARAMETER CASE

In the case where the mean reversion level μ, the mean-reversion speeds ξ_j, the diffusion coefficients κ_{jq}, and the market price of risk processes θ_q, are all constant, the key pricing formula for a pure discount bond equation (6) evaluates to

$$B(M,t) = \exp\left\{-\tau\left[R(\infty) - w(\tau) - \sum_{j=1}^{J} H(\xi_j\tau)X_j(t)\right]\right\} \tag{22}$$

with

$$R(\infty) = \mu + \sum_{q=1}^{Q}\theta_q\sum_{j=1}^{J}\frac{\kappa_{jq}}{\xi_j} - \frac{1}{2}\sum_{q=1}^{Q}\left(\sum_{j=1}^{J}\frac{\kappa_{jq}}{\xi_j}\right)^2 \tag{23a}$$

$$w(\tau) = \sum_{j=1}^{J} H(\xi_j\tau)\left[\sum_{q=1}^{Q}\theta_q\frac{\kappa_{jq}}{\xi_j} - \sum_{q=1}^{Q}\sum_{i=1}^{J}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}\right] + \frac{1}{2}\sum_{i=1}^{J}\sum_{j=1}^{J}H\left(\left(\xi_i+\xi_j\right)\tau\right)\sum_{q=1}^{Q}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j} \tag{23b}$$

where

$$\tau \equiv M - t \tag{23c}$$

$$H(x) = \frac{1-e^{-x}}{x}. \tag{23d}$$

Note from the form of (22) that the spot rate from t to M depends upon calendar time only through the state variables, $X_1(t)$, $X_2(t)$, being otherwise a function of residual term to maturity, $M-t$.

5. THE STATE SPACE MODEL AND THE KALMAN FILTER

In this section we are concerned with one and two factor versions of the constant parameter case of our subclass of models. We derive the state space model formulation of the term structure model and present the Kalman filter algorithm. This is used in the evaluation of the exact likelihood function of the observed interest rates and the computation of the unobserved state variables and parameters of the model. Our presentation draws on the notation and set-up of Lund (1994) below in discussing the Kalman filter. The theoretical yield curve is given by

$$R(t+\tau,t) \equiv -\log B(t+\tau,t)/\tau = A_0(\tau) - A_1(\tau)' X(t) \tag{24}$$

where $A_0(\tau) = R(\infty) - w(\tau)$ and $A_1(\tau) = H(\xi_j\tau)$ is a $J\times 1$ vector (where the superscript denotes transpose). The scalar $A_0(\tau)$ and the vector $A_1(\tau)$ are functions of the time to maturity

τ and the parameters of the model. We have N observed interest rates at time t_k, for $k=1,2,\ldots,n$, which are denoted by $R_k = (R_{1k},\ldots,R_{Nk})$, where $R_{ik} = -\log B(t_k + \tau_i, t_k)/\tau_i$.

We assume that measurement errors in interest rates are additive and normally distributed. The measurement equation is then given by

$$R_k = d(\psi) + Z(\psi)X_k + \varepsilon_k \quad ; \quad \varepsilon_k \sim N(0, H(\psi)) \tag{25}$$

where ψ contains the unknown parameters of the model including the parameters from the distribution of the measurement errors. The i'th row of the matrices $d\ (N \times 1)$ and $Z(N \times J)$ are given by $A_0(\tau_i;\psi)$ and $-A_1(\tau_i;\psi)'$, respectively. The error terms ε_k are measurement errors to allow for noise in the sampling process of the data. The variance-covariance matrix of the measurement errors can take various forms. Typically it is assumed in empirical work that either $H = h * I$ or we have maturity specific variances $H = h_1, \ldots h_N$ along the diagonal. The first assumption has the advantage of reducing the computational burden in the Kalman filter (see below). But as Geyer and Pichler (1996) point out, having maturity specific variances takes into consideration that trading activity is going to vary across maturities and therefore the bid-ask spread will differ across maturities.

The transition equation is the exact discrete-time distribution of the state variables obtained from the solution of (2) (see Bergstrom (1984) and Lund (1994)) and is a VAR(1) model

$$X_k = \Phi(\psi)X_{k-1} + \eta_k \tag{26}$$

where $\Phi(\psi) = e^{-\xi_J(t_k - t_{k-1})}$. The error term η_k is normally distributed with $E[\eta_k] = 0$ and $Cov[\eta_k] = V(\psi)$, where V is given in Bergstrom (1984, Theorem 3) and Lund (1994). The measurement and transition equations represent the state space formulation of our model. We now present the Kalman filter algorithm and the exact likelihood function.

Let $\hat{X}_{k/k-1}$ and $\hat{X}_k$ denote the optimal estimator (in a mean square error sense, MSE) of the unknown state vector X_k based on the available information (i.e. the observed interest rates) up to time t_{k-1} and t_k, respectively. The optimal estimator is the conditional mean of X_k in both cases, denoted $E_{k-1}[\cdot]$ and $E_k[\cdot]$, respectively. The prediction step is given by

$$\hat{X}_{k/k-1} = E_{k-1}(X_k) = \Phi \hat{X}_{k-1} \qquad (27)$$

with mean square error (MSE) matrix

$$\Sigma_{k/k-1} = E_{k-1}\left[\left(X_k - \hat{X}_{k/k-1}\right)\left(X_k - \hat{X}_{k/k-1}\right)'\right] = \Phi\Sigma_{k-1}\Phi' + V \qquad (28)$$

In the update step the additional information given by R_k is used to obtain a more precise estimator of X_k:

$$\hat{X}_k = E_k(X_k) = \hat{X}_{k/k-1} + \Sigma_{k/k-1}Z'F_k^{-1}v_k \qquad (29)$$

$$\Sigma_k = E_k\left[\left(X_k - \hat{X}_k\right)\left(X_k - \hat{X}_k\right)'\right] = \Sigma_{k/k-1} - \Sigma_{k/k-1}Z'F_k^{-1}Z\Sigma_{k/k-1}$$

$$= \left(\Sigma_{k/k-1}^{-1} + Z'H^{-1}Z\right)^{-1} \qquad (30)$$

where

$$v_k = R_k - \left(d + Z\hat{X}_{k/k-1}\right) \qquad (31)$$

$$F_k = Z\Sigma_{k/k-1}Z' + H \qquad (32)$$

(cf. Harvey (1989, Ch.3)). This new estimate of X_k is called the *filtered* estimate. The aim of the Kalman filter is to obtain information about X_k from the observed interest rates. The Kalman filter also has the advantage of being able to evaluate the likelihood function using the prediction error decomposition. The log-likelihood function is given by (apart from a constant)

$$\log L(R_1,\ldots,R_n;\psi) = -\frac{1}{2}\sum_{k=1}^{n}\log|F_k| - \frac{1}{2}\sum_{k=1}^{n}v_k'F_k^{-1}v_k \qquad (33)$$

where v_k and F_k are given by equations (31) and (32). We can also use the formulae of Harvey (1989, p.108) for computing the inverse and determinant of F_k given by

$$F_k^{-1} = H^{-1} - H^{-1}Z\left(\Sigma_{k/k-1}^{-1} + Z'H^{-1}Z\right)^{-1}Z'H^{-1}$$

$$|F_k| = |H| \cdot \left|\Sigma_{k/k-1}\right| \cdot \left|\Sigma_{k/k-1}^{-1} + Z'H^{-1}Z\right|.$$

In the case of $H = h * I$ these formulae can be simplified and the computational burden reduced which is especially important if the number of maturities used in the estimation is large. Finally, the Kalman filter recursions are started by setting the initial state vector X_0 and covariance matrix Σ_0 to their unconditional mean and covariance.

6. EMPIRICAL RESULTS

6.1. Data description

The data used in our empirical work consists of constructed zero coupon yields obtained from the First National Bank of Chicago, London. The data cover the period April 1987 to December 1996, a total of 507 weekly observation dates and at each date we have N-interest rates. The following maturities: 3 and 6 months, 1, 2, 3, 5, 7 and 10 years were chosen $(N = 8)$. We use weekly data on a Wednesday following, for example, Lund (1997) to avoid missing observations and week-day effects.

6.2. Estimation results

The application of the Kalman filter to the one and two factor models of our subclass to US data are now discussed. The estimation results are presented in Table 1 which contains the parameter estimates of $\mu, \xi_j, \kappa_{jq}, \theta_q$ (note $\kappa_{11} = c_1$; $\kappa_{12} = 0$; $\kappa_{21} = c_2\rho$; $\kappa_{22} = c_2\sqrt{1-\rho^2}$) and the estimated standard deviations of the measurement errors ($\sqrt{h_1}, \ldots, \sqrt{h_N}$). Standard errors are computed using the formulae in Hamilton (1994, p.389). The Table also contains the BIC Information Criterion and log-likelihood value.

In both the one and two factor models, the mean reversions ξ_j and diffusion parameters c_j are significant and have plausible values. In the two factor model the market prices of risk θ_q are not significant whereas they are in the one factor model. The long run average rate μ is significant in both models and has plausible values. Most of the measurement errors are highly significant. The log-likelihood value increases strongly as we move to a two factor model.

The standard deviations of the measurement errors in the one factor model for the 3 month rate is 36 basis points and 22 basis points for the 6 month rate. This compares to 17 basis points and 4 basis points in the two factor model. Overall the standard deviations are much lower in the two factor model. In particular, they are 17 basis points for the 1 year rate, 28 basis points for the 2 year rate, 19 basis points for the 3 year rate, 9 basis points for the 5 year rate, less than one basis point for the 7 year rate and 8 basis points for the 10 year rate. This compares to a range of 4 to 73 basis points for the one factor model in these maturities.

The mean reversion parameters imply mean half-lives for the interest rate process (i.e. the expected time for the process to return halfway to its long term mean) for the one factor model of 3.6 years and for the two factor model 1.3 years for the first factor and 10.6 years for the second factor. The correlation coefficient is -84 percent and highly significant. Calibration of an arbitrage-free analogue of this model to market prices of interest rate caps and swaptions yielded similiar large values for the correlation parameter. The log-likelihood values for the one factor model is 20494 and for the two factor model 24397. Based on the BIC Information Criterion (BIC) we find that moving from a one factor to the two factor the BIC improves by 16%. The likelihood ratio test of the one factor versus the two factor gives a value of 7806 and one can reject the null hypothesis of a one factor model at the 5% significance level.

We also look at the factor loadings for the two factor model as a function of maturity which should help determine the nature of the factors calculated by the Kalman filter. Litterman and Scheinkman (1991) using principal components analysis investigated a number of US yields and identified three factors which they interpreted as changes in level, steepness and curvature.

Factor loadings correspond to orthogonal Brownian motions whereas innovations in our state variables are correlated. We therefore choose to produce factor loadings by re-expressing W_1 and W_2 in terms of uncorrelated Brownian motions Y_1 and Y_2 in such a way that dY_2 does not impact the term structure at a particular maturity $\tau *$. In Litterman and Scheinkman (1991, Table 2) they find that factor two has approximately zero impact on the term structure at the 5 year maturity and this is imposed here for comparison with their graphs $(\tau * = 5)$.

The factor loadings are derived in the Appendix and are given below for factor one, $\gamma_1(\tau)$ and factor two, $\gamma_2(\tau)$.

$$\gamma_1(\tau) = \frac{H(\xi_1\tau)c_1\beta\alpha_{22}}{\alpha_{22}H_1c_1 - \alpha_{21}H_2c_2} + \frac{H(\xi_2\tau)c_2\beta\alpha_{21}}{\alpha_{21}H_2c_2 - \alpha_{22}H_1c_1} \quad (34)$$

$$\gamma_2(\tau) = \frac{-H(\xi_1\tau)c_1H_2c_2}{\alpha_{22}H_1c_1 - \alpha_{21}H_2c_2} - \frac{H(\xi_2\tau)c_2H_1c_1}{\alpha_{21}H_2c_2 - \alpha_{22}H_1c_1} \quad (35)$$

where

$$\beta = \sqrt{H_1^2c_1^2 + 2\rho H_1H_2c_1c_2 + H_2^2c_2^2}$$

$$H_j = H(\xi_j\tau^*)$$

$$\alpha_{21} = -\frac{H_1c_1\rho + H_2c_2}{\beta\sqrt{1-\rho^2}} \; ; \; \alpha_{22} = -\alpha_{21}\frac{H_1c_1 + H_2c_2\rho}{H_1c_1\rho + H_2c_2}$$

$$\tau^* = 5.$$

[Figure 1 about here]

Figure 1 plots the factor loadings of the two factor model as a function of maturity. The first factor's impact on yield changes has a increasing positive effect on the maturities up to 4 years then has an equal impact on the remaining maturities. We conclude as Litterman and Scheinkman identified (see also Geyer and Pichler (1996)) that the first factor could represent a *level* factor. The second factor has a strong influence on short term rates up to 5 years and lowers them and then has a positive impact on longer maturities by raising them. We conclude that the second factor could represent a *steepness* factor as identified by Litterman and Scheinkman. Overall the model generates factor loadings in line with their results.

In figures 2 to 17 we have plotted the estimated time series implied by the measurement equation for the one and two factor models. The estimates of the unobserved state variables at time t are based on information available at that time (i.e. the filtered estimates).

[Figures 2 - 17 about here]

The simulation results of the one and two factor models imply in particular that the two factor model tracks the 3 and 6 month rates with smaller error. The one factor model tracks the 1 year rate more closer than the two factor model. The two factor model clearly dominates the one factor model for the rest of the maturities (2 to 10 years) and especially in the long end of the curve. For the longer maturities the one factor model is more volatile. Overall the two factor model's performance is superior.

7. CONCLUSIONS

In this paper we have been concerned with a subclass of the general linear Gaussian model of Langetieg (1980). We have confirmed that the arbitrage-based version of models in this subclass are precisely the multifactor "Generalized Vasicek" family discussed by Babbs (1993). The subclass has the advantage of reducing the number of coefficient functions in the model by twice the number of state variables so long as the state variables are unobservable. We have shown also that the double-decay model of Beaglehole and Tenney (1991) can be re-expressed as a special case of our model, and that our formulation may be preferable in that the obvious intuitive interpretation of the mean-reversion speeds is shown to be, in fact, illusory!. For our empirical work, the model is expressed in a state space formulation which allows us to take into account both the cross-sectional and time-series restrictions on the data and that the observed yield curve contains measurement errors. Estimates are obtained for one and two factor models using US data over the period 1987-1996. We find overall that the two factor model is able to explain most of the variation in the US term structure, based on the smallness of the measurement errors and on simulation results compared to the one factor model. The model has also been applied to UK Gilt-Edged market data in Babbs and Nowman (1997a) and to a range of other currencies in Babbs and Nowman (1997b).

APPENDIX

Unique pricing of interest rate claims under incomplete markets equilibrium.

In addition to the state variable and short rate behaviour expressed in (1) and (3), and of deterministic market prices of risk, as assumed in the main text, we assume that the continuously-compounded money-market account, whose balance at time t is given by:

$$S_0(t) \;=\; \exp\left\{\int_0^t r(u)\,du\right\} \tag{A1}$$

is continuously available for trade.

Using the results of Babbs and Selby (1993), it can be verified that our model is consistent with general equilibrium, and that the price processes of pure discount bonds and interest rate derivatives are uniquely determined, independent of the market prices of risks attaching to all risks not explicitly modelled in our assumptions.[14] In particular, the price at time t, $B(M,t)$, of a pure discount bond maturing at M, is given by:

$$B(M,t) \;=\; E_t^*\left[\exp\left\{-\int_t^M r(u)\,du\right\}\right] \tag{A2}$$

where the asterisk on the conditional expectation operator signifies that the expectation is taken under any "risk pricing measure" (RPM), under which the processes $Z_1^*,\ldots,Z_Q^*$ defined by (5) are independent standard Brownian motions.

Derivation of bond pricing formula (6)

Substituting (1) into *(A2)*, we obtain:

$$B(M,t) \;=\; \exp\left\{-\int_t^M \mu(u)\,du\right\} E_t^*\left[\exp\left\{\sum_{j=1}^{J}\int_t^M X_j(u)\;du\right\}\right] \tag{A3}$$

Now, the state variable dynamics (2) can be re-expressed in the form:

$$X_j(u) \;=\; G_j'(u)\left\{\frac{X_j(t)}{G_j'(t)} \;+\; \sum_{q=1}^{Q}\int_t^u \frac{\kappa_{jq}(s)}{G_j'(s)}\,dZ_q\right\} \tag{A4}$$

whence

$$\int_t^M X_j(u)du = \frac{G_j(M)-G_j(t)}{G_j'(t)}X_j(t) + \sum_{q=1}^{Q}\int_t^M G_j'(u)\int_t^u \frac{\kappa_{jq}(s)}{G_j'(s)}dZ_q du \tag{A5}$$

Applying Ito's lemma to

$$G_j(u)\int_t^u \frac{\kappa_{jq}(s)}{G_j'(s)}dZ_q$$

we obtain:

$$d\left\{G_j(u)\int_t^u \frac{\kappa_{jq}(s)}{G_j'(s)}dZ_q\right\} = G_j'(u)\int_t^u \frac{\kappa_{jq}(s)}{G_j'(s)}dZ_q du + G_j(u)\frac{\kappa_{jq}(u)}{G_j'(u)}dZ_q \tag{A6}$$

Re-arranging (A6), and integrating over $[t, M]$, yields

$$\int_t^M G_j'(u)\int_t^u \frac{\kappa_{jq}(s)}{G_j'(s)}dZ_q du = G_j(M)\int_t^M \frac{\kappa_{jq}(u)}{G_j'(u)}dZ_q du - \int_t^M G_j(u)\frac{\kappa_{jq}(u)}{G_j'(u)}dZ_q$$

$$= \int_t^M \{G_j(M)-G_j(u)\}\frac{\kappa_{jq}(u)}{G_j'(u)}dZ_q \tag{A7}$$

Substituting (A7) into (A5) and summing over j,

$$\sum_{j=1}^{J}\int_t^M X_j(u)du = \sum_{j=1}^{J}\frac{G_j(M)-G_j(t)}{G_j'(t)}X_j(t) + \sum_{q=1}^{Q}\int_t^M \sum_{j=1}^{J}\{G_j(M)-G_j(u)\}\frac{\kappa_{jq}(u)}{G_j'(u)}dZ_q \tag{A8}$$

which, by (7b),

$$= \sum_{j=1}^{J}\frac{G_j(M)-G_j(t)}{G_j'(t)}X_j(t) + \sum_{q=1}^{Q}\int_t^M \sigma_q(M,u)dZ_q \tag{A9}$$

which, by (5),

$$= \sum_{j=1}^{J}\frac{G_j(M)-G_j(t)}{G_j'(t)}X_j(t) + \sum_{q=1}^{Q}\int_t^M \sigma_q(M,u)dZ_q^* - \sum_{q=1}^{Q}\int_t^M \theta_q(u)\sigma_q(M,u)du \tag{A10}$$

Substituting *(A10)* into *(A3)* and evaluating the expectation gives (6) as required.

Expressing Beaglehole and Tenney's (1991) model in terms of (1) and (3)

We commence by removing the interdependence of the state variables from (18a). Define:

$$\alpha(t) \equiv \exp\left\{-\int_0^t \xi_1(s)-\xi_2(s)\, ds\right\}\left[A-\int_0^t \exp\left\{\int_0^u \xi_1(s)-\xi_2(s) ds\right\}\xi_1(u) du\right] \qquad (A11)$$

where A is some convenient constant such that α never vanishes.[15] Then,

$$d(\alpha y + r) = \alpha' y\, dt + \alpha dy + dr$$

$$= \{\alpha \xi_2 m - \xi_1(\alpha y + r)\}dt + \{\alpha \kappa_{21} + \kappa_{11}\}dZ_1 + \alpha \kappa_{22}\, dZ_2$$

whose solution is:

$$\alpha(t)y(t) + r(t) = \exp\left\{-\int_0^t \xi_1(s) ds\right\}\left[A\, y(0) + r(0) + \int_0^t \exp\left\{\int_0^u \xi_1(s) ds\right\}\alpha(u)\xi_2(u)m(u) du\right.$$

$$\left. + \int_0^t \exp\left\{\int_0^u \xi_1(s) ds\right\}\left(\{\alpha(u)\kappa_{21}(u) + \kappa_{11}\}dZ_1 + \alpha(u)\kappa_{22}(u) dZ_2\right)\right]$$

(A12)

The solution to (18b) is:

$$y(t) = \exp\left\{-\int_0^t \xi_2(s) ds\right\}\left[y(0) + \int_0^t \exp\left\{\int_0^u \xi_2(s) ds\right\}\xi_2(u)m(u) du\right.$$

$$\left. + \int_0^t \exp\left\{\int_0^u \xi_2(s) ds\right\}\left(\kappa_{21}\, dZ_1 + \kappa_{22} dZ_2\right)\right] \qquad (A13)$$

By inspection of *(A12)*-*(A13)*, it is immediate that one alternative set of state variables for the model is obtained by extracting the stochastic parts of the RHS of *(A12)* and *(A13)*. Combining these with carefully chosen deterministic functions leads to the state variable pair X_1, X_2 where:

$$X_1(t) = -\exp\left\{-\int_0^t \xi_1(s) ds\right\}\left[A\{y(0) - m(0)\} + \{r(0) - m(0)\}\right]$$

$$-\exp\left\{-\int_0^t \xi_1(s) ds\right\}\int_0^t \exp\left\{\int_0^u \xi_1(s) ds\right\}\left(\{\alpha(u)\kappa_{21}(u) + \kappa_{11}(u)\}dZ_1 + \alpha(u)\kappa_{22}(u) dZ_2\right) \qquad (A14a)$$

$$X_2(t) = \alpha(t)\exp\left\{-\int_0^t \xi_2(s) ds\right\}\left[y(0) - m(0) + \int_0^t \exp\left\{\int_0^u \xi_2(s) ds\right\}\left(\kappa_{21}(u) dZ_1 + \kappa_{22}(u) dZ_2\right)\right]$$

(A14b)

To demonstrate this, substitute *(A14a)*-*(A14b)* into *(A12)*-*(A13)* and eliminate y to obtain:

$$\mu(t) \equiv r(t) + X_1(t) + X_2(t) = \alpha(t)y(t) + r(t) - \alpha(t)y(t) + X_1(t) + X_2(t)$$

$$= \exp\left\{-\int_0^t \xi_1(s)\,ds\right\}\left[\{A+1\}m(0) + \int_0^t \exp\left\{\int_0^u \xi_1(s)\,ds\right\}\alpha(u)\xi_2(u)m(u)\,du\right]$$

$$- \alpha(t)\exp\left\{-\int_0^t \xi_2(s)\,ds\right\}\left[m(0) + \int_0^t \exp\left\{\int_0^u \xi_2(s)\,ds\right\}\xi_2(u)m(u)\,du\right] \qquad (A15)$$

which is deterministic. Moreover, expressing (*A14a*)-(*A14b*) in differential form gives:

$$dX_1 = -\xi_1 X_1\,dt - (\alpha\kappa_{21} + \kappa_{11})dZ_1 - \alpha\kappa_{22}\,dZ_2 \qquad (A16a)$$

$$dX_2 = -\xi_1\left(1+\frac{1}{\alpha}\right)X_2\,dt + \alpha\kappa_{21}\,dZ_1 + \alpha\kappa_{22}\,dZ_2 \qquad (A16b)$$

which are of the form (3).

Finally, note that if ξ_1 and ξ_2 are constants, then we are free to choose[16]

$$A = \frac{\xi_1}{\xi_2 - \xi_1} \qquad (A17)$$

which makes α constant at that same level. If, moreover, m is constant, then μ as given by (A15) is constant also, and at the same level. Thus a Beaglehole and Tenney model with constant parameters translates into our form with constant parameters also.

Derivation of (22)-(23d)

Specialising (7a)-(7b) to the constant parameter case,

$$\sigma_q(M,u) = \sum_{j=1}^{J} \frac{1 - e^{-(M-u)\xi_j}}{\xi_j}\kappa_{jq} \qquad (A18)$$

whence

$$\int_t^M \theta_q(u)\sigma_q(M,u)\,du = \theta_q \sum_{j=1}^{J} \frac{\kappa_{jq}}{\xi_j}\int_t^M 1 - e^{-(M-u)\xi_j}\,du$$

$$= \theta_q \sum_{j=1}^{J} \frac{\kappa_{jq}}{\xi_j}\,\tau\,\left[1 - H(\xi_j\tau)\right] \qquad (A19)$$

and

$$\int_t^M \sigma_q^2(M,u)\,du = \sum_{i=1}^{J}\sum_{j=1}^{J}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}\int_t^M 1-e^{-(M-u)\xi_i}-e^{-(M-u)\xi_j}+e^{-(M-u)(\xi_i+\xi_j)}\,du$$

$$=\sum_{i=1}^{J}\sum_{j=1}^{J}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}\,\tau\left[1-H(\xi_i\tau)-H(\xi_j\tau)+H((\xi_i+\xi_j)\tau)\right] \qquad (A20)$$

where τ and $H(\,)$ are given by (23c)-(23d) respectively. Moreover,

$$\frac{G_j(M)-G_j(t)}{G_j'(t)}=\tau\ H(\xi_j\tau) \qquad (A21)$$

Substituting (A20)-(A21) into (6), we obtain (22), with:

$$R(\infty) = \mu + \sum_{q=1}^{Q}\theta_q\sum_{j=1}^{J}\frac{\kappa_{jq}}{\xi_j} - \frac{1}{2}\sum_{q=1}^{Q}\sum_{i=1}^{J}\sum_{j=1}^{J}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}$$

$$= \mu + \sum_{q=1}^{Q}\theta_q\sum_{j=1}^{J}\frac{\kappa_{jq}}{\xi_j} - \frac{1}{2}\sum_{q=1}^{Q}\left(\sum_{j=1}^{J}\frac{\kappa_{jq}}{\xi_j}\right)^2 \qquad (A22a)$$

$$w(\tau) = \sum_{j=1}^{J}H(\xi_j\tau)\left[\sum_{q=1}^{Q}\theta_q\frac{\kappa_{jq}}{\xi_j} - \frac{1}{2}\sum_{q=1}^{Q}\sum_{i=1}^{J}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}\right] - \frac{1}{2}\sum_{i=1}^{J}H(\xi_i\tau)\sum_{q=1}^{Q}\sum_{j=1}^{J}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}$$

$$+\frac{1}{2}\sum_{i=1}^{J}\sum_{j=1}^{J}H((\xi_i+\xi_j)\tau)\sum_{q=1}^{Q}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}$$

$$=\sum_{j=1}^{J}H(\xi_j\tau)\left[\sum_{q=1}^{Q}\theta_q\frac{\kappa_{jq}}{\xi_j}-\sum_{q=1}^{Q}\sum_{i=1}^{J}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j}\right]+\frac{1}{2}\sum_{i=1}^{J}\sum_{j=1}^{J}H((\xi_i+\xi_j)\tau)\sum_{q=1}^{Q}\frac{\kappa_{iq}\kappa_{jq}}{\xi_i\xi_j} \qquad (A22b)$$

as required by (23a)-(23b).

Derivation of (34)-(35)

From (22)

$$R(\tau) = R(\infty) - w(\tau) - \sum_{j=1}^{J}H(\xi_j\tau)X_j(t) \qquad (A23)$$

therefore, as t varies

$$dR(\tau) = \sum_{j=1}^{J}H(\xi_j\tau)\xi_jX_j\,dt - \sum_{j=1}^{J}H(\xi_j\tau)c_j\,dW_j \qquad (A24)$$

Choose Y_1 so that dY_1 is associated with the entirety of the diffusion term in (A24) when $\tau = \tau^*$. Thus

$$dY_1 = \sum_{j=1}^{J} \frac{H_j c_j dW_j}{\beta} \tag{A25}$$

where

$$\beta = \sqrt{H_1^2 c_1^2 + 2\rho H_1 H_2 c_1 c_2 + H_2^2 c_2^2} \tag{A26}$$

and $H_j = H(\xi_j, \tau^*)$.

Now choose Y_2 to be a standard Brownian motion orthogonal to Y_1; thus we set

$$dY_2 = \alpha_{21} dW_1 + \alpha_{22} dW_2 \tag{A27}$$

so that

$$\alpha_{21}^2 + \alpha_{22}^2 + 2\rho\alpha_{21}\alpha_{22} = 1 \tag{A28}$$

$$\frac{\alpha_{21} H_1 c_1}{\beta} + \frac{\alpha_{21} H_2 c_2 \rho}{\beta} + \frac{\alpha_{22} H_1 c_1 \rho}{\beta} + \frac{\alpha_{22} H_2 c_2}{\beta} = 0 \tag{A29}$$

Solving (A28), (A29) (and using the symmetry of standard Brownian motions to choose negative roots) gives

$$\alpha_{22} = -\alpha_{21} \frac{H_1 c_1 + H_2 c_2 \rho}{H_1 c_1 \rho + H_2 c_2} \tag{A30}$$

$$\alpha_{21} = -\frac{H_1 c_1 \rho + H_2 c_2}{\beta\sqrt{1-\rho^2}} \; . \tag{A31}$$

Expressing dW_1 and dW_2 in terms of dY_1 and dY_2 gives

$$dW_1 = \frac{\beta\alpha_{22}}{\alpha_{22} H_1 c_1 - \alpha_{21} H_2 c_2} dY_1 - \frac{H_2 c_2}{\alpha_{22} H_1 c_1 - \alpha_{21} H_2 c_2} dY_2 \tag{A32}$$

$$dW_2 = \frac{\beta\alpha_{21}}{\alpha_{21} H_2 c_2 - \alpha_{22} H_1 c_1} dY_1 - \frac{H_1 c_1}{\alpha_{21} H_2 c_2 - \alpha_{22} H_1 c_1} dY_2 \tag{A33}$$

Substituting these into (A24) we obtain:

$$dR(\tau) = \sum_{j=1}^{J} H(\xi_j \tau)\xi_j X_j dt \; - \; \gamma_1(\tau)dY_1 - \gamma_2(\tau)dY_2$$

where

$$\gamma_1(\tau) \;=\; \frac{H(\xi_1\tau)c_1\beta\alpha_{22}}{\alpha_{22}H_1c_1 - \alpha_{21}H_2c_2} + \frac{H(\xi_2\tau)c_2\beta\alpha_{21}}{\alpha_{21}H_2c_2 - \alpha_{22}H_1c_1} \qquad \text{(A34)}$$

$$\gamma_2(\tau) \;=\; \frac{-H(\xi_1\tau)c_1H_2c_2}{\alpha_{22}H_1c_1 - \alpha_{21}H_2c_2} - \frac{H(\xi_2\tau)c_2H_1c_1}{\alpha_{21}H_2c_2 - \alpha_{22}H_1c_1} \qquad \text{(A35)}$$

as required.

Notes

1. Strictly speaking, the assumption of deterministic market prices of risk restricts us further, to what Langetieg termed the "multivariate elastic random walk model".

2. This case corresponds to that in which the matrix B in Langetieg (1980) is diagonal. His matrix Ψ, introduced on p.78 and which plays a major part in the subsequent analysis, is then also diagonal, with *jth* entry satisfying: $\Psi_j(\nu - t) = B_j(\nu)\Psi_j(\nu - t)$ where B_j signifies the *jth* entry on the diagonal of B.

3. This contrasts with multi-factor CIR models where, for which, there is no closed form formula for the bond price unless innovations in the state variables are uncorrelated. In practice we feel that this confers additional flexibility on Gaussian models.

4. We view such an identification as undesirable for our purposes, since it would presuppose that the spot rate concerned could be observed without measurement error (see, for example, Lund (1994)).

5. See, for example Bergstrom (1983, 1990), Chambers (1991) and Nowman (1991, 1993) for further information on the econometric methods.

6. The possibility $Q < J$ is realised if the matrix of the instantaneous correlations ρ_{jk} is at all times of less than full rank. Babbs (1993) illustrates the potential usefulness of allowing $Q < J$ in the arbitrage-based version of "Generalized Vasicek" models.

7. The balance on such an account at time t is given by (A1) in the Appendix.

8. See derivation in the Appendix.

9. Equations (5a) and (9) in the published text of Babbs (1993) contained some typographical errors. His equation (9) should read:

$$B(M,t) = \frac{B(M,0)}{B(t,0)} \exp\left\{ \sum_{j=1}^{J} \{G_j(M) - G_j(t)\} \left(Y_j(t) + \sum_{i=1}^{J} \sum_{q=1}^{Q} \int_0^t G_i(u)\lambda_{iq}(u)\lambda_{jq}(u)\,du \right) - \frac{1}{2} \sum_{i=1}^{J} \sum_{j=1}^{J} \{G_i(M)G_j(M) - G_i(t)G_j(t)\} \sum_{q=1}^{Q} \int_0^t \lambda_{iq}(u)\lambda_{jq}(u)\,du \right\}$$

10. Of course, the substitutions necessary to yield (15a) require that w_j never vanishes. This seems a reasonable requirement, since in practice we would almost invariably take w_j constant; moreover, were w_j identically zero, X_j would have no influence on r on the other state variables and hence be totally redundant (see the remark in italics on p87 of Langetieg (1980)).

11. Except in the case where $\xi_2 - \xi_1$ vanishes. We shall neglect the case $\xi_1 = 0$ which as casual inspection of (18a) reveals corresponds to the degenerate and non-stationary model in which r follows a driftless random walk.

12. Except under the constant parameters non-stationary case where ξ_2 vanishes, $\xi_1 \neq 0$.

13. $\kappa^*_{21} = \left(1 - \frac{\xi_1}{\xi_2}\right)\kappa_{11} + \frac{\xi_1}{\xi_2}\kappa_{21}; \quad \kappa^*_{22} = \frac{\xi_1}{\xi_2}\kappa_{22}.$

14. Babbs and Webber (1994) illustrates the technique, for a model that is considerably more complex.

15. Formally, the model is constructed in an economy with a finite end-date, T. It is therefore trivial that such an A exists.

16. It is (A17) which requires us to exclude the exceptional case where $\xi_1 = \xi_2$, as noted in footnote [11]. The degenerate and non-stationary case $\xi_1 = 0$ was set aside in footnote [11].

References

Andersen, T. G. and J. Lund. (1997). "Estimating Continuous Time Stochastic Volatility Models of the Short Term Interest Rate", *Journal of Econometrics*, 77, 343-377.

Babbs, S. H. (1990). "The Term Structure of Interest Rates: Stochastic Processes and Contingent Claims", unpublished PhD, Imperial College, University of London.

Babbs, S. H. (1993). "Generalized Vasicek Models of the Term Structure", in J. Janssen and C. H. Skiadas, eds: *Applied Stochastic Models and Data Analysis: Proceedings of the Sixth International Symposium,* World Scientific, Vol I, 49-62.

Babbs, S. H. and M. J. P. Selby. (1993). "Pricing by Arbitrage in Incomplete Markets", unpublished paper.

Babbs, S. H. and N. Webber. (1994). "A Theory of the Term Structure with an Official Short Rate", Pre-print: 94/49, Financial Options Research Centre, University of Warwick.

Babbs, S. H. and K. B. Nowman. (1997a). "An Application of Generalized Vasicek Term Structure Models to the UK Gilt-Edged Market: A Kalman Filtering Analysis, unpublished paper.

Babbs, S. H. and K. B. Nowman. (1997b). "Empirical Analysis of Generalized Vasicek Term Structure Models", unpublished paper in preparation.

Ball, C. A. and W. N. Torous. (1996). "Unit Roots and the Estimation of Interest Rate Dynamics", *Journal of Empirical Finance*, 3, 215-238.

Barone, E., D. Cuoco and E. Zautzik. (1991). "Term Structure Estimation using the Cox, Ingersoll, and Ross Model: the Case of Italian Treasury Bonds, *Journal of Fixed Income*, December, 87-95.

Beaglehole, D. R. and M. S. Tenney. (1991). "General Solutions of Some Interest Rate-Contingent Claim Pricing Equations", *Journal of Fixed Income*, 1, 69-83.

Bergstrom, A. R. (1983). "Gaussian Estimation of Structural Parameters in Higher-Order Continuous Time Dynamic Models", *Econometrica,* 51,117-152.

Bergstrom, A. R. (1984). "Continuous Time Stochastic Models and Issues of Aggregation Over Time, in Z. Griliches and M. D. Intriligator, eds: Handbook of Econometrics, Volume II (Elsevier Science, Amsterdam).

Bergstrom, A. R. (1990). *Continuous Time Econometric Modelling*, Oxford University Press, Oxford.

Brenner, R. J., Harjes, R. H. and K. F. Kroner. (1996). "Another Look at Models of the Short-Term Interest Rate", *Journal of Financial and Quantitative Analysis, 31*, 85-107.

Brown, S. J. and P. H. Dybvig. (1986). "The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates", *Journal of Finance*, 41, 617-630.

Brown, R. H. and S. M. Schaefer. (1994). "The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model, *Journal of Financial Economics*, 35, 3-42.

Broze, L., O. Scaillet and J-M Zakoian. (1995). "Testing for Continuous Time Models of the Short-Term Interest Rate", *Journal of Empirical Finance*, 2, 199-223.

Chambers, M. J. (1991). "Discrete Models for Estimating General Linear Continuous Time Systems", *Econometric Theory*, 7, 531-542.

Chan, K. C., G. A. Karolyi, F. A. Longstaff, and A. B. Sanders. (1992). "Comparison of Models of the Short-Term Interest Rate", *Journal of Finance,* 47, 1209-1227.

Chen, R. and L. Scott. (1993). "Maximum Likelihood Estimation for a Multifactor Equilibrium Model of the Term Structure of Interest Rates", *Journal of Fixed Income*, December, 14-31.

Chen, R. and L. Scott. (1995). "Multi-Factor Cox-Ingersoll-Ross Models of the Term Structure: Estimates and Tests From a Kalman Filter Model", unpublished paper.

Dahlquist, M. (1996). "On Alternative Interest Rate Processes", *Journal of Banking and Finance, 20,* 1093-1119.

De Munnik, J. and P. Schotman. (1994). "Cross Sectional Versus Time Series Estimation of Term Structure Models: Empirical Results for the Dutch Bond Market", *Journal of Banking and Finance*, 18, 997-1025.

Duffie, D. and R. Kan. (1996). "A Yield-Factor Model of Interest Rates", *Mathematical Finance*, 6, 379-406.

Duan, J. C. and J. G. Simonato. (1995). "Estimating and Testing Exponential-Affine Term Structure Models by Kalman Filter", unpublished paper.

Geyer, A. L. J. and S. Pichler. (1996). "A State-Space Approach to Estimate and Test Multi-Factor Cox-Ingersoll-Ross Models of the Term Structure", unpublished paper.

Gibbons, M. R. and K. Ramaswamy. (1993). "A Test of the Cox, Ingersoll, and Ross Model of the Term Structure, *Review of Financial Studies*, 6, 619-658.

Hamilton, J. D. (1994). *Time Series Analysis*, Princeton University Press, Princeton.

Harvey, A. C. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge University Press, Cambridge.

Hull, J. and A. White. (1990). "Pricing Interest-Rate Derivative Securities", *Review of Financial Studies*, 3, 573-592.

Hull, J. and A. White. (1994). "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models, *Journal of Derivatives*, 2, 37-48.

Langetieg, T. C. (1980). "A Multivariate Model of the Term Structure", *Journal of Finance*, 35, 71-91.

Litterman, R. and J. Scheinkman (1991). "Common Factors Affecting Bond Returns", *Journal of Fixed Income*, 1, 54-61.

Lund, J. (1994). "Econometric Analysis of Continuous-Time Arbitrage-Free Models of the Term Structure of Interest Rates", The Aarhus School of Business.

Lund, J. (1997). "A Model for Studying the Effect of EMU on European Yield Curves", The Aarhus School of Business.

Jegadeesh, N. and G. G. Pennacchi. (1996). "The Behavior of Interest Rates Implied by the Term Structure of Eurodollar Futures", *Journal of Money, Credit, and Banking, 28*, 426-446.

Nowman, K. B. (1991). "Open Higher-Order Continuous Time Dynamic Model with Mixed Stock and Flow Data and Derivatives of Exogenous Variables", *Econometric Theory, 7*, 404-408.

Nowman, K. B. (1993). "Finite-Sample Properties of the Gaussian Estimation of an Open Higher-Order Continuous-Time Dynamic Model with Mixed Stock and Flow Data", in G. Gandolfo ed.: *Continuous Time Econometrics: Theory and Applications* (Chapman-Hall, London).

Nowman, K. B. (1997a). "Gaussian Estimation of Single-Factor Continuous Time Models of the Term Structure of Interest Rates", *forthcoming Journal of Finance, 52, Sept.*

Nowman, K. B. (1997b). "Continuous Time Short Rate Interest Rate Models, *forthcoming Applied Financial Economics.*

Pearson, N. D. and T. S. Sun. (1994). "Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll and Ross Model, *Journal of Finance*, 49, 1279-1304.

Pennacchi, G. G. (1991). "Identifying the Dynamics of Real Interest Rates and Inflation: Evidence Using Survey Data", *Review of Financial Studies*, 4, 53-86.

Rogers, L. C. G. and W. Stummer. (1994). "How Well do One-Factor Models Fit Bond Prices", unpublished paper.

Table 1

Estimates of One and Two Factor Generalized Vasicek Models: 1987 - 1996

	One Factor Model	**Two Factor Model**
ξ_1	0.1908 (0.0017)	0.5529 (0.0108)
ξ_2		0.0652 (0.0120)
c_1	0.0132 (0.0004)	0.0195 (0.0023)
c_2		0.0186 (0.0018)
ρ		-0.8360 (0.0440)
μ	0.0594 (0.0024)	0.0728 (0.0168)
θ_1	0.6483 (0.0192)	-0.0849 (1.0361)
θ_2		0.0963 (1.2321)
h_1	0.0036 (0.00005)	0.0017 (0.00002)
h_2	0.0022 (0.00003)	0.0004 (0.00006)
h_3	0.0004 (0.00008)	0.0017 (0.00004)
h_4	0.0037 (0.00002)	0.0028 (0.00004)
h_5	0.0042 (0.00003)	0.0019 (0.000006)
h_6	0.0052 (0.00005)	0.0009 (0.00003)
h_7	0.0062 (0.00006)	0.00008 (0.00007)
h_8	0.0073 (0.00006)	0.0008 (0.00003)
LogL	20494	24397
BIC	-40914	-48695

Figure 1: Factor loadings of the two factor model

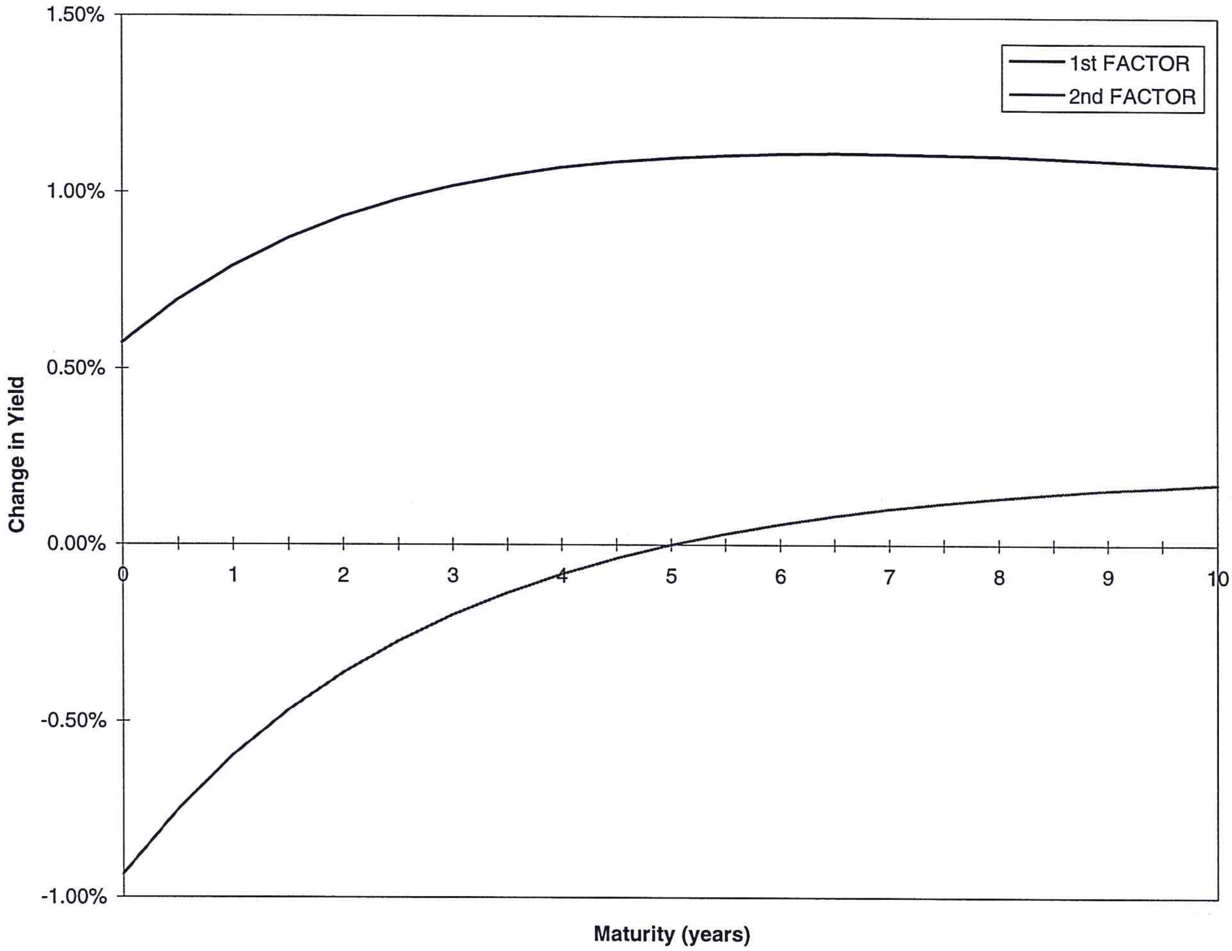


Figure 2: Comparison of the observed and estimated 3-month interest rate (one factor model)

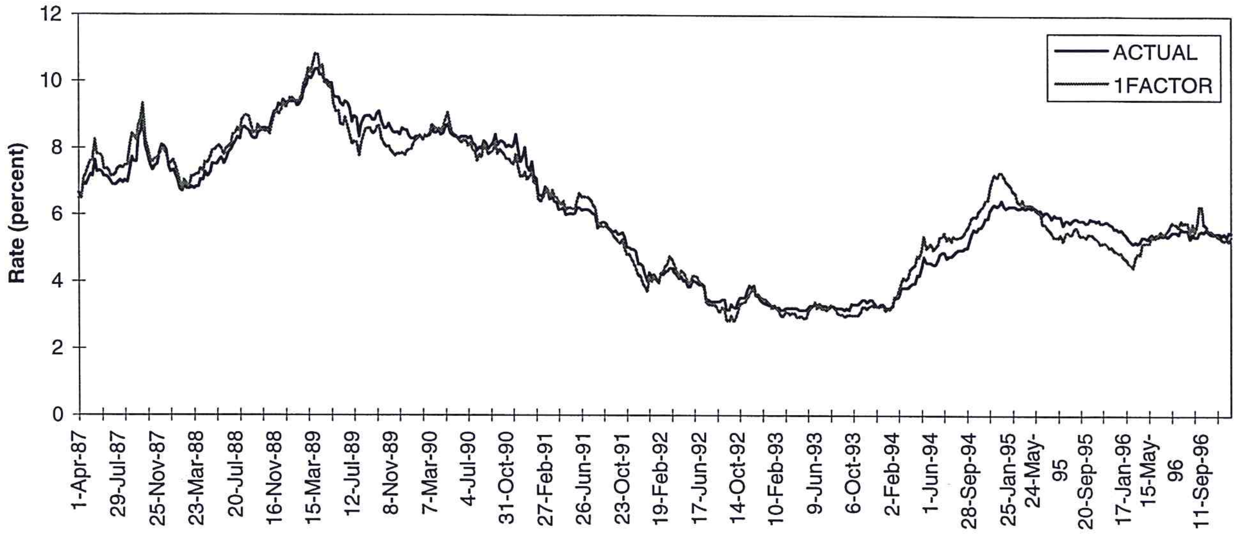


Figure 3: Comparison of the observed and estimated 3-month interest rate (two factor model)

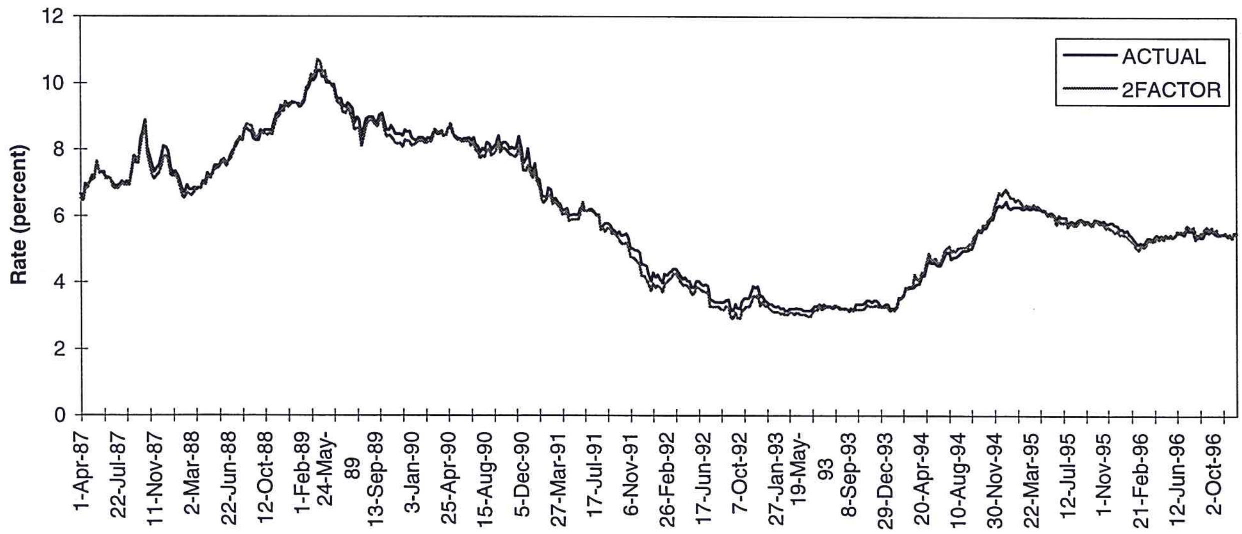


Figure 4: Comparison of the observed and estimated 6-month interest rate (one factor model)

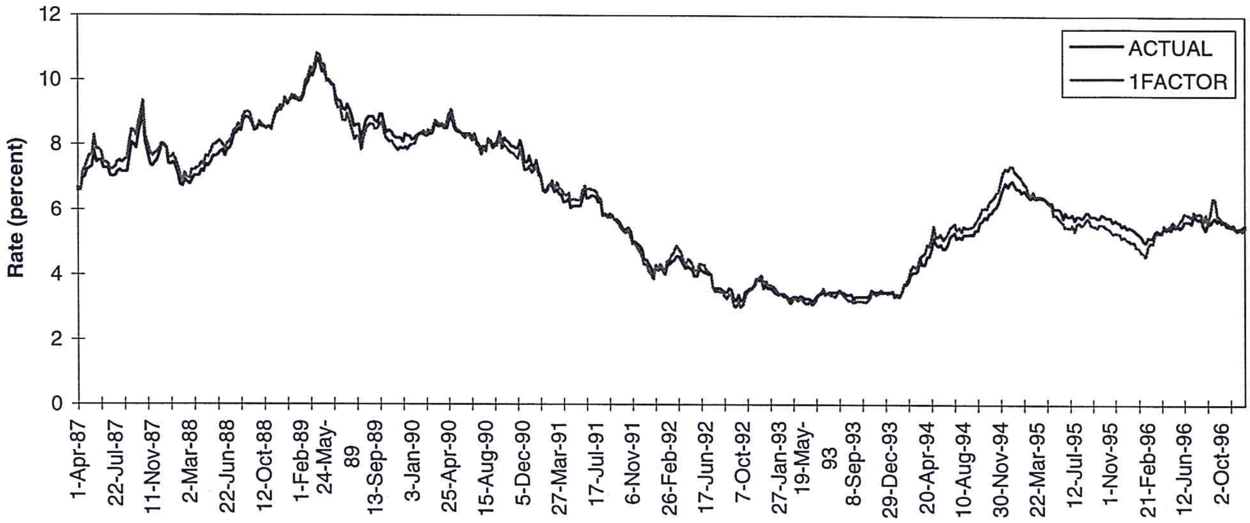


Figure 5: Comparison of the observed and estimated 6-month interest rate (two factor model)

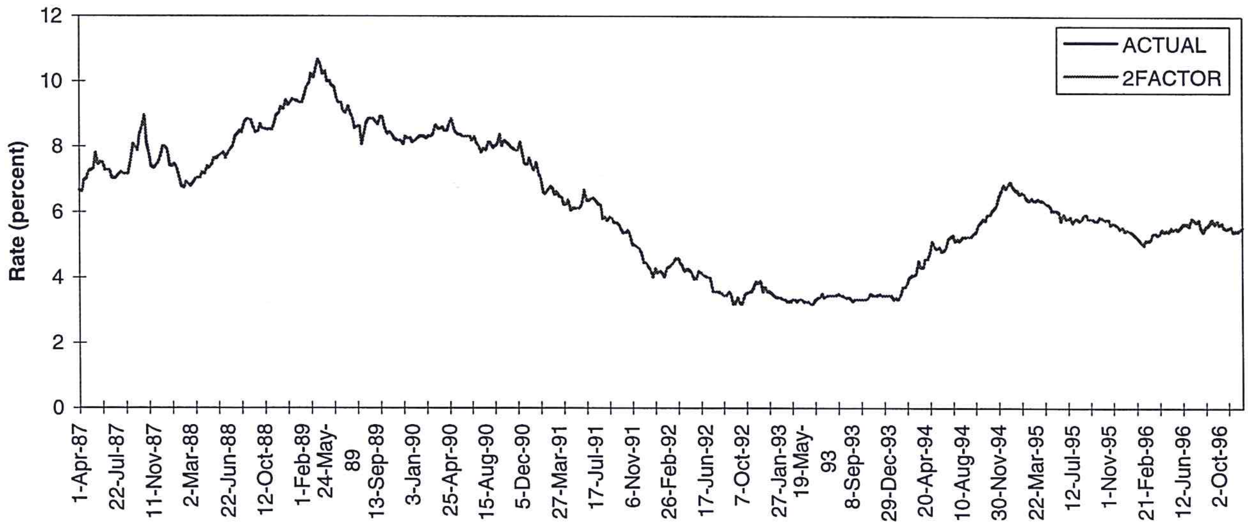


Figure 6: Comparison of the observed and estimated 1-year interest rate (one factor model)

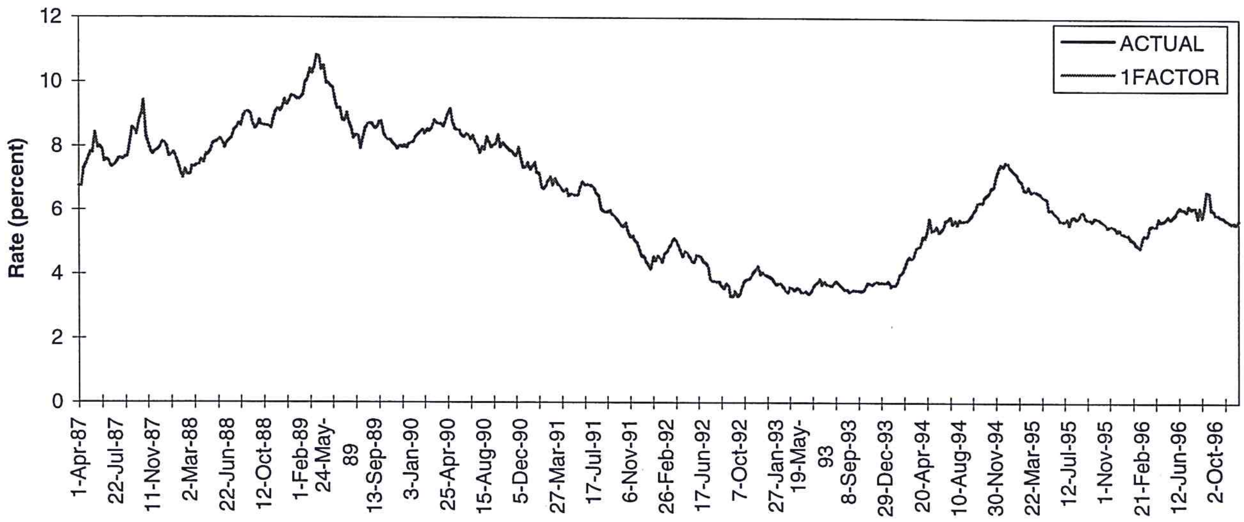


Figure 7: Comparison of the observed and estimated 1-year interest rate (two factor model)

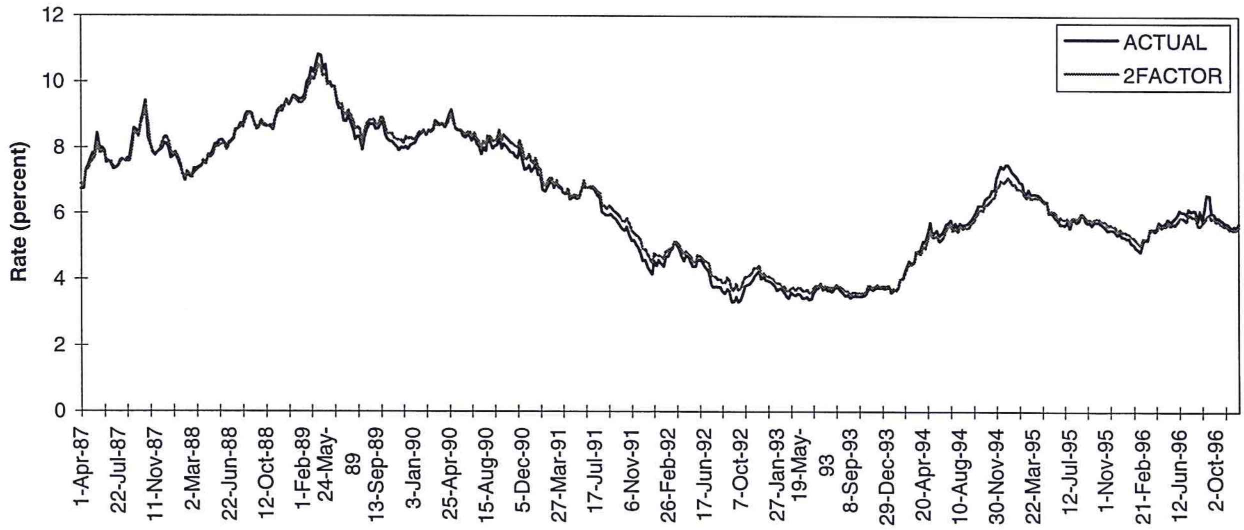


Figure 8: Comparison of the observed and estimated 2-year interest rate (one factor model)

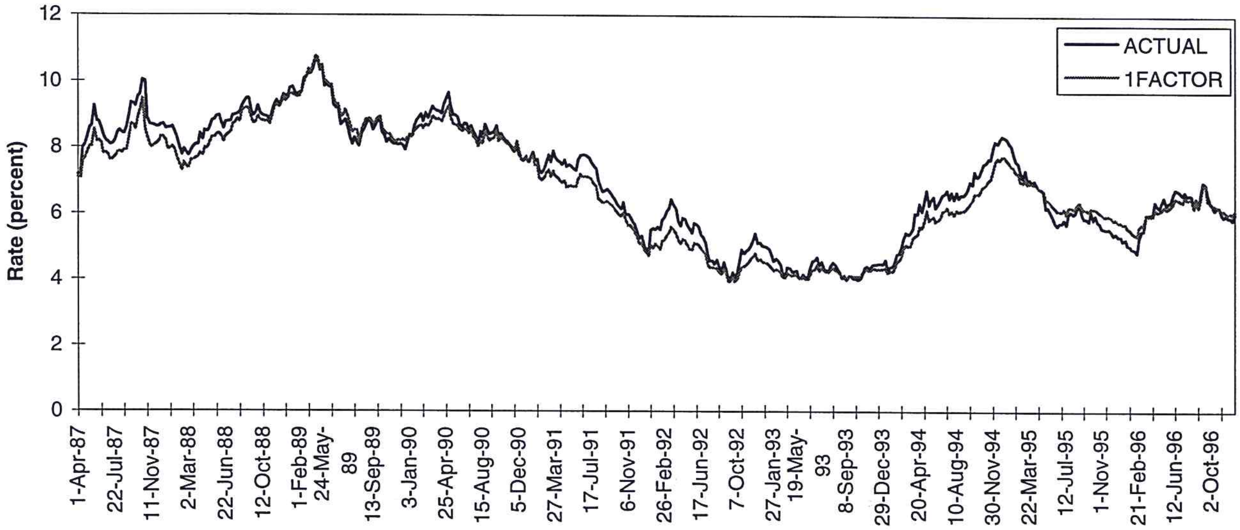


Figure 9: Comparison of the observed and estimated 2-year interest rate (two factor model)

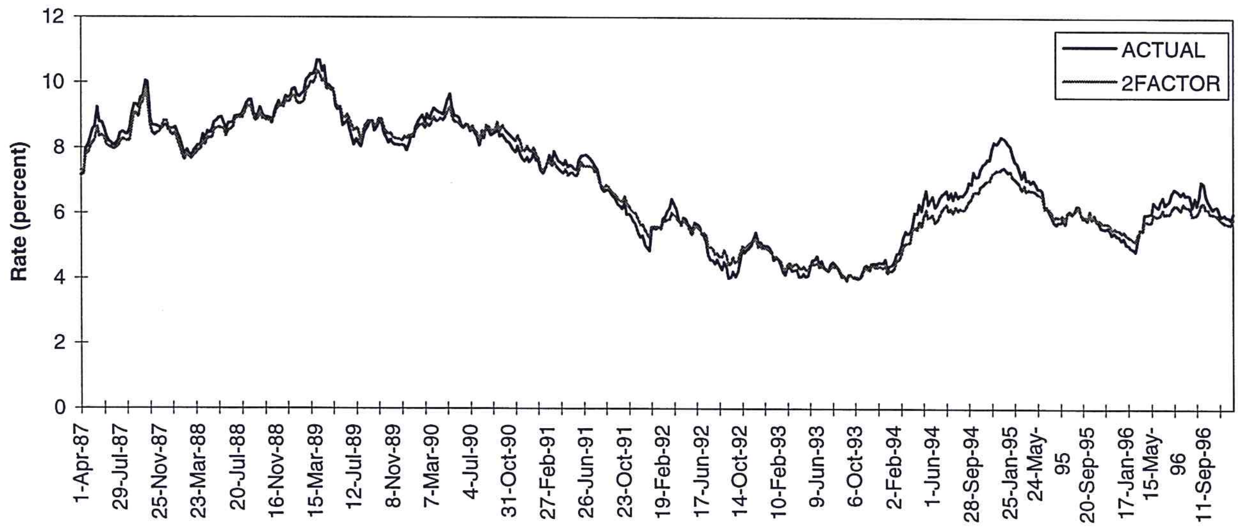


Figure 10: Comparison of the observed and estimated 3-year interest rate (one factor model)

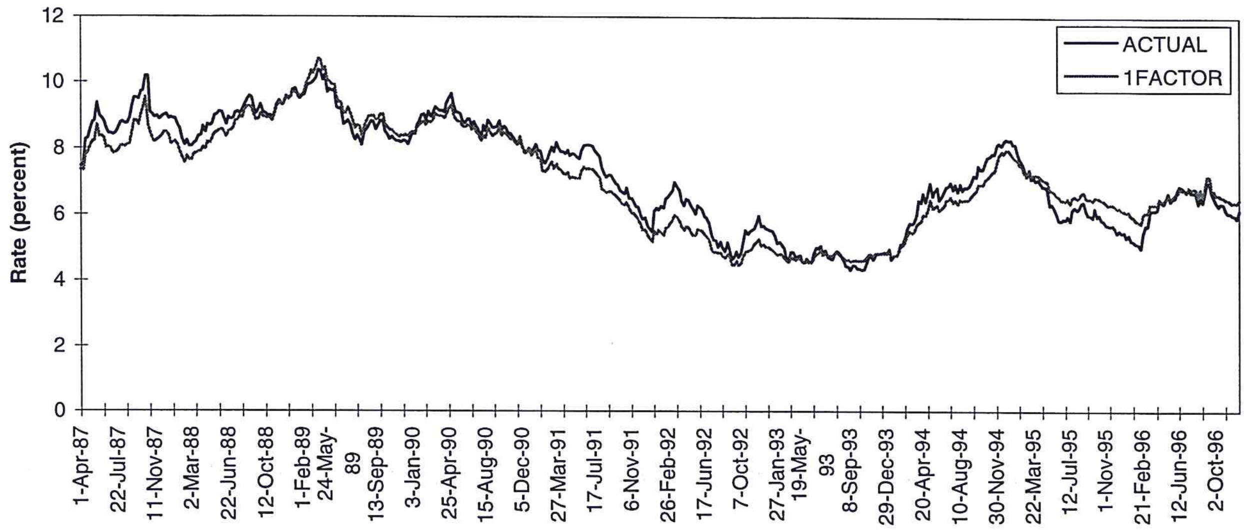
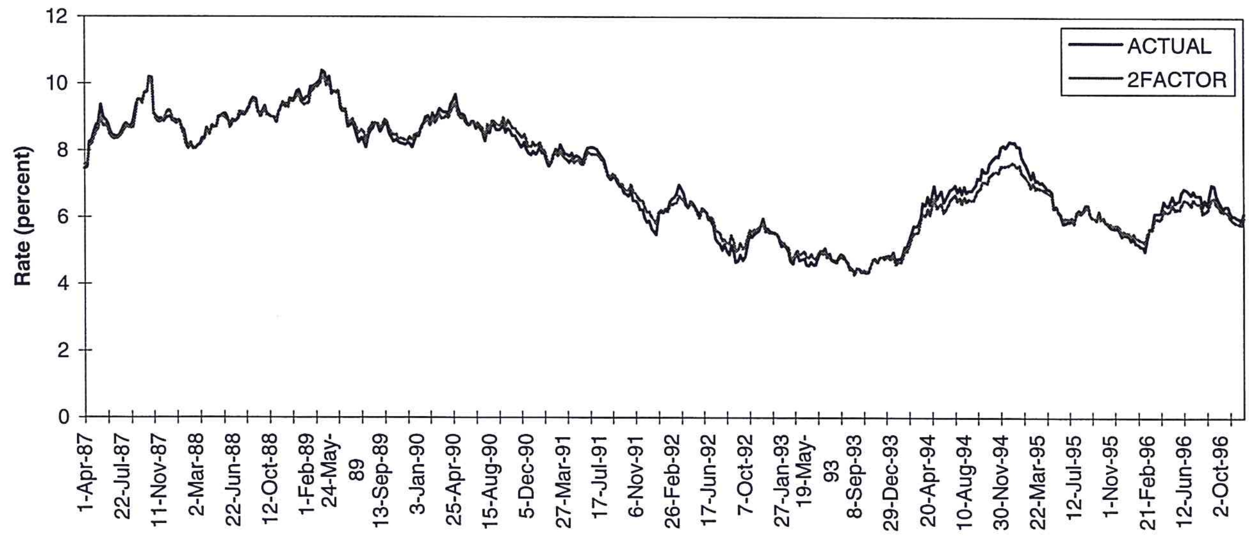


Figure 11: Comparison of the observed and estimated 3-year interest rate (two factor model)



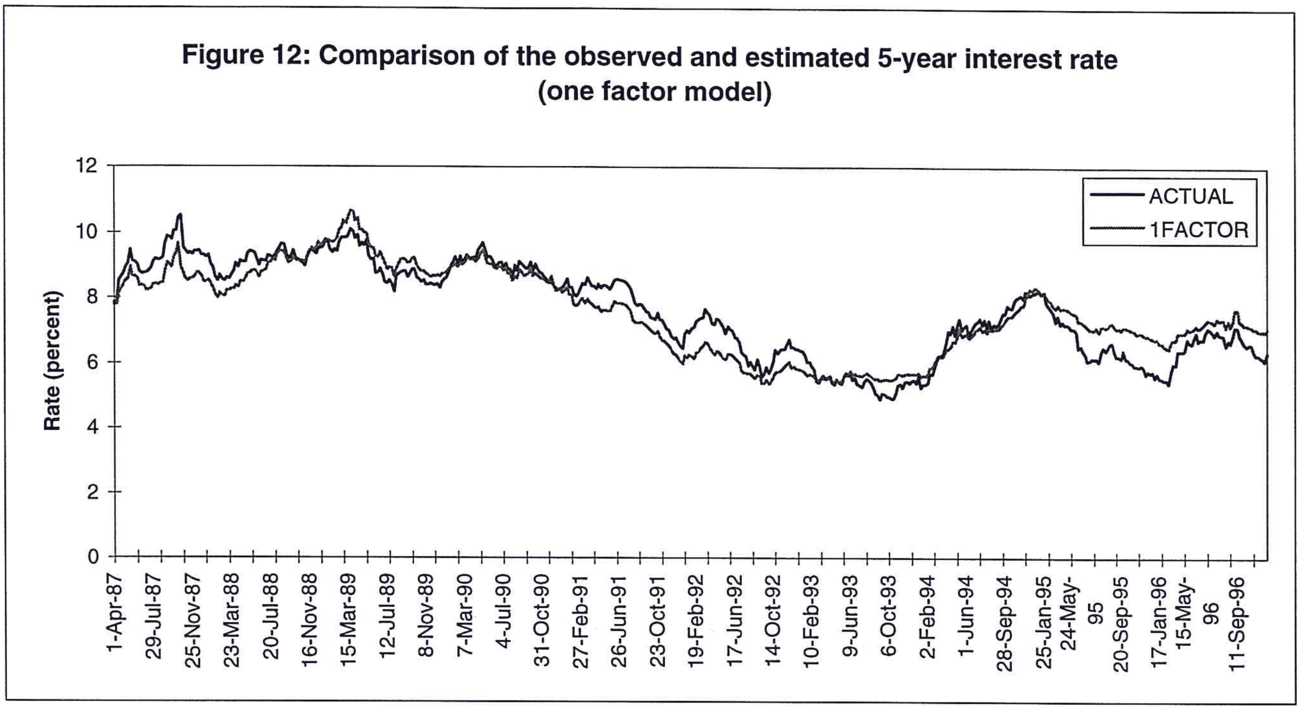


Figure 12: Comparison of the observed and estimated 5-year interest rate
(one factor model)
Rate (percent)
ACTUAL
1FACTOR
0
2
4
6
8
10
12
1-Apr-87
29-Jul-87
25-Nov-87
23-Mar-88
20-Jul-88
16-Nov-88
15-Mar-89
12-Jul-89
8-Nov-89
7-Mar-90
4-Jul-90
31-Oct-90
27-Feb-91
26-Jun-91
23-Oct-91
19-Feb-92
17-Jun-92
14-Oct-92
10-Feb-93
9-Jun-93
6-Oct-93
2-Feb-94
1-Jun-94
28-Sep-94
25-Jan-95
24-May-95
20-Sep-95
17-Jan-96
15-May-96
11-Sep-96

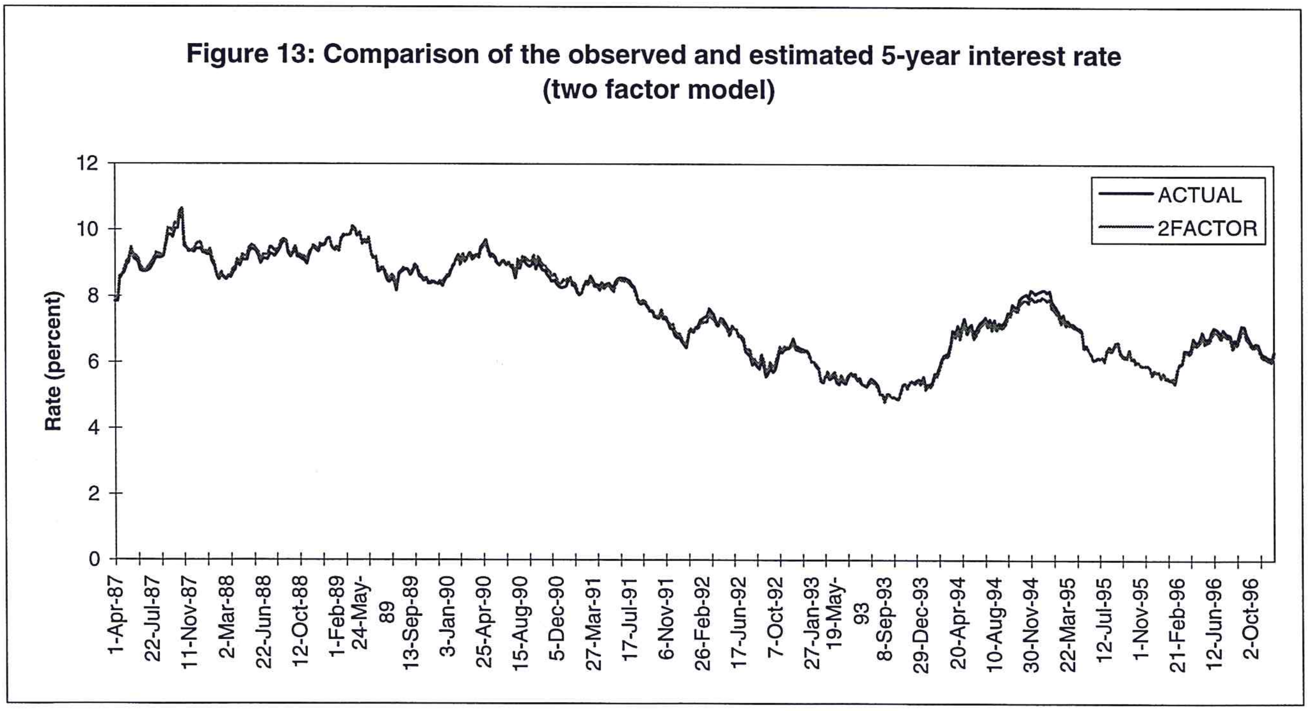


Figure 13: Comparison of the observed and estimated 5-year interest rate
(two factor model)
Rate (percent)
ACTUAL
2FACTOR
0
2
4
6
8
10
12
1-Apr-87
22-Jul-87
11-Nov-87
2-Mar-88
22-Jun-88
12-Oct-88
1-Feb-89
24-May-89
13-Sep-89
3-Jan-90
25-Apr-90
15-Aug-90
5-Dec-90
27-Mar-91
17-Jul-91
6-Nov-91
26-Feb-92
17-Jun-92
7-Oct-92
27-Jan-93
19-May-93
8-Sep-93
29-Dec-93
20-Apr-94
10-Aug-94
30-Nov-94
22-Mar-95
12-Jul-95
1-Nov-95
21-Feb-96
12-Jun-96
2-Oct-96

Figure 14: Comparison of the observed and estimated 7-year interest rate (one factor model)

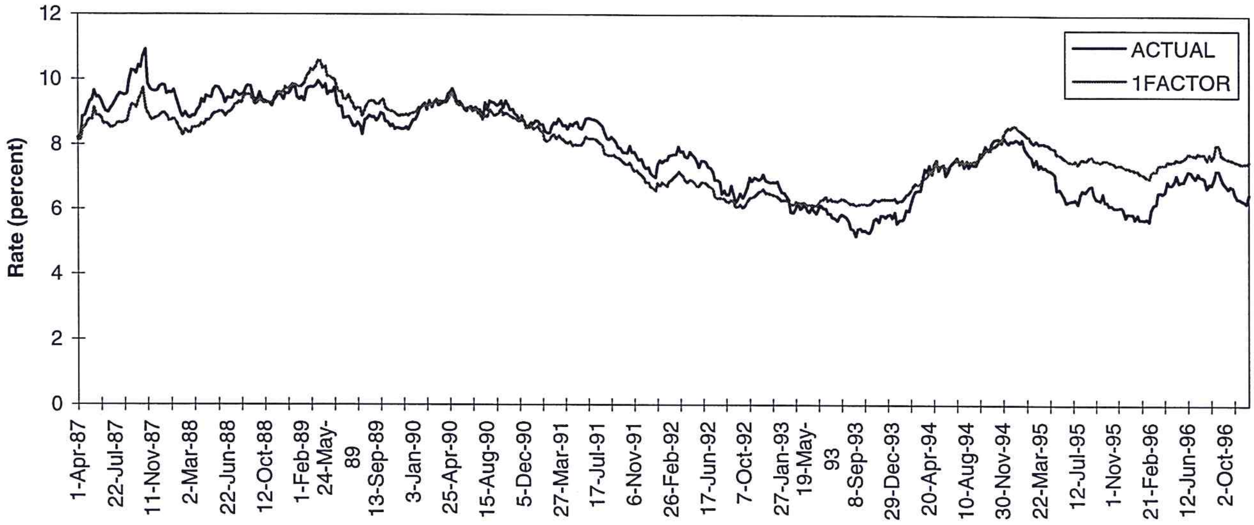


Figure 15: Comparison of the observed and estimated 7-year interest rate (two factor model)

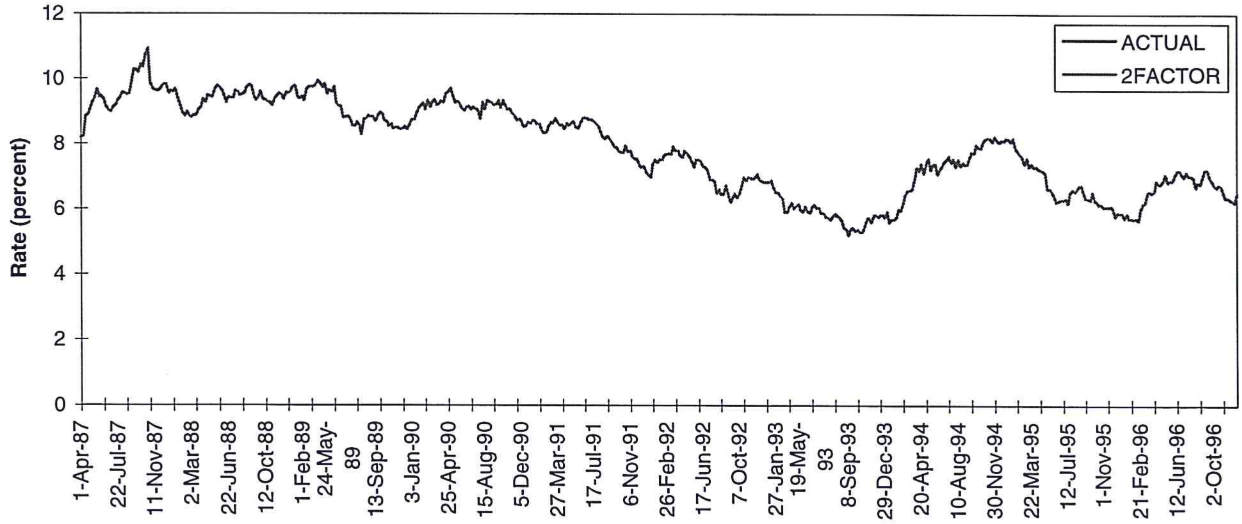


Figure 16: Comparison of the observed and estimated 10-year interest rate (one factor model)

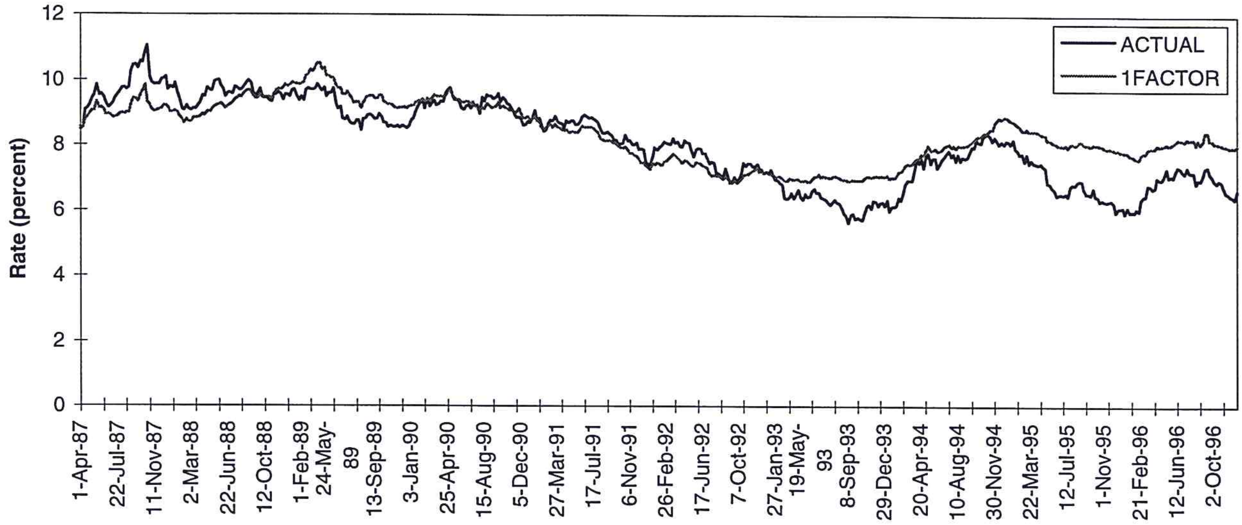


Figure 17: Comparison of the observed and estimated 10-year interest rate (two factor model)