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The Extension Theorem and a Unified Approach to No-Arbitrage Pricing *

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Abstract

The paper examines an important result in arbitrage pricing - the extension of the pricing rule from the marketed subspace to the whole market. We give a simple exposition of the Extension Theorem, showing that the absence of portfolios that cost nothing to purchase and converge to a strictly positive claim is a necessary and sufficient condition for the existence of a strictly positive and continuous extension of the pricing rule. As a consequence it is demonstrated that no arbitrage implies the existence of an equivalent martingale measure provided the right class of trading strategies is chosen. Finally, we show that the Extension Theorem unifies pricing between the incomplete and the complete market case.

JEL classification code: G12, D40

Keywords: Extension Theorem, incomplete market, arbitrage, strictly positive pricing rule, state price, price region

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Introduction

The paper examines an important result in arbitrage pricing - the extension of the pricing rule from the marketed subspace to the whole market. We show that the strictly positive extension only fails to exist in trivial cases, that is when the continuity of prices implies arbitrage in the closure of the marketed subspace. Our result implies the equivalence between no arbitrage and the existence of an equivalent martingale measure, provided that the class of trading strategies is chosen carefully. Finally, we demonstrate that the Extension Theorem gives a complete description of the no arbitrage prices for any claim *outside* the marketed subspace. Consequently, we show that the no arbitrage price of any collection of securities is determined in the same way regardless of market completeness.

Why is the Extension Theorem important? Suppose that we start with an incomplete market where prices preclude arbitrage. Mathematically, there is a strictly positive linear pricing rule restricted to the incomplete market. A strictly positive extension of the restricted pricing rule implies that one can leave the prices of marketed securities unchanged, complete the market by adding new securities, and not give rise to arbitrage opportunities, provided of course that the new securities are priced accordingly to the extended pricing rule. The extension result is even more striking when we realize that a pricing rule defined on the whole market determines all state prices. The extension property then means that any no arbitrage price in incomplete market is supported by a state price from a complete market in which no arbitrage opportunities exist.

The work on the Extension Theorem in infinite dimensional spaces dates back to Ross (1978), who is the first to show that no arbitrage is synonymous with a strictly positive linear pricing function defined on the marketed subspace. The initial motivation for a positive extension of this pricing function is somewhat unclear. According to Ross[1], "the advantage of this extension is that the domain of the pricing function does not depend on the set of marketed assets." However, if this was the only goal then any (not just positive) extension would do.

Harrison and Kreps (1979) provide very convincing justification for the

[1]The quote comes from Dybvig and Ross (1987). The original article does not give explicit motivation.

Extension Theorem by demonstrating a one-to-one correspondence between strictly positive pricing functions and equivalent martingale measures. Furthermore, they show that if a new claim is added to the market, the enlarged market will be *viable* if and only if the price of the new claim is determined by a strictly positive extension of the original pricing function.

Building on this work Kreps (1981) reduces the requirement of viability to the 'no free lunch' condition. Unfortunately, the no free lunch condition in itself is rather technical and one struggles to give it intuitive meaning. Worse still, subsequent research found it difficult to reduce the requirement of no free lunch to the simple no arbitrage condition.

It is worth mentioning a few later papers which help to clarify the relationship between no arbitrage and the equivalent martingale measure. Dalang et al. (1990) showed that in a market with a finite number of securities and finitely many trading dates no arbitrage is sufficient for the existence of an equivalent martingale measure. In contrast, Back and Pliska (1991) give an example with infinitely many trading dates where the absence of arbitrage does not give us a strictly positive extension of the pricing functional.

Coming back to the roots of the problem, Clark (1993) gives a nice exposition of Ross' definition of arbitrage and points out that arbitrage is an algebraic notion whereas the pricing functional must also satisfy some topological properties, namely continuity. As it turns out this is the crucial observation, which, unfortunately, the author does not explore to its fullest.

We take Clark's point and search for a topological version of no arbitrage condition. Mathematically, this version of no arbitrage requires that there is a continuous and strictly positive extension of the pricing functional to the closure of the marketed subspace. For such an extension to exist there must be *no approximate arbitrage,* that is no sequence of marketed claims with zero price may converge to a claim in the positive cone with the origin deleted. Since prices are by assumption continuous, whenever such sequence exists there is arbitrage in the closure of the marketed subspace and the strictly positive extension fails to exist for a trivial reason. On the other hand, the absence of approximate arbitrage in L^p spaces with $1 < p < +\infty$ is sufficient to guarantee a strictly positive and continuous extension of the pricing rule which was originally defined only on the marketed subspace.

The idea of the proof is summarized here. The pricing function on the marketed subspace defines a yet smaller subspace of marketed claims with zero price, denoted $M_0(p)$. If there is no arbitrage this subspace must be disjoint from the positive cone with the origin deleted X_{++}.

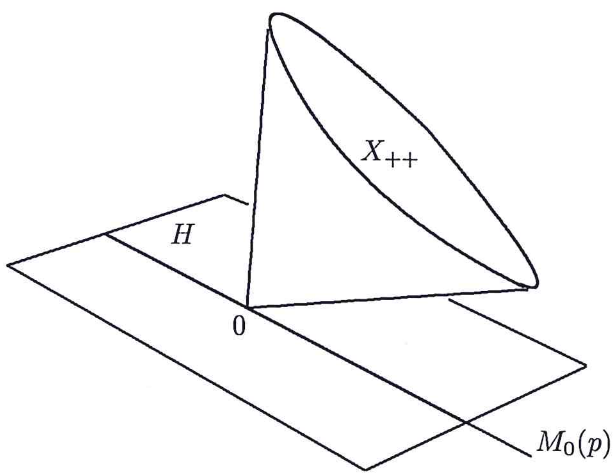

Figure 1: Illustration to the Extension Theorem

Naturally, as shown in the diagram, one would expect that there is a hyperplane H containing $M_0(p)$ and still disjoint from X_{++}, and this is indeed true in finite dimensional spaces. The separating hyperplane H is then interpreted as the subspace of all claims with zero price generated by the extended pricing rule. The fact that H is disjoint from X_{++} then guarantees that the extended pricing rule is strictly positive.

In infinite dimensional spaces there are two separate problems. Firstly, the zero marketed subspace need not be closed. This creates pathological cases whereby there is no arbitrage in the marketed subspace but by the continuity of prices arbitrage is implied in the closure of the marketed subspace. Instead of $M_0(p)$ one needs to work with its closure, hence the no approximate arbitrage condition. Secondly, the separation theorem that works so nicely for finite dimensional spaces is not generally available in infinite dimension. Reflexive spaces, such as L^p, $1 < p < \infty$, are a special case where the desired version of the separation theorem is valid.

The paper was originally motivated by an attempt to price several/all securities jointly knowing only their dividend process. This is an interesting application of the Extension Theorem which shows that there is little difference between the pricing in complete and in incomplete market. It is demonstrated that no arbitrage prices are always generated by complete market state prices. Moreover prices of all securities jointly lie in a convex region and the dimension of this region is equal to the number of linearly independent securities that are being priced. This goes to say that no arbitrage prices in a complete market are *not unique* in general. Only after one has fixed prices of a sufficient number of securities will the prices of the remaining securities be unique.

The last argument implies that the divide between complete and incomplete markets presented in finance literature is misleading. It is more precise to distinguish between the pricing of redundant and non-redundant securities. The former has a unique price, the price of the latter lies in an interval. Both may coexist within the same market.

The paper is organized as follows: The first part reviews the fundamentals of no-arbitrage pricing. In the second part we derive the Extension Theorem in an abstract model of a security market. In the third part we apply the theorem to the pricing of arbitrary number of claims when some prices are predetermined and discuss implications of this result. A simple numerical example highlights some of the points raised in this section. Section four concludes.

1 Axiomatic theory of no-arbitrage pricing

In this section we describe the axiomatic theory of no arbitrage pricing in the treatment of Clark (1993). The abstract model of security market presented here has its precursors in Harrison and Kreps (1979) and Kreps (1981). It can be naturally interpreted as a one period model thinking of claims as dividends. The second and more interesting interpretation comes from the multiperiod dynamic model[2].

We will have a topological vector space X of all claims. The space of all continuous linear forms on X (strong dual) is denoted X^*. The vector space X will be endowed with a natural ordering $\geq$ which defines the *positive cone*[3]

[2]See Harrison and Kreps, and for more structural detail Černý and Hodges (1998), section 6.

[3]In no arbitrage pricing we work with natural (canonical) ordering. Thus e.g. positive cone in $\mathbb{R}^n$ would be formed by n-tulles with non-negative coordinates, positive cone in

$X_+ = \{x \in X : x \geq 0\}$. We require X_+ to be closed.

We say that a contingent claim x is better than or equal to a contingent claim y, $x \geq y$, if and only if $x - y \in X_+$. Similarly a contingent claim x is strictly better than y, $x > y$, if and only if $x - y \in X_{++} \equiv X_+ \backslash \{0\}$ where X_{++} contains all strictly positive claims[4].

Suppose there is a collection $\{m_i\}_{i \in I}$ of marketed claims, then the marketed subspace M is the linear span of $\{m_i\}_{i \in I}$

$$M = \left\{ x \in X : x = \sum_{i \in I} \lambda_i m_i \text{ for some } \{\lambda_i\}_{i \in I} \right\}.$$

It is understood that only a finite number of λ_i are non-zero.

Assume that each marketed claim has a market price p_i, the possible prices of a marketed claim m are naturally given as

$$p(m) = \left\{ \sum_{i \in I} \lambda_i p_i \ : m = \sum_{i \in I} \lambda_i m_i \text{ for some } \{\lambda_i\}_{i \in I} \right\} \tag{1}$$

We will say that $p(m) > 0$ if $q > 0$ for all $q \in p(m)$.

Having the pair (M, p) we proceed with the characterization of a market equilibrium. The following definition clarifies the interpretation of the strictly positive cone X_{++}.

Definition 1 *We say that there is* no arbitrage *if the price of all strictly positive marketed claims is positive*

$$x \in M, x > 0 \textit{ implies } p(x) > 0.$$

In other words, arbitrage is a strictly positive claim with zero or negative price.

Axiom 1 (Incentive to trade) *There is a strictly positive marketed claim.*

L^p by non-negative random variables etc. Here we will not require the added generality of having a positive cone which is wider than the one defined by canonical ordering. For further reference on such generalization see Černý and Hodges (1998).

[4]The reader should bear in mind that the term 'strictly positive' is used in the sense of the canonical ordering on X and not in the intuitive sense. Note that a claim $(1, 0, 0) \in \mathbb{R}^3$ is strictly positive in the canonical ordering on $\mathbb{R}^3$ but at the same time it is equal to zero with positive probability, therefore 'strictly positive in X' is not to be confused with strictly positive with probability one.

If there were no strictly positive marketed claims then arbitrage pricing would not place any restriction on existing asset prices. Thus from now on we assume that Axiom 1 holds.

The next theorem restates the key result of Ross explaining the link between no arbitrage and the existence of a positive pricing functional.

Definition 2 *Let M be a linear subspace of X . Denote $M_+ = M \cap X_+$ and $M_{++} = M \cap X_{++}$. A linear functional $p : M \to \mathbb{R}$ is* positive *if $p(m) \geq 0$ for all $m \in M_+$. We say that p is* strictly positive *if $p(m) > 0$ for all $m \in M_{++}$.*

Theorem 1 *Suppose that (M, p) gives no arbitrage. Then $p : M \to \mathbb{R}$ is a strictly positive linear functional, i.e. $p(m)$ is unique for all $m \in M$, $p(m_1 + m_2) = p(m_1) + p(m_2)$ for all $m_1, m_2 \in M$ and $p(m) > 0$ for all strictly positive $m \in M$.*

Proof The theorem and its proof are stated in Clark (1993). ■

2 Continuous extension of the pricing rule

As we have seen, no arbitrage guarantees that each marketed contingent claim has a unique price. This price is determined by a strictly positive linear functional on the linear space of marketed contingent claims. However, positivity of p does *not* imply that p can be continuously extended, and even if it can be continuously extended, in general the extension will not be strictly positive. Below we give a simple necessary and sufficient condition for the existence of a continuous strictly positive extension of the linear pricing rule p from the marketed subspace to the whole market. First, however, a useful definition and a proposition.

Definition 3 *For a given strictly positive pricing functional p on M we say that*

$$M_0(p) = \{m \in M : p(m) = 0\}$$

is a zero investment marketed subspace[5].

It is standard in arbitrage literature to say that a marketed claim m is *feasible* if $p(m) \leq 0$. The set of all feasible claims is denoted $F(p)$.

[5]The term 'zero investment portfolio' was introduced by Ingersoll (1987), alternatively one could use the term 'zero cost marketed subspace'.

Proposition 1 *Let m_0 be a marketed strictly positive claim and suppose that there is no arbitrage. Then $M = M_0(p) \oplus \mathbb{R}m_0$ and $F(p) = M_0(p) - \mathbb{R}_+ m_0$.*

Proof By $\mathbb{R}m_0$ is meant the span of m_0. Symbol '$\oplus$' denotes direct sum of two linear subspaces. Since there is no arbitrage $m_0 \in M_{++}$ must have positive price, $p(m_0) > 0$. Now take an arbitrary claim $m \in M$ and set $\lambda = \frac{p(m)}{p(m_0)}$. Then $p(m - \lambda m_0) = 0$ showing that $x = m - \lambda m_0 \in M_0(p)$. Since λ is determined uniquely the decomposition $m = x + \lambda m_0$ is unique which proves the first statement. If, in addition $m \in F(p)$, then $p(m) \leq 0$ and $\lambda \leq 0$ which proves the second statement. ■

Note that no arbitrage implies $M_0(p) \cap X_{++} = \emptyset$.

The following theorem leads to a topological equivalent of no arbitrage condition.

Theorem 2 *Let there be no arbitrage in (M, p). Then p can be extended to a continuous strictly positive linear functional on the closure of M if and only if* $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$.

Proof a) Suppose $\bar{p}$ is a continuous, strictly positive linear functional on $\mathrm{cl}M$ such that $\bar{p}|M = p$. Define $N_0 = \{x \in \mathrm{cl}M : \bar{p}(x) = 0\}$, surely $N_0 \cap X_{++} = \emptyset$. In addition since $\bar{p}$ is continuous $N_0 \supset \mathrm{cl}M_0(p)$. Therefore $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$, which completes the first part of the proof.
b) Suppose now that $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$. Let m_0 be a strictly positive marketed claim. From the Proposition 1 we have $M = M_0(p) \oplus \mathbb{R}m_0$. Since $m_0 \in X_{++}$ it must be that $\mathrm{cl}M_0(p) \cap \mathbb{R}m_0 = \{0\}$ otherwise we have a contradiction with $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$. Therefore $\mathrm{cl}M = \mathrm{cl}M_0(p) \oplus \mathbb{R}m_0$ and one can extend p to $\bar{p}$ on $\mathrm{cl}M$ by setting $\bar{p}(\mathrm{cl}M_0) = 0$, $\bar{p}(m_0) = p(m_0)$. Clearly $\bar{p}$ is continuous since its kernel is not dense in M ($m_0 \notin \mathrm{cl}M_0$).
It remains to be shown that $\bar{p}$ is strictly positive. For the purpose of contradiction suppose there is $y \in \mathrm{cl}M \cap X_{++}$ such that $\bar{p}(y) \leq 0$. We know that there is a unique decomposition $y = x + \lambda m_0$ where $x \in \mathrm{cl}M_0(p)$. Thus

$$0 \geq \bar{p}(y) = \bar{p}(x) + \lambda\bar{p}(m_0) = \lambda\bar{p}(m_0)$$

which implies $\lambda \leq 0$. Then, however, $x = y - \lambda m_0 \in X_{++}$ contradicts $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$. ■

Definition 4 *An element of* $\mathrm{cl}M_0(p) \cap X_{++}$ *is called* approximate arbitrage[6].

Approximate arbitrage means that there is a sequence of marketed claims with zero price that converges to a strictly positive claim. Since we assume that prices are continuous, the limit of such a sequence represents an arbitrage opportunity, even though formally no arbitrage opportunities exist in the marketed subspace. Thus approximate arbitrage is the generalization of no arbitrage for continuous prices.

Proposition 2 *When the zero investment marketed subspace $M_0(p)$ is closed then the absence of arbitrage is equivalent to the absence of approximate arbitrage.*

Proof No arbitrage implies $M_0(p) \cap X_{++} = \emptyset$. When $M_0(p)$ is closed this is equivalent to $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$ and thus there is no approximate arbitrage. ■

An important class of financial models where arbitrage and approximate arbitrage automatically coincide is represented by models where a finite number of securities is traded at a finite number of dates. Here the zero investment marketed subspace is closed by Stricker's lemma, see Schachermayer (1992).

As we have demonstrated in Theorem 2 the absence of approximate arbitrage is the appropriate generalization of no arbitrage for continuous prices. The next theorem demonstrates that the absence of approximate arbitrage already guarantees a strictly positive and continuous extension of the pricing rule to the whole market in L^p spaces, $1 < p < \infty$.

Theorem 3 (Extension Theorem) *Suppose that $X = L^p(\Omega, \mathcal{F}, P)$, $1 < p < \infty$. There is a strictly positive continuous linear extension of the pricing rule on the marketed subspace to the whole market if and only if there is no approximate arbitrage*[7].

[6]Duffie (1996) uses this term for a slightly modified free lunch. Dybvig and Ross (1987) give this name to the asymptotic result in Ross' Arbitrage Pricing Theory. Here we adopt the term approximate arbitrage because it matches very well the mathematical content of our definition. One might also use the name *virtual arbitrage* since the arbitrage opportunity is implied from the continuity of prices.

[7]Schachermayer (1992) shows a similar result for L^1. His result, however, depends on a particular information structure – the marketed subspace arises from trading a finite

Proof a) If $\tilde{p}$ is a continuous and strictly positive extension of p then $M_0(\tilde{p})$ is closed by continuity, $M_0(\tilde{p}) \supseteq M_0(p)$ by the extension property and hence $M_0(\tilde{p}) \supseteq \mathrm{cl}M_0(p)$. By strict positivity $M_0(\tilde{p}) \cap X_{++} = \emptyset$ and hence $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$.
b) Let us denote $N \equiv \mathrm{cl}M_0(p)$. If there is no approximate arbitrage then $N \cap X_+ = \{0\}$ or, alternatively, $N \cap X_{++} = \emptyset$.
1) Recall that X_+ is the cone of all non-negative random variables in L^p. Let us take $K = \{x \in X_+ : ||x|| = 1\}$ and let J be the closed convex hull of K. For $x_1, x_2 \in K$ and $0 \leq \lambda \leq 1$ one can easily show that $||\lambda x_1 + (1-\lambda)x_2|| \geq (\frac{1}{2})^p$. Hence $0 \notin J$ which implies $J \subset X_{++}$ and thus $N \cap J = \emptyset$. Now L^p, $1 < p < \infty$ is a reflexive space, J is a bounded closed convex set, N is closed convex set and the two sets are disjoint. By Theorem 19 C in Holmes[8] N and J can be strictly separated by a continuous linear functional, i.e. there is $\varphi \in X^*$ such that $\varphi(N) = 0$ and $\varphi(J) > 0$. However, this implies $\varphi(X_{++}) > 0$.
2) Finally it remains to be seen that $\tilde{p} = \frac{p(m_0)}{\varphi(m_0)}\varphi$ is an extension of the original pricing rule p. Recall from Lemma 1 that $M = M_0(p) \oplus \mathbb{R}m_0$. Thus it suffices to show that $\tilde{p}|M_0(p) = p|M_0(p) \equiv 0$ and $\tilde{p}(m_0) = p(m_0)$ where $\varphi|M$ denotes the restriction of functional φ onto the domain M. The former equality follows from the fact that $\tilde{p}(N) = 0$ by construction of φ and $N \supseteq M_0(p)$ whereas the latter is a trivial consequence of the re-normalization of φ. Since φ is strictly positive and continuous so is $\tilde{p}$ which completes the proof. ■

The theorem above simplifies and generalizes extension results available so far. Firstly we introduce no approximate arbitrage condition $\mathrm{cl}M_0(p) \cap X_{++} = \emptyset$ instead of the less straightforward no free lunch condition introduced in Kreps (1981). Secondly we show that in L^p spaces with $1 < p < \infty$

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number of securities at a finite number of dates, and as a consequence of this setup the zero marketed subspace is closed.

[8] For completeness we provide the proof of that part of the theorem which is relevant to us and which Holmes leaves as an exercise: It is known that the unit ball $U(X)$ in a normed reflexive space is weakly compact (Theorem 16 F). Furthermore for convex sets 'closed' is equivalent to 'weakly closed' (Corollary 12 A). J is closed, convex and bounded, therefore weakly compact. N is convex, closed and therefore weakly closed. The separation theorem for one closed and one compact convex set (Corollary 11 F) asserts that N and J can be strictly separated by a weakly continuous functional ψ, however such functional is continuous in the original topology on X as well (Theorem 12 A).

no approximate arbitrage actually implies the existence of a strictly positive extension of the pricing functional defined in an incomplete market. This result does not require separability of X as in Kreps and Clark, nor a finite number of trading dates and finite number of securities as in Schachermayer. Remarkably, our result directly applies to *any* intertemporal model with no intermediate consumption, whether with finite or infinite number of trading dates and regardless of the number of securities traded. This is achieved by taking $X \equiv L^p(\Omega, \mathcal{F}_T, P)$ where $\mathcal{F}_T$ is the element of the filtration corresponding to the final period.

A striking consequence of our Extension Theorem is the equivalence between no arbitrage and an equivalent martingale measure when the zero marketed subspace is closed. As Duffie writes on page 121:

> There seems no obvious method to deduce the existence of an equivalent martingale measure from the absence of arbitrage.

Here we have such method readily available. We have mentioned earlier that in the models with finite number of trading periods and finite number of securities the arbitrage and approximate arbitrage coincide because the zero investment marketed subspace is closed. In models with infinite number of trading dates the closedness of $M_0(p)$ depends on how large class of trading strategies one allows. Back and Pliska (1991) show that with bounded trading strategies and infinite number of trading periods one can have approximate arbitrage even when there is no arbitrage. One can imagine, however, that with a careful choice of trading strategies the resulting zero investment marketed subspace turns out to be closed and we have the desired equivalence between no arbitrage and the equivalent martingale measure.

3 Valuation of non-redundant claims

So far the main concern was to extend the pricing rule from the marketed subspace to the complete market. However the Extension Theorem can be used also in the opposite direction, i.e. to find no arbitrage price for a given claim, say $x \in X$.

Taking an arbitrary strictly positive linear functional φ defined on the whole X the value $\varphi(x)$ gives a no arbitrage price. More importantly, the

Extension Theorem 3 asserts that by taking all possible strictly positive linear functionals the value $\varphi(x)$ one will cover all possible no arbitrage prices for the claim x.

Taking x as an Arrow-Debreu security, i.e. a security that pays one unit in one state of the world and zero units otherwise, $\varphi(x)$ becomes its corresponding state price. This gives motivation for the following definition:

Definition 5 *A continuous strictly positive functional on X is called a* no arbitrage state price functional[9]. *The set of all such state price functionals is denoted X^*_{++}*

$$X^*_{++} = \{\varphi \in X^* : \varphi(x) > 0 \textit{ for all } x \in X_{++}\}.$$

Proposition 3 *X^*_{++} is a convex cone in X^*.*

Proof Let us take $\lambda_1 > 0, \lambda_2 > 0$. If $\varphi_1 \in X^*_{++}$ and $\varphi_2 \in X^*_{++}$ then

$$\lambda_1\varphi_1(x) + \lambda_2\varphi_2(x) > 0 \text{ for all } x \in X_{++}.$$

Hence $\lambda_1\varphi_1 + \lambda_2\varphi_2 \in X^*_{++}$. ■

Finally we provide a characterization of no arbitrage price region of several claims jointly when prices of claims in the marketed subspace M are predetermined. The following theorem is important for several reasons. Firstly it tells us how to find all no arbitrage prices for a given set of claims. Secondly, it shows that market completeness or incompleteness does not affect the way prices are determined. Finally it demonstrates that a claim is priced by arbitrage if and only if it is redundant[10].

Theorem 4 (Pricing Theorem) *Suppose X is an L^p space , $1 < p < +\infty$. Let us have a closed marketed subspace M in which prices are given by*

[9]In practical applications it is unusual to work with abstract linear functionals. For different representations of complete market pricing rules see e.g. Dybvig and Ross (1987), page 104.

[10]This result is obtained by Jacka (1992) via martingale theory in L^1. As a precursor, Kreps (1981) claims (pg. 30) that even non-redundant claims can in general have their price determined uniquely. Here we show that in $L^p, 1 < p < \infty$ this situation cannot occur.

a strictly positive and continuous linear functional[11] ψ. *Let there be further* m *claims* $y_1, y_2, \ldots, y_m$, *no arbitrage prices of which we want to find. Then*

a) the no arbitrage price region P *for these claims is given as*

$$P = \left\{ (\varphi(y_1), \ldots, \varphi(y_m)) \in \mathbb{R}^m : \varphi \in X^*_{++} \text{ and } \varphi | M = \psi \right\},$$

b) P *is a convex set in* $\mathbb{R}^m$,

c) defining $N \equiv \operatorname{span}(M \cup \{y_1, y_2, \ldots, y_m\})$ *the dimension of the price region* P *satisfies*

$$\dim P = \operatorname{codim}_N M \tag{2}$$

which is the codimension of the marketed subspace in the enlarged marketed subspace N.

d) the price of y_i *is uniquely determined by arbitrage if and only if* y_i *is redundant, i.e.* $y_i \in M$

Proof a) We will show that no arbitrage in N implies no approximate arbitrage in N as well. Then the assertion follows from the Extension Theorem.

If there is no arbitrage in N then by Theorem 1 there is a strictly positive functional φ in N. In addition, φ has to price correctly all claims in M, $\varphi | M = \psi$. This implies $(N \supset) N_0(\varphi) \supset M_0(\psi)$. Note, however, that $\operatorname{codim}_N M_0(\psi) \leq m + 1$ and hence $\operatorname{codim}_{N_0(\varphi)} M_0(\psi) \leq m + 1$. In other words $N_0(\varphi) = M_0(\psi) \oplus L$ where $\dim L$ is finite. Since ψ is continuous and M is closed the zero investment marketed subspace $M_0(\psi)$ is closed in X. Then also $N_0(\varphi) = M_0(\psi) \oplus L$ is closed because L is finite dimensional. Now φ is strictly positive and therefore $N_0(\varphi) \cap X_{++} = \emptyset$ – there is no approximate arbitrage in N.

b) The convexity of the no arbitrage price region follows from the convexity of generalized state prices and the part a). Namely, if $p_1, p_2 \in P$ then by assertion a) there exist functionals $\varphi_1, \varphi_2 \in X^*_{++}, \varphi_i(y) = p_i$, that price correctly all claims in M. Of course, $\lambda\varphi_1 + (1 - \lambda)\varphi_2 \in X^*_{++}$ prices claims in M correctly, too, and therefore by assertion a)

$$(\lambda\varphi_1 + (1 - \lambda)\varphi_2)(y) = \lambda p_1 + (1 - \lambda)p_2 \in P.$$

[11] This assumption is not restrictive. Suppose we start with an incomplete and not necessarily closed market M in which there is no approximate arbitrage. By Theorem 2 there is a continuous strictly positive functional on $\mathrm{cl}M$ consistent with original prices in M. We simply take $\mathrm{cl}M$ and the extended functional as our starting point.

c) To prove the last statement we will first demonstrate that the cone of no arbitrage state price functionals X^*_{++} is open.

i) Let us take $K = \{x \in X_+ : ||x|| = 1\}$ and let J be the closed convex hull of K. For $x_1, x_2 \in K$ and $0 \leq \lambda \leq 1$ one can easily show that $||\lambda x_1 + (1-\lambda)x_2|| \geq (\frac{1}{2})^p$. Hence $0 \notin J$ and consequently $J \subset X_{++}$. Take an arbitrary $\varphi \in X^*_{++}$ and denote $\varepsilon = \inf_{x \in J} \varphi(x)$. We claim that $\varepsilon > 0$. For the purpose of contradiction suppose that $\varepsilon = 0$. Then there is a sequence $x_n \in J$ such that $\lim \varphi(x_n) = 0$. Since J is closed, convex and bounded, from the reflexivity of X follows that J is weakly sequentially compact (Holmes, Theorem 16F and Corollary 18 A). Hence there is a subsequence x_k converging weakly to $x \in J$ implying $\varphi(x) = 0$. Hoewever, $x \in J \subset X_{++}$ contradicts $\varphi \in X^*_{++}$.

Thus $\varphi(K) \geq \varepsilon > 0$. Taking an arbitrary functional $\psi \in X^*$ such that $||\psi|| < \frac{\varepsilon}{2}$ and $x \in X_{++}$ we have

$$(\varphi + \psi)(x) > ||x|| \left(\varphi(\frac{x}{||x||}) - ||\psi|| \right) > ||x||(\varepsilon - \frac{\varepsilon}{2}) > 0$$

which means that $\varphi + \psi \in X^*_{++}$ for any $||\psi|| < \frac{\varepsilon}{2}$. Since φ is arbitrary this means that X^*_{++} is open (in the norm topology on X^*).

ii) Let us first assume that $\mathrm{codim}_N M = m$. Applying Hahn-Banach theorem to the subspace $\mathrm{span}(M \cup \{y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m\})$ and the point y_j one can find linear functionals $\psi_j, j = 1, \ldots, m$ such that $\psi_j(M) = 0$ for all j and $\psi_j(y_i) = \delta_{ij}$ (Kronecker's delta). By the Extension Theorem the pricing rule on M can be extended to a strictly positive functional φ_0 that correctly prices securities in M. Note that functionals $\varphi_0 + \lambda\psi_j$ too price these securities correctly and moreover for $|\lambda|$ sufficiently small $\varphi + \lambda\psi_j$ will be a strictly positive functional by the result in i). Thus the price vectors $\varphi_0(y) + \lambda\psi_j(y)$ give no arbitrage prices for securities $y = (y_1, \ldots, y_m)$ consistent with the predetermined prices of securities x. By construction $\psi_j(y)$ are linearly independent vectors in $\mathbb{R}^m$ and recall that $\mathrm{dim}P$ is defined as the dimension of the affine hull of $P - \varphi_0(y)$ (which is a linear subspace) thus the dimension of the price region P is at least $\mathrm{rank}(\psi_1(y), \ldots, \psi_m(y)) = m$, and of course it cannot be more than m.

iii) In a general case a certain number of vectors y_i, say $m - l$, will lie in the marketed subspace M. However, for any $\tilde{\varphi}$ that prices correctly claims in M

the difference $\tilde{\varphi}(y_i) - \varphi_0(y_i)$ will be zero, hence if $y_i \in M$ it will not contribute to the dimensionality of the price region.
That leaves l claims that do not belong to the marketed subspace and these we partition into two groups — the first $\tilde{m}$ claims $c_1' = (y_1, \ldots, y_{\tilde{m}})$ that are linearly independent and the remaining $l - \tilde{m}$ claims $c_2' = (y_{\tilde{m}+1}, \ldots, y_l)$ that can be expressed as a linear combination of the first $\tilde{m}$ claims, $c_2 = Dc_1$ with $D \in \mathbb{R}^{(l-\tilde{m})\times\tilde{m}}$.
First of all it is clear that there cannot be more than $\tilde{m}$ linearly independent vectors of the type $\psi(y_1), \ldots, \psi(y_l)$. If there were more, one could find a non-trivial linear combination that annuls the first $\tilde{m}$ coordinates, $\sum_i \lambda_i \psi_i(c_1) = 0$. However, such a linear combination annuls the remaining $l - \tilde{m}$ coordinates as well since

$$\sum_i \lambda_i \psi_i(c_2) = \sum_i \lambda_i \psi_i(Dc_1) = \sum_i \lambda_i D\psi_i(c_1) = D\left(\sum_i \lambda_i \psi_i(c_1)\right) = 0.$$

On the other hand one can find $\tilde{m}$ linearly independent prices of the desired form by the procedure described in ii). Thus $\dim P = \tilde{m}$ and by construction $\tilde{m} = \mathrm{codim}_N M$.

d) Suppose there is just one security to be priced, say security y_i and denote its no arbitrage price region $P \subset \mathbb{R}^1$. We set $N = \mathrm{span}(M \cup \{y_i\})$ and have $\mathrm{codim}_N M = 0$ if and only if $y_i \in M$. By definition the no arbitrage price of y_i is unique if and only if $\dim P = 0$. Then the assertion c) implies that y is uniquely priced by arbitrage if and only if y is redundant. ■

The fact that the no arbitrage state price functionals are the same for both complete and incomplete markets may come as a surprise, given that the pricing in incomplete markets is conventionally regarded as much more complicated. The difference between complete and incomplete markets, however, stems from the way we like to *formulate* pricing problems rather then from an essential difference between the two cases.

The finance literature applies the term 'pricing in complete markets' to situations where the claims with predetermined prices span the whole market, $M = X$. Since the price of any additional claim is determined uniquely this creates an impression that in a complete market prices of all claims are determined uniquely (law of one price).

In incomplete markets, on the other hand, even if we assume that all marketed securities have predetermined prices, these are not sufficient to span the whole market and thus one is usually forced to price *non-redundant* securities. Hence pricing in incomplete markets usually involves calculation of a price range which is obviously more difficult than the calculation of a unique price.

However, pricing of non-redundant securities can easily arise in complete markets as well. Think of a situation where $X = \text{span}(M \cup \{y_1\})$ and y_1 is not redundant, i.e. $y_1 \notin M$. Then the price of y_1 is not unique despite the fact that markets are complete! The contrast is even stronger when no claims, save for a numeraire, have predetermined prices because then no marketed claim will have unique no arbitrage price.

In fact, we have the following distinction: regardless of market completeness it is easy to price redundant claims since their price is determined uniquely. In the case of a non-redundant claim theorem 4 shows that its no arbitrage price lies in an interval, again regardless of market completeness. Of course, finding a price interval is a much more challenging task than calculating a unique price, but whether a claim is redundant depends mainly on how many prices are predetermined and not on market completeness.

3.1 Illustrative example

To illustrate our point about pricing in complete and incomplete markets we take a simple one period economy with s states of the world, $X = \mathbb{R}^s$ with the standard positive cone $\mathbb{R}^s_+$. State price functionals will be represented by vectors in $\mathbb{R}^s$ and it is easy to see that the set of complete market state price vectors

$$\tilde{X}^*_{++} = \{q \in \mathbb{R}^s : q'x > 0 \text{ for all } x \in \mathbb{R}^s_+ \backslash \{0\}\}$$

satisfies

$$\tilde{X}^*_{++} = \text{int}\mathbb{R}^s_+.$$

Suppose there is a particular marketed strictly positive claim that pays one unit of account in each state and the price of this claim is one (there is riskless borrowing with no interest). In our earlier notation $M = \text{span}(\mathbf{1})$ and $\psi(\mathbf{1}) = 1$ where $\mathbf{1} \in \mathbb{R}^s$ is vector of ones. This gives one restriction for the state prices $\mathbf{1}'q = 1$. In particular this restriction is satisfied by vectors

$e_1, e_2, \ldots, e_s \in \mathbb{R}^s$ that form a standard orthonormal basis of $\mathbb{R}^s$.We will denote $\text{co}[e_1, e_2, \ldots, e_s]$ the *convex hull* of $e_1, e_2, \ldots, e_s$.

It is easily seen that the state prices which correctly price the claim $\mathbf{1}$ occupy the relative interior of the s-simplex $\text{co}[e_1, e_2, \ldots, e_s]$ and then the set of all admissible state prices is

$$Q_{++} = \text{rel-int}(\text{co}[e_1, e_2, \ldots, e_s]).$$

Suppose that other m claims to be priced are ordered in a matrix $y = (y_1, \ldots, y_m)$. Then we have the following pricing formula

$$P = y'Q_{++} = y'\text{rel-int}(\text{co}[e_1, e_2, \ldots, e_s]) = \text{rel-int}(\text{co}[y'e_1, y'e_2, \ldots, y'e_s]). \tag{3}$$

With this result at hand let us take a concrete example of no arbitrage pricing with three states. Then $X_+ = \mathbb{R}^3_+$ is positive octant and $\tilde{X}^*_{++} = \text{Int}\mathbb{R}^3_+$ is interior of positive octant. Q_{++} is the relative interior of the triangle $\text{co}[(1,0,0),(0,1,0),(0,0,1)]$. Consider two pairs of securities to be priced

$$y = \begin{pmatrix} 1 & 5 \\ 2 & 3 \\ 3 & 1 \end{pmatrix} \qquad \tilde{y} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \\ 3 & 3 \end{pmatrix}.$$

Applying the pricing formula (3) we find

$$P = \text{rel-int}(\text{co}[(1,5),(2,3),(3,1)]) = \text{rel-int}(\text{co}[(1,5),(3,1)])$$

and

$$\tilde{P} = \text{int}(\text{co}[(1,5),(2,3),(3,3)]).$$

The no-arbitrage price regions are depicted below.

In the first example markets are not complete, since $\text{rank}\,(\mathbf{1}, y) = 2$ instead of three. Although none of the two securities which are being priced is redundant on its own, the prices of the two securities jointly have to move together and this is confirmed from the price region P. Knowing the price of the first security (say $p_1 = 2$) the price of the second security is determined uniquely ($p_2 = 3$). The dimensionality of the price region P is also confirmed from the formula (2) where we take $N = \text{span}(\mathbf{1}, y_1, y_2)$

$$\dim P = \text{codim}_N M = \dim N - \dim M = \text{rank}\,(\mathbf{1}, y) - \text{rank}\,(\mathbf{1}) = 2 - 1 = 1.$$

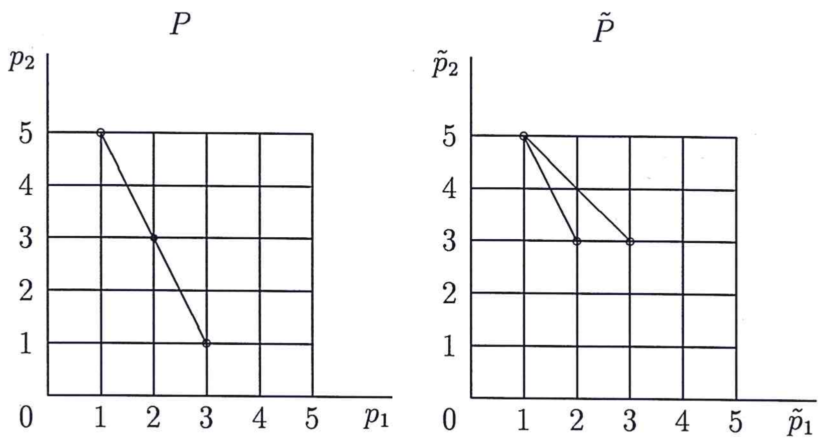

Figure 2: No-arbitrage price regions P and $\tilde{P}$

In the second example markets are complete, $\operatorname{rank}(\mathbf{1}, \tilde{y}) = 3$, and this gives fully dimensional price region

$$\dim \tilde{P} = \operatorname{rank}(\mathbf{1}, \tilde{y}) - \operatorname{rank}(\mathbf{1}) = 3 - 1 = 2.$$

Note that in both cases prices of the two securities are constrained but not unique.

4 Conclusions

The aim of the paper was, among others, to show how the Extension Theorem in Arbitrage Pricing Theory provides a unifying viewpoint from which the pricing of securities in complete and in incomplete markets appears the same. We have demonstrated that what makes the difference in no-arbitrage pricing is not market completeness but the size of the subspace with predetermined prices.

Apart from breaking the stereotypical view of complete and incomplete markets we hope, more generally, that our note will incourage researchers to develop new techniques that take into account the price indeterminacy which is inherently present in the no-arbitrage framework.

The paper also contains new technical results. We replace free lunch with more intuitive approximate arbitrage and without additional assumptions show that in L^p spaces, $1 < p < \infty$, no approximate arbitrage condition implies the existence of the strictly positive and continuous extension of the pricing functional. This leads to the equivalence between the absence of arbitrage and the existence of an equivalent martingale measure, provided that the zero marketed subspace is closed, which, we believe, can be achieved by a careful choice of trading strategies.

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