Improved Testing for the Efficiency of Asset Pricing Theories in Linear Factor Models

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Abstract

This paper suggests a refinement of the standard T^2 test statistic used in testing asset pricing theories in linear factor models. The test is designed to have improved power characteristics and to deal with the empirically important case where there are many more assets than time periods. This is necessary because the case of too few time periods invalidates the conventional T^2. Furthermore, the test is shown to have reasonable power in cases when common factors are present in the residual covariance matrix.

1 Introduction

The purpose of this paper is to suggest a new procedure for testing asset pricing theories in linear factor models. Conventionally, an F test based on Hotelling's T^2 statistic is the standard procedure; this was suggested by Jobson and Korkie (1982), investigated by Mackinlay (1987), and further investigated by Gibbons, Ross, and Shanken (1989). A broad discussion of this test and its properties are contained in Campbell, Lo, and MacKinlay (1997) (hereafter, CLM).

The T^2 test is known to suffer from two clear defects. Firstly, it can only be applied to universe of stocks where the number of stocks, N, is less than the number of observations, T. Empirically, this is very restrictive, as practitioners often set N to 20,000 (stocks) and T to 60 (five years of monthly data) in global stock selection models. Secondly, it is known to lack power; in particular, the procedure uses the unconstrained maximum likelihood estimator of the $(N \times N)$ covariance matrix of the idiosyncratic errors in the linear factor model. In many cases there are known restrictions on this covariance matrix.

The procedure we advocate uses a restricted covariance matrix to gain power and, by construction, will be valid for all N and T greater than K, where K is the number of factors in the linear model. Furthermore, our test, a Wald test, has an exact distribution which, like T^2, is parameter free and depends only on N, T, and K. In fact, the distribution of our test statistic is of some statistical interest being the N fold convolution of the $F(1, T-K-1)$ distribution divided by N. Since the characteristic function of the F distribution is known, see Phillips (1982), it is possible to compute an expression for the characteristic function of our test statistic. We cannot derive a closed form expression for the density of the test statistic but we can compute it numerically very easily, as well as its power function, for the usual alternative hypotheses.

In the next section, we described the Hotelling T^2 test and motivate and derive our test statistic. In section 3, we present some critical values for our test and compare them with those of the T^2 test. Some power calculations and empirical calculations are contained in section 4. Our conclusions follow in section 5.

2 New Test Statistic for Linear Factor Models

The popularity of linear factor models (LFMs) in financial economics stems from the arbitrage pricing theory (APT) introduced by Ross (1976) which allows for explanations of risk premia using multiple risk factors. Under the assumption that markets are competitive and frictionless and that there are K factors, the APT is applied to the system of N asset returns;

$$\begin{aligned} \mathbf{R}_t &= \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{f}_t + \boldsymbol{\varepsilon}_t \qquad (1) \\ E[\boldsymbol{\varepsilon}_t|\mathbf{f}_t] &= 0 \end{aligned}$$

$$E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'|\mathbf{f}_t] \;=\; \boldsymbol{\Sigma}$$

where $\mathbf{R}_t \equiv (R_{1,t}\ R_{2,t} \cdots R_{N,t})'$ is a vector of asset returns, $\boldsymbol{\alpha}\equiv (\alpha_1\ \alpha_2 \cdots \alpha_N)'$ is a vector of intercept of the factor model, $\boldsymbol{\beta} \equiv (\boldsymbol{\beta}_1\ \boldsymbol{\beta}_2 \cdots \boldsymbol{\beta}_N)'$ is an $(N \times K)$ matrix of factor sensitivities, where $\boldsymbol{\beta}_j$ is an $(K \times 1)$ vector of factor sensitivities for asset j, $\mathbf{f}_t \equiv (f_{1,t}\ f_{2,t} \cdots f_{K,t})'$ is a vector of common factor realizations, and $\boldsymbol{\varepsilon}_t \equiv (\varepsilon_{1,t}\ \varepsilon_{2,t} \cdots \varepsilon_{N,t})'$ is an *iid* vector of disturbance terms.

In this model, the disturbance terms are uncorrelated across assets since the factors are assumed to account for all the common variation in asset returns. Furthermore, following the arguments in Ross (1976) the model implies,

$$E(\mathbf{R}) = \mathbf{i}\lambda_0 + \boldsymbol{\beta}\boldsymbol{\lambda}_k \tag{2}$$

where λ_0 is the riskfree return or zero-beta portfolio return, $\boldsymbol{\lambda}_K$ is a $(K \times 1)$ vector of factor risk premia, and $\mathbf{i}$ is an $(N \times 1)$ vector, $\mathbf{i} \equiv (1 \cdots 1)'$. Note that the exact parametric restriction implied by equation (2) is justified by the findings of Connor (1984), Dybvig (1985) and Grinblatt and Titman (1985), who show that theoretical deviations of approximate factor pricing models from exact factor pricing are likely to be negligible under plausible conditions. Although the APT is known to apply for more general processes than those described by equation (1), see, for example, Chamberlain and Rothschild (1983) and Reisman (1992), equations (1) and (2) remain a model of central importance to financial economics, not least because of its links with factor analysis, and its widespread adoption by practitioners.

2.1 Conventional F Statistic for Linear Factor Models

Among several versions of the exact factor pricing model, we consider the case that factors are portfolios of traded assets and a riskfree asset exists.[1]Then, a testable version for the exact factor pricing model is

$$\mathbf{r}_t = \mathbf{a} + \boldsymbol{\beta}\mathbf{r}_t^f + \boldsymbol{\epsilon}_t \tag{3}$$

where $\mathbf{r}_t$ is now a $(N \times 1)$ vector of excess returns for N assets and $\mathbf{r}_t^f$ is a $(K \times 1)$ vector of factor portfolio excess returns, and $\boldsymbol{\epsilon}_t \equiv (\epsilon_{1,t}\ \epsilon_{2,t} \cdots \epsilon_{N,t})'$ is a $(N \times 1)$ vector of disturbance terms whose variance-covariance matrix is $E(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t') = \boldsymbol{\Sigma}$.

When we assume that returns conditional on the factor realizations are iid jointly multivariate normal, from the maximum likelihood (or equivalently OLS) estimators we have conditionally on the factor realisations,

$$\widehat{\mathbf{a}}\,\tilde{}\,N(\mathbf{a}, \frac{1}{T}(1 + \widehat{\boldsymbol{\mu}}_K'\widehat{\boldsymbol{\Omega}}_K^{-1}\widehat{\boldsymbol{\mu}}_K)\boldsymbol{\Sigma}), \tag{4}$$

where

$$\widehat{\boldsymbol{\mu}}_K \;=\; \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t^f,$$

$$\widehat{\boldsymbol{\Omega}}_K = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{r}_t^f - \widehat{\boldsymbol{\mu}}_K)(\mathbf{r}_t^f - \widehat{\boldsymbol{\mu}}_K)'.$$

See Gibbons, Ross, and Shanken (1989) for the one factor model and chapter 6 of CLM for the multi-factor model.

The Wald test statistic for the hypothesis $H_0 : \mathbf{a} = \mathbf{a_0}$ is

$$S_1^I = T[1 + \widehat{\boldsymbol{\mu}}_K' \widehat{\boldsymbol{\Omega}}_k^{-1} \widehat{\boldsymbol{\mu}}_K]^{-1}(\widehat{\mathbf{a}} - \mathbf{a_0})' \boldsymbol{\Sigma}^{-1} (\widehat{\mathbf{a}} - \mathbf{a_0})\tilde{\ } \chi^2(N). \tag{5}$$

The equation (5) says that the null hypothesis is distributed as chi-square with N degrees of freedom. With the replacement of the sample estimate of $\boldsymbol{\Sigma}$ in equation (5), the usual multivariate F-test statistic is

$$S_1 = \frac{T-N-K}{N}[1 + \widehat{\boldsymbol{\mu}}_K' \widehat{\boldsymbol{\Omega}}_k^{-1} \widehat{\boldsymbol{\mu}}_K]^{-1}(\widehat{\mathbf{a}} - \mathbf{a_0})' \widehat{\boldsymbol{\Sigma}}^{-1} (\widehat{\mathbf{a}} - \mathbf{a_0}), \tag{6}$$

where

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{r}_t - \widehat{\mathbf{a}} - \widehat{\boldsymbol{\beta}}\mathbf{r}_t^f)(\mathbf{r}_t - \widehat{\mathbf{a}} - \widehat{\boldsymbol{\beta}}\mathbf{r}_t^f)'$$

and $\widehat{\mathbf{a}}$ and $\widehat{\boldsymbol{\beta}}$ are the maximum likelihood estimators of $\mathbf{a}$ and $\boldsymbol{\beta}$, respectively. It can be shown that the test statistic, S_1, is unconditionally distributed central F with N degrees of freedom in the numerator and $(T-N-K)$ degrees of freedom in the denominator; see chapters 5 and 6 of CLM for a detailed explanation.

Thus, for the test of APT models, i.e., the null hypothesis of $H_0^a : \mathbf{a} = 0$ against the alternative hypothesis $H_1^a : \mathbf{a} \neq 0$, the usual multivariate F-test statistic is

$$S_1 = \frac{T-N-K}{N} c \widehat{\mathbf{a}}' \widehat{\boldsymbol{\Sigma}}^{-1} \widehat{\mathbf{a}} \tag{7}$$

where $c = (1 + \widehat{\boldsymbol{\mu}}_K' \widehat{\boldsymbol{\Omega}}_K^{-1} \widehat{\boldsymbol{\mu}}_K)^{-1}$.

As previously mentioned, we face a problem when T is smaller than N. Because of the time-varying properties of risk factors, most practitioners tend to estimate LFM's using monthly data for 5 to 7 years. In this case, the degrees of freedom, $(T-N-K)$, in the denominator of the conventional F test statistic can become negative, the covariance matrix $\widehat{\boldsymbol{\Sigma}}$ becomes singular, and thus we cannot apply the F test, see equation (7).

Although asset returns are widely known to be skewed and fat-tailed and as such this may seem to restrict the usefulness of the linear factor model with normal errors, it should be pointed out that in many actual and potential appications; (i) the data are monthly, so that time aggregation makes the data more normal, (ii) most of the non-normal results are to do with returns, rather than residual returns. It is obvious that the errors could be normal and returns could be (unconditionally) non-normal.

It is further assumed that returns are iid; much has been written on this topic but it is worth noting that we only require our errors to be iid. Thus, factors could provide a variety of time-series behaviour; indeed since they contribute to the risk premia of individual assets, we may well expect tham to be serially correlated.

2.2 Sum of F Statistic for Linear Factor Models

A solution to the problem of the standard procedure based on the Hotelling's statistics is to impose zero restrictions on the off-diaonal elements of the covariance matrix of the errors, $\mathbf{\Sigma}$. The new test statistic based on this assumption is well defined even in situations where $N > T$. In what follows we assume the Wald testing principle.

Note that the standard assumption of linear factor models is that $\mathbf{\Sigma}$ is diagonal, see Ross (1976). Even if $\mathbf{\Sigma}$ is not diagonal, then its off-diagonal elements will be the result of missing factors whose exposure is hopefully small. Imposing a (false) zero restriction may reduce estimation error, especially in the case when N is large. Therefore, under the assumption that the risk premium at time t is described by the estimated value of $\boldsymbol{\beta}\mathbf{r}_t^f$ at time t, the appropriate test statistic, S_2^N, to test the efficiency of such a theory, i.e., $H_0 : \mathbf{a} = 0$, is

$$\begin{aligned} S_2^N &= Tc\widehat{\mathbf{a}}'\mathbf{\Sigma}^{-1}\widehat{\mathbf{a}} \qquad (8) \\ &= Tc\sum_{j=1}^{N}\frac{\widehat{a}_j^2}{\sigma_j^2} \end{aligned}$$

For the unknown true σ_j^2, the new test statistic is still distributed with chi-square with N degrees of freedom as explained in equation (5). Now let us consider the sample estimate $\widehat{\sigma}_j^2$ instead of σ_j^2 in the new test statistic S_2^N. The relationship between the unknown σ_j^2 and the sample estimate, $\widehat{\sigma}_j^2$ is

$$\begin{aligned} \widehat{\sigma}_j^2 &= \frac{1}{T-K-1}\sum_{t=1}^{T}(r_{j,t} - \widehat{a}_j - \widehat{\boldsymbol{\beta}}_j\mathbf{r}_t^f)^2 \qquad (9) \\ &\sim \sigma_j^2\frac{\chi^2(T-K-1)}{T-K-1}. \end{aligned}$$

Therefore, we have

$$\frac{\widehat{\sigma}_j^2}{\sigma_j^2} \sim \frac{\chi^2(T-K-1)}{T-K-1}. \qquad (10)$$

When $\widehat{a}_j^2/\sigma_j^2$ in equation (8) is divided by $\widehat{\sigma}_j^2/\sigma_j^2$ for each j, we have the following test statistic. This is a Wald test in that we directly compute the distribution of the tested restrictions, replacing unknown parameters by maximum likelihood estimates.

Diagonality for the covariance matrix follows from the maintained hypothesis. Thus we get,

$$\begin{aligned} S_2^* &= Tc\sum_{j=1}^{N} \frac{\widehat{a}_j^2}{\sigma_j^2} \Big/ \frac{\widehat{\sigma}_j^2}{\sigma_j^2} \qquad (11) \\ &= Tc\sum_{j=1}^{N} \frac{\widehat{a}_j^2}{\sum_{t=1}^{T}(r_{j,t} - \widehat{a}_j - \widehat{\boldsymbol{\beta}}_j \mathbf{r}_t^f)^2/(T-K-1)} \\ &\sim \sum_{j=1}^{N} F_j(1, T-K-1) \end{aligned}$$

where $F_j(1, T-K-1)$ are independent variable with 1 degree of freedom in the numerator and $(T-K-1)$ degrees of freedom in the denominator.

When T is very large compared with K, each $F_j(1, T-K-1)$ converges in distribution to a chi-square distribution with 1 degree of freedom. Therefore, in this case, S_2^* converges to a chi-square distribution with N degrees of freedom. Furthermore, our test, as a Wald test, will have the usual large T fixed N optimality properties associated with local power.

As suggested by a referee, an alternative to our procedure is to consider the above problem as a Neyman and Scott (1948) problem and use 'new likelihood' methods such as the Cox-Reid conditional likelihood and McCullagh-Tibshirani adjusted score and likelihood to generate alternative estimators and tests; see, Ghosh (1994). We have not explored linkages and comparison between these approaches and our approach.

To obtain a statistic that is asymptotically scaled for large T, we suggest as our new statistic

$$\begin{aligned} S_2 &= S_2^*/N \qquad (12) \\ &\sim \frac{1}{N}\sum_{j=1}^{N} F_j(1, T-K-1). \end{aligned}$$

When T is very large compared with K, S_2 is distributed as $\chi^2(N)/N$. This is the same asymptotic result as the case of the conventional test statistic of equation (7), since in that case we have $\lim_{T\to\infty} F(N, T-N-K) \xrightarrow{d} \chi^2(N)/N$, where d means convergence in distribution. For other relationships for S_2 and between S_1 and S_2, note that $S_1 = S_2$ if $N = 1$. If N gets large we get by the weak law of large numbers that S_2 converges in probability to $E[F(1, T-K-1)]$ which equals to $\frac{T-K-1}{T-K-3}$ for $T-K > 3$. For S_1, if N gets large T needs to get large as well so that $T-K-N$ remains positive. If T and N get large together such that $T-K-N > 0$, we find that $p\lim S_1 = p\lim S_2 = 1$.

The new statistic can be applied to any case when the number of observations, T, is larger than the number of factors plus one, $K+1$. This is a major advantage of our new test statistic since in many cases, as we have discussed, we have the number of assets larger than the number of observations. On the other hand, the conventional F statistic requires $T-K-N>0$, which does not always happen in practice as we explained above.

The probability density function of our new test statistic is discussed and its characteristic function derived; see Appendix. However, a closed form of the pdf of the new test statistic is not suggested. This might be thought of as a disadvantage of the new test statistic. By inspection, its density depends only upon T, N, and K. In the following section, we tabulate the new statistic for various T, N, and K with Monte Carlo simulations.

2.3 Missing Factors and the New Test Statistic

Suppose that the true model is the linear factor model represented in equation (3). Then for any P, $0<P<K$, and $Q=K-P$, we can rewrite the model as

$$\mathbf{r}_t=\mathbf{a}+\boldsymbol{\beta}_1\mathbf{r}_t^{f_1}+\boldsymbol{\beta}_2\mathbf{r}_t^{f_2}+\boldsymbol{\epsilon}_t \tag{13}$$

where $\boldsymbol{\beta}_1$ $(N\times P)$ is the first P columns of $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_2$ $(N\times Q)$ is the last Q columns of $\boldsymbol{\beta}$. In addition, $\mathbf{r}_t^{f_1}$ $(P\times 1$ vector$)$ is the first P rows of $\mathbf{r}_t^f$ and $\mathbf{r}_t^{f_2}$ $(Q\times 1$ vector$)$ is the last Q rows of $\mathbf{r}_t^f$.

Let us assume that we include only the first P factors in the LFM. That is,

$$\mathbf{r}_t=\mathbf{a}_P+\boldsymbol{\beta}_1\mathbf{r}_t^{f_1}+\boldsymbol{\nu}_t \tag{14}$$

where $\mathbf{a}_P=\mathbf{a}+\boldsymbol{\beta}_2\boldsymbol{\mu}^{f_2}$, $\boldsymbol{\nu}_t=\boldsymbol{\epsilon}_t+\boldsymbol{\beta}_2(\mathbf{r}_t^{f_2}-\boldsymbol{\mu}^{f_2})$, and $\boldsymbol{\mu}^{f_2}=E(\mathbf{r}_t^{f_2})$. Then the variance-covariance matrix conditioning on the P factors are

$$\boldsymbol{\Sigma}_P=\boldsymbol{\beta}_2\boldsymbol{\Omega}_Q\boldsymbol{\beta}_2'+\boldsymbol{\Sigma}, \tag{15}$$

where $\boldsymbol{\Omega}_Q$ is the variance-covariance matrix of the Q factor returns. Note that in this discussion, the factor returns are usually restricted to be orthogonal to each other and to have unit variance, that is, $\boldsymbol{\Omega}_K=\mathbf{I}_K$, $\boldsymbol{\Omega}_Q=\mathbf{I}_Q$,and $\boldsymbol{\Omega}_P=\mathbf{I}_P$. Therefore, we have

$$\widehat{\mathbf{a}}_P\tilde{\ }N(\mathbf{a}+\boldsymbol{\beta}_2\boldsymbol{\mu}^{f_2},\frac{1}{T}(1+\widehat{\boldsymbol{\mu}}_P'\widehat{\boldsymbol{\mu}}_P)\boldsymbol{\Sigma}_P) \tag{16}$$

where $\boldsymbol{\Sigma}_P=\boldsymbol{\beta}_2\boldsymbol{\beta}_2'+\boldsymbol{\Sigma}$.

Since we do not know the true factors in the LFM, for the null hypothesis of $H_0:\mathbf{a}_P=0$ against the alternative hypothesis $H_1:\mathbf{a}_P\neq 0$, the usual multivariate F-test statistic is

$$S_1=\frac{T-N-P}{N}[1+\widehat{\boldsymbol{\mu}}_P'\widehat{\boldsymbol{\mu}}_P]^{-1}\widehat{\mathbf{a}}_P'\widehat{\boldsymbol{\Sigma}}_P^{-1}\widehat{\mathbf{a}}_P \tag{17}$$

where

$$\widehat{\boldsymbol{\mu}}_P = \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t^{f_1}$$

$$\widehat{\boldsymbol{\Sigma}}_P = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{r}_t - \widehat{\mathbf{a}}_P - \widehat{\boldsymbol{\beta}}_1\mathbf{r}_t^{f_1})(\mathbf{r}_t - \widehat{\mathbf{a}}_P - \widehat{\boldsymbol{\beta}}_1\mathbf{r}_t^{f_1})'.$$

For our new test statistic, we may not simply assume $\widehat{\mathbf{a}}_P'\boldsymbol{\Sigma}_P^{-1}\widehat{\mathbf{a}}_P = \sum_{j=1}^{N}\frac{\widehat{a}_j^2}{\sigma_j^2}$ as in equation (8), since the variance-covariance matrix of $\boldsymbol{\Sigma}_P$ in equation (16) is no longer a diagonal matrix. However, as is the often the case, the factor loadings, the elements in the vector $\boldsymbol{\beta}_2$, are small and if the number of missing factors, Q, is also small, we can assume $\boldsymbol{\beta}_2\boldsymbol{\beta}_2'$ is negligible. The assumption gives us approximately the same test statistic as follows, i.e.,

$$S_{2,P} \sim \frac{1}{N}\sum_{j=1}^{N}F_j(1, T-P-1). \tag{18}$$

In this case, S_1 is still distributed correctly under H_0 up to a scalar. If the above assumptions are not reasonable, then power calculations can be carried out to evaluate both tests relative performance, see Mackinlay (1987), for an analysis of S_1.

Our arguments in the non-diagonal case can be seen as a procedure of imposing false constraints in a model to improve efficiency or power. Thus we (falsely) assume that the nuisance (off-diagonal) parameters in the covariacne matrix are zero even when they not. To assess the merits of this approach requires a simulation study which we provide in the following section.

3 Simulations for the New Test Statistic

In the previous section, we investigated the new test statistic, S_2, when all the population covariances were zero in equation (2). It seems difficult to get an analytic derivation of $pdf(y)$ and we do not suggest a closed form solution for the new statistics. Note that an F test based on Hotelling's T^2 statistics has a finite sample distribution that is known in closed form and tabulated.

In this section, we calculate $pdf(y)$ via simulations. Since the closed form solution is not known, we carry out extensive simulations to tabulate the statistics; simulations for test statistics and power tests for given critical values. All the simulation results reported in this study are obtained with 10,000 replications.

3.1 Simulated Tabulations of the New Test Statistic

Our first simulation procedure for the calculation of test statistics for given critical values is as follows. We first generate N variables each of which has $T-K$ observations from the $N(0,1)$ distribution, which is denoted as variable z. Then for each variable, the square of the last observation is divided by the sum of the square of the first $T-K-1$ observations. More formally, we generate the j^{th} $F(1,T-K-1)$ variable x_j

$$x_j = \frac{z_{T-K,j}^2}{\sum_{m=1}^{T-K-1} z_{m,j}^2/(T-K-1)} \tag{19}$$

where $z\tilde{}N(0,1)$. We use the same method for all N variables. Then, the N variables are averaged. We repeat this procedure 10,000 times to obtain 10,000 statistics and report the result together with the conventional F distribution of S_1 with $T-K-N$ degrees of freedom in the numerator and N degrees of freedom in the denominator.

We use the number of observations, $T=60$ and 120, and $T=600$ to investigate the properties of the new statistics when T become very large. The number of assets, N, varies from 10 to $1,000$ in our settings. As explained before, when $T-N-K \leq 0$, ordinary F statistics are not available, whilst the new statistic is available. We report the new statistics for large N such as 500 and 1,000 to see how the new statistics behave. Finally, the number of factors is set to 1, 2, 3, 4, 5, and 10.[2]We believe these values of the number of factors cover most cases in the tests of linear factor models. See section 6.4 of CLM for a summary on this topic.

Table 1 reports the simulation results for seven cases. All the results in table 1 are supported by the analytical discussion in the previous section. First, the results show that both the new and conventional F statistics approach $\chi^2(N)/N$ when T becomes large. For example, when $T=600$ (panel G), the conventional F statistics and the new statistics become close to each other; compare the results in panel G with those of panels A and B. Secondly, when N is large, by the weak law of large numbers, the new statistic approaches $\frac{T-K-1}{T-K-3}$ for $T-K>3$. The last row of each panel shows values of $\frac{T-K-1}{T-K-3}$ for given T and K. Table 1 also reports critical values when $N>T$, which cannot be obtained with the conventional F statistic. This is one of the most useful advantage of the new statistic.

3.2 The Power of the New Tests

In this subsection, we represent three simulations for the power of the new tests and compare the results with those of the conventional F tests. The power of a test is the probability that the null hypothesis is rejected when an alternative hypothesis is true, and is a useful tool to discriminate between different test statistics. The choice of parameter values for the power calculations depend upon ones view about what are interesting alternatives. In this study, we use the non-centrality parameter values in chapter 5 of CLM, and real data for the missing factors in the previous section, details of which are explained below.

As explained above, the distribution of the conventional S_1 for the alternative hypothesis is

$$S_1(\delta_1)\tilde{}F(N,T-N-K;\delta_1) \tag{20}$$

where δ_1 is the non-centrality parameter of the F distribution which is defined as

$$\delta_1 = T[1 + \widehat{\boldsymbol{\mu}}_k' \widehat{\boldsymbol{\Omega}}_k^{-1} \widehat{\boldsymbol{\mu}}_k]^{-1} \mathbf{a}' \boldsymbol{\Sigma}^{-1} \mathbf{a} \tag{21}$$

see MacKinlay (1987) for a detailed explanation. On the other hand, the distribution of S_2 for the alternative hypothesis is

$$S_2(\boldsymbol{\delta}_2)\tilde{\ } \frac{1}{N} \sum_{j=1}^{N} F_j(1, T-K-1; \delta_{2,j}) \tag{22}$$

where $\boldsymbol{\delta}_2 = (\delta_{2,1}, \delta_{2,2}, ...\delta_{2,N})$ is the vector of the non-centrality parameters of the new test statistic distribution. The $\delta_{2,j}$ are defined by

$$\delta_{2,j} = T[1 + \widehat{\boldsymbol{\mu}}_k' \widehat{\boldsymbol{\Omega}}_k^{-1} \widehat{\boldsymbol{\mu}}_k]^{-1} \frac{a_j^2}{\sigma_j^2} \tag{23}$$

Here we only consider the case of $K = 1$ for the market portfolio, but similar results can be obtained for $K > 1$. In the $K = 1$ case for the alternative hypothesis, the distribution of the conventional $S_1(\delta_1)$ is $F(N, T-N-1; \delta_1)$ where $\delta_1 = T[1 + \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}]^{-1} \mathbf{a}'\boldsymbol{\Sigma}^{-1}\mathbf{a} = T\left[1 + \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}\right]^{-1} \left[\frac{\widehat{\mu}_p^2}{\widehat{\sigma}_p^2} - \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}\right]$, where p represents the tangency portfolio (the mean-variance efficient portfolio); see Gibbons, Ross, and Shanken (1989) for the derivation of this equation. In addition, the distribution of S_2 for the alternative hypothesis is $S_2(\boldsymbol{\delta}_2)\tilde{\ } \frac{1}{N} \sum_{j=1}^{N} F_j(1, T-2; \delta_{2,j})$ where $\delta_{2,j} = T[1 + \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}]^{-1} \frac{a_j^2}{\sigma_j^2}$.

Let us first consider the case of $H_0^1 : a_j = 0$, $j = 1, ..., N$ against $H_1^1 : a_j = \mu$, for $j = 1$, and $a_j = 0$, for $j = 2, ..., N$. This is the case when one asset, asset 1, of the N has a non-zero alpha under H_1^1. We first generate a chi-square variable $\chi^2(1)$ from the standard normal variable $z\tilde{\ }N(0,1)$. Then the non-central variable $x_1^*(\delta_1)$ for the alternative hypothesis of $S_1^*(\delta_1)$ is generated with

$$x_1^*(\delta_1) = \frac{\left[\chi^2(1;\delta_1) + \sum_{i=2}^{N} \chi_i^2(1)\right]/N}{\sum_{j=1}^{T-N-1} \chi_j^2(1)/(T-N-1)} \tag{24}$$

where δ_1 is the non-centrality parameter of the first chi-square variable which can be represented as $\delta_1 = T[1 + \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}]^{-1} \frac{a_1^2}{\sigma_1^2}$. On the other hand, the non-central variable $x_2^*(\delta_2)$ for the alternative hypothesis of $S_2^*(\boldsymbol{\delta}_2)$ is generated with

$$x_2^*(\boldsymbol{\delta}_2) = \frac{1}{N}\left[\frac{\chi^2(1;\delta_{2,1})}{\sum_{m=1}^{T-2} \chi_{m,1}^2(1)/(T-2)} + \sum_{j=2}^{N} \frac{\chi_j^2(1)}{\sum_{m=1}^{T-2} \chi_{m,j}^2(1)/(T-2)}\right] \tag{25}$$

where $\delta_{2,1} = T[1 + \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}]^{-1} \frac{a_1^2}{\sigma_1^2}$. For the value of $\frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}$, we use 0.013 as in chapter 5 of CLM, which is equivalent to the market portfolio with the annualised expected excess return of 8% and the annualised standard deviation of 20%. In addition, the values for $\frac{a_1^2}{\sigma_1^2}$ are set to the values of 0, 0.004, 0.008, ..., 0.396. The power of our test statistic, $S_2^*(\boldsymbol{\delta}_2)$, is calculated for critical values of 1%, 2.5%, 5%, and 10%. That is, numbers of variables larger than the test statistics obtained with the critical values of 0.5%, 1%, 2.5%, 5%, and 10% in $S_2^*(\boldsymbol{\delta}_2)$ are calculated. The power of $S_1^*(\delta_1)$ is calculated using the same method. We use the number of observations, $T = 120$ and 60. The number of assets, N, varies from 10 to 110 for $T = 120$ and from 10 to 50 for $T = 60$. The number of factors is set to 1 and 5.

Figure 1 represents the power function of the new and conventional tests at 1%, 5%, and 10%, for case 1 ($T = 120$, $N = 30$, and $K = 1$), and case 2 ($T = 120$, $N = 90$, and $K = 1$).[3] The figures show that the new test has more power than the conventional F test. Especially, note that when N is close to T, the power of the conventional F test is very low, while the new test has more power.

Figure 1 is based on the assumption that only one asset, a_1, is different from zero. The next simulations consider the case that $H_0^N : a_j = 0,\ j = 1, ..., N$ against $H_1^N : a_j = \mu,\ j = 1, ..., N$. That is, all assets have a common value different from zero in the alternative hypothesis. Since under H_0, the distribution of S_2 does not depend upon Σ, it is convenient, under H_1^N, in what follows to set $\Sigma = \mathbf{I}_N \sigma_p^2$ and set $a_j = \frac{\mu}{\sigma_p}\ \forall j$. The results for the conventional F test are reported in page 207, chapter 5, CLM. Here we use exactly the same method as CLM. The non-centrality parameter, δ, is calculated with

$$\delta = T[1 + \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}]^{-1}(\frac{\mu_p^2}{\sigma_p^2} - \frac{\mu_m^2}{\sigma_m^2}) \tag{26}$$

where p represents the tangency portfolio and m is the market portfolio. CLM set the values of $[1 + \frac{\widehat{\mu}_m^2}{\widehat{\sigma}_m^2}]^{-1}(\frac{\mu_p^2}{\sigma_p^2} - \frac{\mu_m^2}{\sigma_m^2})$ to 0.01, 0.02, 0.03, 0.04. These numbers are obtained by assuming that the tangency portfolios have the annualised expected excess return of 8.5%, 10.2%, 11.6%, 13.0% and the annualised standard deviation of 16% for all four tangency portfolios, and that the market portfolio has the annualised expected excess return of 8% and the annualised standard deviation of 20%. The non-centrality parameter, δ, can be used to generate the non-central variable $x_1(\delta)$ for the $F(N, T - N - 1; \delta)$ as follows;

$$x_1(\delta) = \frac{\chi^2(N; \delta)/N}{\chi^2(T - N - 1)/(T - N - 1)} \tag{27}$$

where $\chi^2(N; \delta) = \sum y_j^2$, and $y\tilde{\ }N(\sqrt{\delta/N}, 1)$. Therefore, $\mu = \sqrt{\delta/N}$. On the other hand, the non-central variable $x_2(\delta)$ for the new test statistic under the alternative hypothesis is

$$x_2(\delta) = \frac{1}{N}\sum_{j=1}^{N} \frac{\chi_j^2(1; \delta/N)}{\chi^2(T - 2)/(T - 2)} \tag{28}$$

where $\chi^2_j(1;\delta/N) = y^2_j$, and $y\tilde{\ } N(\sqrt{\delta/N}, 1)$.

The simulation results are reported in table 2. The numbers in the conventional F test power are not exactly the same as those in the table 5.2 of CLM, but the difference is marginal and reflects the differences in parameter information required to specify the distribution of S_2 under H_1^N. The table shows that in general the new test has more power than the conventional F test. As expected, when $N = 1$, the results are exactly the same. However, when the number of assets increases, the new test begins to perform relatively better. Interestingly and as hoped, the table shows that the relative power of the new test against the F test is also an increasing function of the number of observations, T.

Finally, we test the power of the new test when there are missing factors as explained in the previous section. We use three factors as explanatory variables; excess market returns and two factor mimicking portfolio (FMP) returns. Factor portfolios may be portfolios of equities corresponding to attractive equity characteristics such as size or growth. The factor portfolios may be actual indices such as the S&P500, FTSE100 (for size) or artificially constructed portfolios long in high growth equities and short in low growth equities (for growth).

The data we use for this power test are the log-returns of the S&P500 index and its individual equities, 3 month US treasury bill, and FMP returns for growth and size. Our choice of factors is intentional, in that a great deal of research on the US stock market has compared the rival merits of these 3 factors, i.e., market, growth, and size. See Capaul, Rowley and Sharpe (1993), Arshanapalli, Coggin and Doukas (1998) Fama and French (1998), and Hall, Hwang, and Satchell (2000) for some recent studies on this topic. Thus it is likely that including a subset of the factors in regression, and by default relegating the other factors to the covariance matrix may have parallels in recent empirical work.

A total number of 120 monthly returns from April 1989 to March 1999 is used. Excess individual equity returns and market returns are calculated by taking the 3 month US treasury bill from individual equity returns and market returns, respectively. For the two factors, we use FMPs for growth and size from Hall, Hwang, and Satchell (2000).[4]Note that all three factor returns are designed to have zero expected returns to prevent the power test results from being affected by missing factors that have negative or positive expected returns. Thus, in these simulations, $\widehat{\boldsymbol{\mu}}_k'\widehat{\boldsymbol{\Omega}}_k^{-1}\widehat{\boldsymbol{\mu}}_k = 0$ as in section 6.6 of CLM and the noncentrality parameter becomes $\delta_M = T\mathbf{a}'\boldsymbol{\Sigma}^{-1}\mathbf{a}$.

Some of the statistical properties of excess market returns and FMP returns are reported in table 3. All of the factor returns have different standard deviations; the standard deviation of the excess market returns are nearly four time larger than that of the growth FMP returns. Correlations between the two FMPs are not significant, but the excess market returns and the growth FMP returns are significantly correlated. Therefore, by missing one or two factors, we can test the power of our new test statistic that is based on the assumption of diagonality of variance-covariance matrix of $\mathbf{\Sigma}_P$ in equation (16).

The simulations are designed as follows. We first randomly select N equities from the 500 equities included in the S&P500 index and then estimate parameters in the following linear factor model;

$$\mathbf{r}_t = \mathbf{a} + \boldsymbol{\beta}_1 r_t^M + \boldsymbol{\beta}_2 r_t^G + \boldsymbol{\beta}_3 r_t^S + \boldsymbol{\epsilon}_t, \tag{29}$$

where $\mathbf{r}_t$ is a $(N \times 1)$ vector of excess returns for N assets, and r_t^M , r_t^G, r_t^S are factor returns, i.e., excess market returns, growth FMP returns, and size FMP returns, and $\boldsymbol{\epsilon}_t$ is a $(N \times 1)$ vector of disturbance terms whose variance-covariance matrix is $E(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \mathbf{\Sigma}$. The mean and standard deviation of the estimates of the coefficients are reposted in table 3. As expected, the coefficients on the excess market returns are close to one and significant, whilst the other two FMP returns are close to zero and not significant.

Then a new set of individual equity returns, $\widetilde{\mathbf{r}}_t$, are generated with the factor returns and the estimates as follows;

$$\widetilde{\mathbf{r}}_t = \widetilde{\mathbf{a}} + \widehat{\boldsymbol{\beta}}_1 r_t^M + \widehat{\boldsymbol{\beta}}_2 r_t^G + \widehat{\boldsymbol{\beta}}_3 r_t^S + \widetilde{\boldsymbol{\epsilon}}_t, \tag{30}$$

where $\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_3$ are maximum likelihood estimates, and $\widetilde{\boldsymbol{\epsilon}}_t \,\tilde{}\, N\left(\left(\begin{array}{c} 0 \\ 0 \\ . \\ 0 \end{array}\right), \left(\begin{array}{cccc} \widehat{\sigma}_1^2 & 0 & . & 0 \\ 0 & \widehat{\sigma}_2^2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \widehat{\sigma}_N^2 \end{array}\right)\right)$

and $\widehat{\sigma}_i^2$ is the estimated variance of disturbance term for equity i, i.e., $\widehat{\sigma}_i^2 = \frac{1}{T}\sum_{t=1}^{T} \epsilon_{it}^2$. By making the variance-covariance matrix diagonal, we assume a setting that there are only three common factors for the explanation of equity returns.

In these simulations, we consider the case that $H_0^M : a_j = 0,\ j = 1, ..., N$ against $H_1^M : a_j = \mu,\ j = 1, ..., N$, where μ is set to 0, 0.083, 0.208, 0.417, 0.625, 0.833, which are 0, 1%, 2.5%, 5%, 7.5%, and 10% in annual term, respectively. For T and N, we use T=60, 120, and N=10, 50, 110. We use equations (7) and (12) for the conventional F test statistic and the new test statistic, respectively. The noncentrality parameter value $\delta_M = T\mathbf{a}'\mathbf{\Sigma}^{-1}\mathbf{a}$ depend on T, $\widetilde{\mathbf{a}}(\mu)$, $\mathbf{\Sigma}$. For the given values of μ and T, the parameter values depend on $\mathbf{\Sigma}$ that changes on the missing factors. See equation (15). We report the noncentrality parameter values in table 4 for each cases.

Table 4 reports the simulated power of the new and conventional tests for five sets of T and N. Our results report two different tests; the power of the tests for increasing noncentrality parameter values and mis-specified factor models. In general, our new test has more power in most cases. Especially, the table shows that as N increases, the power of the conventional F test tends to decrease, whilst that of the new test tend to increase. Similar to the previous two power simulations, the new test has more power as N increases. In addition, our new test provides the results of the power test for the case of T=60 and N=110, which cannot be obtained with the conventional F test. Panel E shows that the power of the new test increases as N increases.

Both the conventional F test and the new test are poor when an important factor is missed (i.e., misspecified models); see the cases of the excess market factor being missed, the excess market and size FMP being missed, and the excess market and the growth FMP being missed. The performance of the new test seems to be relatively better than that of the conventional F test. However, in these cases, both tests tend to accept the null hypothesis too frequently even though the noncentality parameter is zero. The tendency is relatively large for the new test, whilst the conventional F test statistic suffers less significantly from the important missing factor.

Suppose that a factor is missed. Then our simulation imply that $\widehat{\widehat{\mathbf{a}}}\tilde{\ } N(\mathbf{0}, \frac{1}{T}(\sigma_f^2 \widehat{\boldsymbol{\beta}}_f \widehat{\boldsymbol{\beta}}_f' + \widetilde{\boldsymbol{\Sigma}}))$, where σ_f^2 is the variance of the factor returns, $\widehat{\boldsymbol{\beta}}_f$ is the $(N \times 1)$ vector of coefficients on the factor returns as in (30), and $\widetilde{\boldsymbol{\Sigma}}$ is the variance-covariance matrix of $\widetilde{\boldsymbol{\epsilon}}_t$. When $\widehat{\boldsymbol{\beta}}_f$ such as $\widehat{\boldsymbol{\beta}}_2$ and $\widehat{\boldsymbol{\beta}}_3$ in (30) is very small, then the power of the new test is not affected significantly. However, when most of the elements of $\widehat{\boldsymbol{\beta}}_f$ such as $\widehat{\boldsymbol{\beta}}_1$ in (30) are positive and large (see table 3 for the mean and standard deviation of the coefficients), then the variance-covariance matrix of $\widehat{\widehat{\mathbf{a}}}$ is not approximated by a diagonal matrix. In this case, if we do not consider the off-diaogonal elements, then we have smaller values of test statistics, since the large positive off-diagonal elements of $\widehat{\boldsymbol{\beta}}_f \widehat{\boldsymbol{\beta}}_f'$ is disregarded. This is the reason why we have zero power (accept too frequently) for the test levels we used for the null hypothesis. See the first column of the power of the new test in table 4.

Except in the case that important factors in the linear factor model are missed, then the new test performs better than the conventional F test. Our simulations in table 4 show that missing two important factors in equity markets such as size and growth do not degrade the power of the new test found in table 2 and figure 1, suggesting some robustness against non-zero off-diagonal elements. Even when important factors are missed, as the value of the noncentrality parameter increases, the new test become powerful. Therefore, the results in table 4 suggest that if we do not miss the most important factor such as the market, our test is more powerful than the conventional F test.

4 Conclusion

The test procedure we outline is simple to calculate, and simple to provide critical values for power analysis, although specifying parameters under the alternative is slightly more complicated than the conventional model. Its distribution, the average value of F statistics, should be intrinsically interesting to statisticians and the extra power it provides for practitioners should be welcomed.

Footnotes

1. This is the first case of section 6.2, CLM. For a detailed discussion on the estimation and testing of the linear factor model, refer to chapter 6, CLM.
2. The new statistics for other values of the number of factors such as 2 and 4 can be obtained from authors upon request.
3. Only these two cases are reported in this paper. The results for other cases can be obtained from the authors by request.
4. See Hall, Hwang, and Satchell (2000) for the detailed explanation on the construction of factor mimicking portfolios. We do not pursue this discussion further, since our purpose in this study is to investigate the power of the conventional F test and the new test.

Appendix Characteristic Function of the Sum of F-Distribution

The distribution function of the sum of F-distributions is not known. Note that all F-distributions in equation (11) have the same degrees of freedom and S_2 is distributed as the mean of N independent and identically distributed F-distributions. Let x be a variable distributed as $F(1,n)$, where $n \equiv T-K-1$ and denote its probability density function as $pdf(x)$. Then the characteristic function of the $F(1,n)$ distribution can be derived as follows

$$\begin{aligned}\phi(t,n) &= \int_0^\infty pdf(x)e^{itx}dx \\ &= \int_0^\infty \frac{\Gamma(\frac{1}{2}+\frac{1}{2}n)n^{n/2}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}n)} \frac{x^{-1/2}}{(n+x)^{\frac{1+n}{2}}} e^{itx}dx.\end{aligned} \tag{31}$$

Let $y = x/n$ and $ndy = dx$, then

$$\begin{aligned}\phi(t,n) &= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}n)} \int_0^\infty (1+y)^{-\frac{1+n}{2}} y^{-\frac{1}{2}} e^{itny}dy \\ &= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)} \Psi(\frac{1}{2}, 1-\frac{1}{2}n; -nit)\end{aligned} \tag{32}$$

where $\Gamma(\cdot)$ is the gamma function, i is the imaginary number, and $\Psi(\cdot)$ is Tricomi's confluent hypergeometric function. Equation (32) was first shown by Phillips (1982). Tricomi's confluent hypergeometric function is

$$\begin{aligned}\Psi(\frac{1}{2}, 1-\frac{1}{2}n; -nit) &= \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}+\frac{1}{2}n)} \, {}_1F_1(\frac{1}{2}, 1-\frac{1}{2}n; -nit) + \\ &\quad \frac{\Gamma(-\frac{1}{2}n)}{\Gamma(\frac{1}{2})}(-nit)^{n/2} \, {}_1F_1(\frac{1}{2}+\frac{1}{2}n, 1+\frac{1}{2}n; -nit),\end{aligned} \tag{33}$$

where ${}_1F_1(\cdot)$ is Kummer's confluent hypergeometric function which is defined as

$${}_1F_1(a,b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + ... \tag{34}$$

See Abadir (1999) for a detailed explanation on various kinds of hypergeometric functions and their applications to economic theory.

If b in equation (34) is a nonpositive integer, ${}_1F_1(a,b;z)$ and thus $\Psi(\frac{1}{2}, 1-\frac{1}{2}n; -nit)$ is not defined. Note that n is a positive integer since it represents degrees of freedom in the denominator of the $F(1,n)$ distribution and thus we need $1-\frac{1}{2}n$ and $1+\frac{1}{2}n$ in equation (33) to be positive integers. However, since n is a positive integer, both $1-\frac{1}{2}n$ and $1+\frac{1}{2}n$ cannot be kept to be positive integers. More generally, when $\frac{n}{2} \in \mathcal{Z}$, we have a definition called the 'logarithmic case' alternative to Tricomi's confluent hypergeometric function in (33). See Abadir (1999) and Erdélyi (1953, vol.1 pp.260-262 and vol.2 p.9) for the discussion on the logarithmic case.

Let $\phi_j(t,n)$ be defined as a characteristic function of the j^{th} independent $F(1,n)$ variable. Then, the characteristic function of S_2 is

$$\begin{aligned} \Phi(t,n) &= \prod_{j=1}^{N} \phi_j(\frac{t}{N}, n) \\ &= \left[\phi(\frac{t}{N}, n)\right]^N \end{aligned} \tag{35}$$

where $\phi(\cdot)$ is defined in (32). Therefore, the density function of our new test statistic S_2, $pdf(y)$, under the null hypothesis is obtained by

$$pdf(y) = \frac{1}{(2\pi)} \int \left[\phi(\frac{t}{N}, n)\right]^N e^{-ity} dt \tag{36}$$

where y is a variable distributed as the average of the N different $F(1,n)$ distributions. This sum of F-distributions can be used when the variance-covariance matrix $\mathbf{\Sigma}$ is a diagonal matrix.

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Table 1 Comparison of New Test Statistics and F-Statistics

A. Number of Observations T=60, Number of Factors K=1

Number of Assets (N)	Test Statistics	Percentile of Test Statistics				
		0.5%	1.0%	2.5%	5.0%	10.0%
N=10	New Test Statistics	2.719	2.508	2.186	1.931	1.674
	F(10,49) N=10, T-N-K=49	2.985	2.728	2.341	2.033	1.737
N=20	New Test Statistics	2.109	1.987	1.802	1.660	1.487
	F(20,39) N=20, T-N-K=39	2.573	2.393	2.092	1.860	1.614
N=30	New Test Statistics	1.868	1.787	1.643	1.528	1.408
	F(30,29) N=30, T-N-K=29	2.614	2.424	2.109	1.869	1.620
N=40	New Test Statistics	1.733	1.651	1.542	1.448	1.342
	F(40,19) N=40, T-N-K=19	3.044	2.764	2.346	2.033	1.729
N=50	New Test Statistics	1.674	1.607	1.497	1.414	1.321
	F(50,9) N=50, T-N-K=9	5.261	4.424	3.453	2.766	2.201
N=60	New Test Statistics	1.623	1.549	1.455	1.375	1.297
N=70	New Test Statistics	1.591	1.529	1.427	1.361	1.275
N=80	New Test Statistics	1.531	1.479	1.387	1.321	1.246
N=90	New Test Statistics	1.488	1.440	1.374	1.307	1.243
N=100	New Test Statistics	1.477	1.422	1.356	1.300	1.234
N=110	New Test Statistics	1.458	1.407	1.339	1.282	1.225
N=200	New Test Statistics	1.342	1.305	1.257	1.218	1.174
N=500	New Test Statistics	1.218	1.203	1.173	1.150	1.123
N=1000	New Test Statistics	1.167	1.150	1.132	1.115	1.097
N=+infinite	New Test Statistics	1.036	1.036	1.036	1.036	1.036

Notes: For the new test statistics, see equation (12). The new statistics are the average value of N different $F(1, T\text{-}K\text{-}1)$ distributions. The statistics reported in the table are the results of simulations with 10,000 replications. Detail simulation procedures are explained in section 3.1.

The standard errors of the above simulated statistics are 0.001, 0.001, 0.002, 0.002, and 0.003 for the critical values of 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% respectively.

B. Number of Observations T=60, Number of Factors K=3

Number of Assets (N)	Test Statistics	Percentile of Test Statistics				
		0.5%	1.0%	2.5%	5.0%	10.0%
N=10	New Test Statistics	2.701	2.444	2.155	1.919	1.681
	F(10,45) *N=10, T-N-K=45*	3.007	2.746	2.350	2.040	1.743
N=20	New Test Statistics	2.131	1.973	1.783	1.649	1.484
	F(20,35) *N=20, T-N-K=35*	2.608	2.423	2.113	1.870	1.624
N=30	New Test Statistics	1.893	1.796	1.657	1.534	1.404
	F(30,25) *N=30, T-N-K=25*	2.681	2.484	2.152	1.900	1.642
N=40	New Test Statistics	1.755	1.670	1.553	1.452	1.351
	F(40,15) *N=40, T-N-K=15*	3.275	2.946	2.465	2.119	1.782
N=50	New Test Statistics	1.657	1.611	1.496	1.408	1.316
	F(50,5) *N=50, T-N-K=5*	7.042	5.776	4.280	3.284	2.508
N=60	New Test Statistics	1.610	1.552	1.460	1.386	1.293
N=70	New Test Statistics	1.578	1.512	1.425	1.352	1.269
N=80	New Test Statistics	1.536	1.481	1.397	1.332	1.264
N=90	New Test Statistics	1.496	1.450	1.376	1.315	1.249
N=100	New Test Statistics	1.483	1.428	1.350	1.293	1.237
N=110	New Test Statistics	1.449	1.400	1.341	1.286	1.225
N=200	New Test Statistics	1.327	1.300	1.255	1.217	1.175
N=500	New Test Statistics	1.219	1.200	1.172	1.150	1.124
N=1000	New Test Statistics	1.162	1.151	1.133	1.117	1.100
N=+infinite	New Test Statistics	1.037	1.037	1.037	1.037	1.037

Notes: For the new test statistics, see equation (12). The new statistics are the average value of N different $F(1, T\text{-}K\text{-}1)$ distributions. The statistics reported in the table are the results of simulations with 10,000 replications. Detail simulation procedures are explained in section 3.1.

The standard errors of the above simulated statistics are 0.001, 0.001, 0.002, 0.002, and 0.003 for the critical values of 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% respectively.

C. Number of Observations T=60, Number of Factors K=5

Number of Assets (*N*)	Test Statistics	Percentile of Test Statistics				
		0.5%	1.0%	2.5%	5.0%	10.0%
N =10	New Test Statistics	2.770	2.539	2.170	1.924	1.674
	F(10,45) *N=10, T-N-K=45*	3.032	2.773	2.368	2.053	1.750
N =20	New Test Statistics	2.053	1.929	1.780	1.645	1.485
	F(20,35) *N=20, T-N-K=35*	2.649	2.461	2.139	1.890	1.636
N =30	New Test Statistics	1.916	1.805	1.649	1.532	1.402
	F(30,25) *N=30, T-N-K=25*	2.747	2.547	2.201	1.932	1.664
N =40	New Test Statistics	1.766	1.666	1.560	1.464	1.353
	F(40,15) *N=40, T-N-K=15*	3.543	3.161	2.610	2.222	1.847
N =50	New Test Statistics	1.662	1.602	1.494	1.417	1.314
	F(50,5) *N=50, T-N-K=5*	11.896	9.265	6.272	4.420	3.135
N =60	New Test Statistics	1.590	1.541	1.460	1.387	1.299
N =70	New Test Statistics	1.561	1.503	1.422	1.350	1.274
N =80	New Test Statistics	1.509	1.460	1.386	1.329	1.255
N =90	New Test Statistics	1.493	1.436	1.367	1.307	1.237
N =100	New Test Statistics	1.473	1.423	1.354	1.303	1.240
N =110	New Test Statistics	1.430	1.393	1.344	1.287	1.226
N =200	New Test Statistics	1.336	1.295	1.254	1.222	1.179
N =500	New Test Statistics	1.220	1.204	1.176	1.154	1.128
N =1000	New Test Statistics	1.162	1.152	1.134	1.116	1.099
N =+infinite	New Test Statistics	1.038	1.038	1.038	1.038	1.038

Notes: For the new test statistics, see equation (12). The new statistics are the average value of N different $F(1, T\text{-}K\text{-}1)$ distributions. The statistics reported in the table are the results of simulations with 10,000 replications. Detail simulation procedures are explained in section 3.1.

The standard errors of the above simulated statistics are 0.001, 0.001, 0.002, 0.002, and 0.003 for the critical values of 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% respectively.

D. Number of Observations T=120, Number of Factors K=1

Number of Assets (N)	Test Statistics	Percentile of Test Statistics				
		0.5%	1.0%	2.5%	5.0%	10.0%
N=10	New Test Statistics	2.638	2.422	2.123	1.886	1.642
	F(10,109) N=10, T-N-K=109	2.682	2.499	2.184	1.934	1.663
N=20	New Test Statistics	2.105	1.942	1.748	1.600	1.452
	F(20,99) N=20, T-N-K=99	2.201	2.070	1.858	1.682	1.496
N=30	New Test Statistics	1.825	1.728	1.581	1.481	1.358
	F(30,89) N=30, T-N-K=89	2.015	1.906	1.722	1.575	1.421
N=40	New Test Statistics	1.681	1.614	1.511	1.419	1.319
	F(40,79) N=40, T-N-K=79	1.959	1.853	1.671	1.538	1.397
N=50	New Test Statistics	1.641	1.571	1.461	1.386	1.287
	F(50,69) N=50, T-N-K=69	1.950	1.839	1.656	1.527	1.388
N=60	New Test Statistics	1.587	1.509	1.412	1.349	1.263
	F(60,59) N=60, T-N-K=59	1.963	1.843	1.673	1.540	1.397
N=70	New Test Statistics	1.512	1.471	1.392	1.320	1.241
	F(70,49) N=70, T-N-K=49	2.024	1.893	1.709	1.571	1.424
N=80	New Test Statistics	1.489	1.439	1.366	1.300	1.233
	F(80,39) N=80, T-N-K=39	2.141	1.985	1.777	1.621	1.459
N=90	New Test Statistics	1.461	1.416	1.340	1.280	1.217
	F(90,29) N=90, T-N-K=29	2.389	2.176	1.914	1.730	1.526
N=100	New Test Statistics	1.425	1.387	1.322	1.268	1.207
	F(100,19) N=100, T-N-K=19	2.998	2.644	2.226	1.957	1.685
N=110	New Test Statistics	1.400	1.366	1.306	1.252	1.196
	F(110,9) N=110, T-N-K=9	5.666	4.531	3.418	2.797	2.207
N=200	New Test Statistics	1.294	1.270	1.230	1.192	1.151
N=500	New Test Statistics	1.192	1.177	1.150	1.128	1.104
N=1000	New Test Statistics	1.137	1.125	1.107	1.093	1.074
N=+infinite	New Test Statistics	1.017	1.017	1.017	1.017	1.017

Notes: For the new test statistics, see equation (12). The new statistics are the average value of N different $F(1, T\text{-}K\text{-}1)$ distributions. The statistics reported in the table are the results of simulations with 10,000 replications. Detail simulation procedures are explained in section 3.1.

The standard errors of the above simulated statistics are 0.001, 0.001, 0.002, 0.002, and 0.003 for the critical values of 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% respectively.

E. Number of Observations T=120, Number of Factors K=3

Number of Assets (N)	Test Statistics	Percentile of Test Statistics				
		0.5%	1.0%	2.5%	5.0%	10.0%
N=10	New Test Statistics	2.528	2.342	2.083	1.860	1.634
	F(10,107) *N=10, T-N-K=107*	2.686	2.502	2.187	1.933	1.663
N=20	New Test Statistics	2.067	1.934	1.743	1.611	1.449
	F(20,97) *N=20, T-N-K=97*	2.209	2.078	1.861	1.684	1.496
N=30	New Test Statistics	1.876	1.750	1.595	1.496	1.378
	F(30,87) *N=30, T-N-K=87*	2.021	1.911	1.728	1.579	1.425
N=40	New Test Statistics	1.696	1.617	1.511	1.418	1.319
	F(40,77) *N=40, T-N-K=77*	1.964	1.857	1.676	1.544	1.401
N=50	New Test Statistics	1.628	1.564	1.466	1.384	1.297
	F(50,67) *N=50, T-N-K=67*	1.965	1.849	1.667	1.532	1.393
N=60	New Test Statistics	1.568	1.516	1.422	1.343	1.261
	F(60,57) *N=60, T-N-K=57*	1.982	1.857	1.683	1.547	1.404
N=70	New Test Statistics	1.526	1.470	1.386	1.322	1.245
	F(70,47) *N=70, T-N-K=47*	2.048	1.913	1.722	1.584	1.434
N=80	New Test Statistics	1.487	1.432	1.357	1.299	1.230
	F(80,37) *N=80, T-N-K=37*	2.192	2.021	1.803	1.641	1.473
N=90	New Test Statistics	1.463	1.412	1.339	1.283	1.216
	F(90,27) *N=90, T-N-K=27*	2.468	2.231	1.957	1.762	1.550
N=100	New Test Statistics	1.439	1.395	1.327	1.271	1.207
	F(100,17) *N=100, T-N-K=17*	3.244	2.815	2.335	2.038	1.739
N=110	New Test Statistics	1.411	1.361	1.303	1.253	1.200
	F(110,7) *N=110, T-N-K=7*	8.013	5.976	4.224	3.322	2.510
N=200	New Test Statistics	1.297	1.267	1.229	1.193	1.153
N=500	New Test Statistics	1.191	1.177	1.151	1.126	1.101
N=1000	New Test Statistics	1.140	1.129	1.110	1.094	1.077
N=+infinite	New Test Statistics	1.018	1.018	1.018	1.018	1.018

Notes: For the new test statistics, see equation (12). The new statistics are the average value of N different $F(1, T\text{-}K\text{-}1)$ distributions. The statistics reported in the table are the results of simulations with 10,000 replications. Detail simulation procedures are explained in section 3.1.
The standard errors of the above simulated statistics are 0.001, 0.001, 0.002, 0.002, and 0.003 for the critical values of 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% respectively.

F. Number of Observations T=120, Number of Factors K=5

Number of Assets (N)	Test Statistics		Percentile of Test Statistics				
			0.5%	1.0%	2.5%	5.0%	10.0%
N=10	New Test Statistics		2.609	2.363	2.126	1.890	1.636
	F(10,105)	*N=10, T-N-K=105*	2.690	2.506	2.189	1.934	1.665
N=20	New Test Statistics		2.067	1.932	1.749	1.612	1.452
	F(20,95)	*N=20, T-N-K=95*	2.214	2.085	1.866	1.690	1.499
N=30	New Test Statistics		1.855	1.754	1.608	1.501	1.378
	F(30,85)	*N=30, T-N-K=85*	2.017	1.912	1.730	1.580	1.429
N=40	New Test Statistics		1.704	1.608	1.507	1.416	1.319
	F(40,75)	*N=40, T-N-K=75*	1.970	1.853	1.675	1.545	1.401
N=50	New Test Statistics		1.647	1.559	1.463	1.381	1.289
	F(50,65)	*N=50, T-N-K=65*	1.973	1.854	1.672	1.539	1.399
N=60	New Test Statistics		1.586	1.519	1.430	1.353	1.267
	F(60,55)	*N=60, T-N-K=55*	2.002	1.871	1.693	1.554	1.408
N=70	New Test Statistics		1.527	1.460	1.383	1.316	1.247
	F(70,45)	*N=70, T-N-K=45*	2.080	1.934	1.738	1.596	1.441
N=80	New Test Statistics		1.486	1.434	1.362	1.298	1.231
	F(80,35)	*N=80, T-N-K=35*	2.222	2.052	1.829	1.663	1.488
N=90	New Test Statistics		1.453	1.417	1.343	1.287	1.219
	F(90,25)	*N=90, T-N-K=25*	2.557	2.308	2.003	1.797	1.574
N=100	New Test Statistics		1.437	1.380	1.321	1.272	1.212
	F(100,15)	*N=100, T-N-K=15*	3.453	3.022	2.471	2.135	1.802
N=110	New Test Statistics		1.411	1.363	1.300	1.253	1.199
	F(110,5)	*N=110, T-N-K=5*	14.562	9.701	6.138	4.491	3.158
N=200	New Test Statistics		1.313	1.282	1.231	1.195	1.156
N=500	New Test Statistics		1.186	1.170	1.147	1.126	1.100
N=1000	New Test Statistics		1.141	1.127	1.110	1.096	1.078
N=+infinite	New Test Statistics		1.018	1.018	1.018	1.018	1.018

Notes: For the new test statistics, see equation (12). The new statistics are the average value of N different $F(1, T\text{-}K\text{-}1)$ distributions. The statistics reported in the table are the results of simulations with 10,000 replications. Detail simulation procedures are explained in section 3.1.

The standard errors of the above simulated statistics are 0.001, 0.001, 0.002, 0.002, and 0.003 for the critical values of 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% respectively.

G. Number of Observations T=600, Number of Factors K=3

Number of Assets (*N*)	Test Statistics		Percentile of Test Statistics				
			0.5%	1.0%	2.5%	5.0%	10.0%
N =10	New Test Statistics		2.590	2.369	2.070	1.835	1.604
	F(10,587)	*N=10, T-N-K=587*	2.556	2.366	2.061	1.837	1.600
N =20	New Test Statistics		1.976	1.873	1.705	1.563	1.420
	F(20,577)	*N=20, T-N-K=577*	2.071	1.930	1.738	1.599	1.442
N =30	New Test Statistics		1.805	1.714	1.572	1.466	1.348
	F(30,567)	*N=30, T-N-K=567*	1.825	1.718	1.579	1.468	1.348
N =40	New Test Statistics		1.661	1.599	1.497	1.401	1.299
	F(40,557)	*N=40, T-N-K=557*	1.719	1.622	1.518	1.422	1.318
N =50	New Test Statistics		1.609	1.538	1.440	1.361	1.276
	F(50,547)	*N=50, T-N-K=547*	1.629	1.564	1.469	1.387	1.286
N =60	New Test Statistics		1.546	1.484	1.394	1.318	1.239
	F(60,537)	*N=60, T-N-K=537*	1.603	1.515	1.418	1.341	1.262
N =70	New Test Statistics		1.510	1.440	1.363	1.296	1.225
	F(70,527)	*N=70, T-N-K=527*	1.546	1.483	1.387	1.318	1.236
N =80	New Test Statistics		1.459	1.410	1.342	1.281	1.213
	F(80,517)	*N=80, T-N-K=517*	1.494	1.445	1.369	1.303	1.226
N =90	New Test Statistics		1.437	1.391	1.318	1.263	1.202
	F(90,507)	*N=90, T-N-K=507*	1.493	1.427	1.344	1.288	1.219
N =100	New Test Statistics		1.395	1.359	1.301	1.250	1.189
	F(100,497)	*N=100, T-N-K=497*	1.470	1.418	1.339	1.282	1.211
N =110	New Test Statistics		1.393	1.351	1.290	1.238	1.184
	F(110,487)	*N=110, T-N-K=487*	1.448	1.405	1.329	1.263	1.197
N =200	New Test Statistics		1.283	1.256	1.215	1.175	1.135
	F(200,397)	*N=200, T-N-K=397*	1.371	1.322	1.268	1.216	1.167
N =500	New Test Statistics		1.174	1.158	1.131	1.109	1.085
	F(500,97)	*N=500, T-N-K=97*	1.541	1.483	1.388	1.316	1.238
N =1000	New Test Statistics		1.123	1.110	1.094	1.079	1.062
N =+infinite	New Test Statistics		1.003	1.003	1.003	1.003	1.003

Notes: For the new test statistics, see equation (12). The new statistics are the average value of *N* different *F* (1, *T*-*K* -1) distributions. The statistics reported in the table are the results of simulations with 10,000 replications. Detail simulation procedures are explained in section 3.1.

The standard errors of the above simulated statistics are 0.001, 0.001, 0.002, 0.002, and 0.003 for the critical values of 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% respectively.

Table 2 Power of the Conventional *F* Test and the New Test of the CAPM for the Hypothesis H_0: a=0, H_1: a=m.

Testing Methods	Alternatives	Number of Observations	Number of Assets				
			N=1	*N=5*	*N=10*	*N=20*	*N=40*
Power of the Conventional *F* Test	m_p=8.5%	*T=60*	0.124	0.076	0.066	0.061	0.052
	s_p=16%	*T=120*	0.189	0.101	0.083	0.073	0.060
		T=240	0.344	0.172	0.135	0.100	0.082
	m_p=10.2%	*T=60*	0.196	0.105	0.084	0.069	0.056
	s_p=16%	*T=120*	0.327	0.172	0.124	0.098	0.073
		T=240	0.598	0.333	0.251	0.169	0.120
	m_p=11.6%	*T=60*	0.272	0.135	0.101	0.079	0.060
	s_p=16%	*T=120*	0.456	0.252	0.171	0.129	0.091
		T=240	0.771	0.497	0.378	0.258	0.170
	m_p=13.0%	*T=60*	0.339	0.169	0.121	0.091	0.064
	s_p=16%	*T=120*	0.575	0.333	0.225	0.161	0.107
		T=240	0.877	0.636	0.507	0.349	0.227
Power of the New Test	m_p=8.5%	*T=60*	0.124	0.077	0.068	0.061	0.057
	s_p=16%	*T=120*	0.189	0.103	0.088	0.078	0.069
		T=240	0.344	0.180	0.132	0.111	0.088
	m_p=10.2%	*T=60*	0.196	0.110	0.087	0.074	0.066
	s_p=16%	*T=120*	0.327	0.172	0.138	0.110	0.088
		T=240	0.598	0.340	0.242	0.188	0.137
	m_p=11.6%	*T=60*	0.272	0.145	0.108	0.089	0.077
	s_p=16%	*T=120*	0.456	0.246	0.196	0.145	0.112
		T=240	0.771	0.507	0.372	0.278	0.192
	m_p=13.0%	*T=60*	0.339	0.183	0.132	0.102	0.086
	s_p=16%	*T=120*	0.575	0.325	0.260	0.188	0.138
		T=240	0.877	0.647	0.502	0.384	0.264

Notes: The table reports simulation results on the case of the null hypothesis $H_0: a_j=0, j=1,\ldots,N$ against $H_1: a_j = m, j=1,\ldots,N$. The non-central variables of the *F* test and the new test are represented in equations (27) and (28) and the non-centrality parameter is defined in equation (26). The above table reports the results of 10,000 replications.

Table 3 Properties of Factors and Their Regression Coefficients

A. Entire Sample Period (April 1989 - March 1999)

		Growth FMP	Size FMP	Excess Market Return
Standard Deviation		0.980	2.457	3.847
Correlation Matrix				
	Growth FMP	1.000		
	Size FMP	0.083	1.000	
	Excess Market Return	0.472	0.168	1.000
Estimates of Coefficients (T=120, N=110)				
	Mean	0.015	-0.271	1.113
	Standard Deviation	0.831	0.463	0.420

B. Second Half Sample Period (April 1994 - March 1999)

		Growth FMP	Size FMP	Excess Market Return
Standard Deviation		1.139	2.493	3.979
Correlation Matrix				
	Growth FMP	1.000		
	Size FMP	0.075	1.000	
	Excess Market Return	0.550	0.102	1.000
Estimates of Coefficients (T=60, N=50)				
	Mean	-0.109	-0.246	0.980
	Standard Deviation	1.062	0.487	0.477

Notes: All factor returns are monthly log-returns and have the expected value of zero.
The entire sample preiod (Panel A) is used for T=120, and the second half sample period (Panel B)
is used for T=60. Estimates of coefficients are the maximum likelihood estimates of the linear factor model in (29).
Mean and standard deviation of the estimates of coefficients are calculated with 110 (Panel A) and 50 (Panel B)
estimates, respectively.

Table 4 Power of the Conventional *F* Test and the New Test of Linear Factor Model for the Null Hypothesis $H_0^M: a=0$ against the Alternative Hypothesis $H_1^M: a=m$ in the Presence of Missing Factors

A. T=60, N=10

Missing Factors			Power of Conventional F Test						Power of New Test					
			Elements of the (*Nx1*) Vector ***a***						Elements of the (*Nx1*) Vector ***a***					
			0	0.083	0.208	0.417	0.625	0.833	0	0.083	0.208	0.417	0.625	0.833
No Missing Factors *(K=3)*	Value of Noncentrality Parameter		0	0.110	0.689	2.757	6.202	11.026	0	0.110	0.689	2.757	6.202	11.026
	Test Level	1%	0.01	0.01	0.01	0.03	0.10	0.23	0.01	0.01	0.02	0.05	0.14	0.34
		5%	0.05	0.05	0.07	0.13	0.27	0.50	0.05	0.06	0.07	0.16	0.33	0.58
		10%	0.10	0.10	0.13	0.22	0.40	0.63	0.10	0.11	0.13	0.25	0.45	0.69
Size FMP *(K=2)*	Value of Noncentrality Parameter		0.00	0.11	0.68	2.73	6.14	10.91	0.00	0.11	0.68	2.73	6.14	10.91
	Test Level	1%	0.01	0.01	0.01	0.03	0.09	0.23	0.01	0.01	0.01	0.04	0.12	0.30
		5%	0.05	0.05	0.06	0.12	0.26	0.48	0.05	0.05	0.06	0.14	0.30	0.55
		10%	0.09	0.09	0.12	0.21	0.39	0.62	0.09	0.10	0.12	0.23	0.43	0.67
Growth FMP *(K=2)*	Value of Noncentrality Parameter		0.00	0.11	0.69	2.75	6.19	11.01	0.00	0.11	0.69	2.75	6.19	11.01
	Test Level	1%	0.01	0.01	0.01	0.03	0.09	0.23	0.01	0.01	0.01	0.04	0.12	0.31
		5%	0.05	0.05	0.06	0.12	0.26	0.49	0.05	0.05	0.07	0.14	0.31	0.56
		10%	0.09	0.09	0.12	0.21	0.40	0.62	0.10	0.10	0.12	0.23	0.44	0.68
Excess Market *(K=2)*	Value of Noncentrality Parameter		0.00	0.03	0.17	0.67	1.50	2.67	0.00	0.03	0.17	0.67	1.50	2.67
	Test Level	1%	0.01	0.01	0.01	0.01	0.01	0.02	0.00	0.00	0.00	0.01	0.03	0.11
		5%	0.04	0.04	0.04	0.05	0.07	0.10	0.01	0.01	0.02	0.05	0.13	0.30
		10%	0.07	0.07	0.08	0.10	0.13	0.19	0.03	0.03	0.04	0.09	0.22	0.44
Growth FMP Size FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.11	0.68	2.72	6.13	10.90	0.00	0.11	0.68	2.72	6.13	10.90
	Test Level	1%	0.01	0.01	0.01	0.03	0.09	0.22	0.01	0.01	0.01	0.03	0.10	0.27
		5%	0.04	0.04	0.06	0.12	0.25	0.48	0.04	0.04	0.06	0.12	0.28	0.53
		10%	0.08	0.09	0.11	0.20	0.38	0.61	0.08	0.09	0.11	0.22	0.41	0.65
Excess Market Size FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.03	0.17	0.67	1.51	2.68	0.00	0.03	0.17	0.67	1.51	2.68
	Test Level	1%	0.01	0.01	0.01	0.01	0.01	0.02	0.00	0.00	0.00	0.01	0.03	0.09
		5%	0.03	0.03	0.03	0.04	0.06	0.09	0.01	0.01	0.02	0.04	0.12	0.28
		10%	0.07	0.07	0.07	0.09	0.12	0.18	0.03	0.03	0.04	0.09	0.21	0.42
Excess Market Growth FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.03	0.18	0.70	1.58	2.80	0.00	0.03	0.18	0.70	1.58	2.80
	Test Level	1%	0.01	0.01	0.01	0.01	0.01	0.02	0.00	0.00	0.00	0.00	0.01	0.06
		5%	0.03	0.03	0.03	0.04	0.06	0.08	0.01	0.01	0.01	0.03	0.08	0.21
		10%	0.07	0.07	0.07	0.08	0.11	0.16	0.02	0.02	0.03	0.06	0.16	0.34

Notes: Randomly selected N excess equity returns included in S&P500 are estimated with three factors in the linear regression model; excess market return, size and growth FMP returns. Then the coefficients and the variance of disturbance terms are used to generate N excess equity returns, which are used to test the power of the F test and our new test in the presence of missing factors. See section 3.2 for a detailed explanation on the simulation proceddure. The numbers in 'Elements of the (*Nx1*) Vector ***a***' are monthly returns equivalent to 0%, 1%, 2.5%, 5%, 7.5%, and 10% in annual term. T=60 represents 60 monthly returns from April 1994 to March 1999. *K* is the number of factors used for the power test. The results on the power of tests are obtained with 10,000 replications.

B. T=60, N=50

Missing Factors			Power of Conventional F Test						Power of New Test					
			Elements of the (*Nx1*) Vector ***a***						Elements of the (*Nx1*) Vector ***a***					
			0	0.083	0.208	0.417	0.625	0.833	0	0.083	0.208	0.417	0.625	0.833
No Missing Factors *(K=3)*	Value of Noncentrality Parameter		0	0.570	3.564	14.255	32.073	57.019	0	0.570	3.564	14.255	32.073	57.019
	Test Level	1%	0.01	0.01	0.01	0.02	0.04	0.08	0.01	0.01	0.02	0.15	0.59	0.95
		5%	0.05	0.06	0.06	0.10	0.17	0.29	0.05	0.06	0.11	0.37	0.82	0.99
		10%	0.10	0.10	0.12	0.17	0.29	0.45	0.11	0.12	0.18	0.50	0.89	1.00
Size FMP *(K=2)*	Value of Noncentrality Parameter		0.00	0.55	3.46	13.84	31.15	55.37	0.00	0.55	3.46	13.84	31.15	55.37
	Test Level	1%	0.01	0.01	0.01	0.02	0.04	0.09	0.01	0.01	0.02	0.13	0.56	0.94
		5%	0.05	0.05	0.06	0.10	0.18	0.31	0.04	0.04	0.08	0.31	0.78	0.98
		10%	0.10	0.10	0.12	0.18	0.30	0.47	0.08	0.09	0.14	0.45	0.86	0.99
Growth FMP *(K=2)*	Value of Noncentrality Parameter		0.00	0.57	3.56	14.23	32.02	56.93	0.00	0.57	3.56	14.23	32.02	56.93
	Test Level	1%	0.01	0.01	0.01	0.02	0.05	0.10	0.01	0.01	0.02	0.14	0.57	0.95
		5%	0.05	0.05	0.06	0.10	0.19	0.32	0.04	0.05	0.09	0.32	0.79	0.99
		10%	0.10	0.10	0.12	0.18	0.31	0.48	0.09	0.09	0.15	0.46	0.87	0.99
Excess Market *(K=2)*	Value of Noncentrality Parameter		0.00	0.12	0.73	2.91	6.54	11.63	0.00	0.12	0.73	2.91	6.54	11.63
	Test Level	1%	0.01	0.01	0.01	0.01	0.02	0.02	0.00	0.00	0.00	0.01	0.16	0.68
		5%	0.05	0.05	0.05	0.05	0.07	0.09	0.00	0.00	0.01	0.06	0.38	0.87
		10%	0.09	0.09	0.10	0.11	0.13	0.16	0.01	0.01	0.02	0.11	0.52	0.93
Growth FMP Size FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.55	3.45	13.81	31.08	55.25	0.00	0.55	3.45	13.81	31.08	55.25
	Test Level	1%	0.01	0.01	0.01	0.02	0.05	0.11	0.00	0.01	0.01	0.10	0.50	0.93
		5%	0.05	0.05	0.06	0.10	0.19	0.34	0.03	0.04	0.06	0.27	0.74	0.98
		10%	0.09	0.10	0.12	0.18	0.31	0.51	0.06	0.07	0.12	0.40	0.84	0.99
Excess Market Size FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.12	0.73	2.91	6.55	11.65	0.00	0.12	0.73	2.91	6.55	11.65
	Test Level	1%	0.01	0.01	0.01	0.01	0.02	0.02	0.00	0.00	0.00	0.01	0.13	0.63
		5%	0.05	0.05	0.05	0.06	0.07	0.09	0.00	0.00	0.00	0.05	0.34	0.85
		10%	0.09	0.09	0.10	0.11	0.13	0.17	0.00	0.01	0.01	0.09	0.48	0.91
Excess Market Growth FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.10	0.61	2.45	5.52	9.81	0.00	0.10	0.61	2.45	5.52	9.81
	Test Level	1%	0.01	0.01	0.01	0.01	0.01	0.02	0.00	0.00	0.00	0.00	0.06	0.46
		5%	0.04	0.04	0.05	0.05	0.06	0.08	0.00	0.00	0.00	0.02	0.21	0.73
		10%	0.09	0.09	0.09	0.10	0.12	0.15	0.00	0.00	0.00	0.05	0.33	0.83

Notes: Randomly selected N excess equity returns included in S&P500 are estimated with three factors in the linear regression model; excess market return, size and growth FMP returns. Then the coefficients and the variance of disturbance terms are used to generate N excess equity returns, which are used to test the power of the F test and our new test in the presence of missing factors.
See section 3.2 for a detailed explanation on the simulation proceddure. The numbers in 'Elements of the (*Nx1*) Vector ***a*** ' are monthly returns equivalent to 0%, 1%, 2.5%, 5%, 7.5%, and 10% in annual term. T=60 represents 60 monthly returns from April 1994 to March 1999.
K is the number of factors used for the power test. The results on the power of tests are obtained with 10,000 replications.

C. T=120, N=10

Missing Factors			Power of Conventional F Test						Power of New Test					
			Elements of the (*Nx1*) Vector ***a***						Elements of the (*Nx1*) Vector ***a***					
			0	0.083	0.208	0.417	0.625	0.833	0	0.083	0.208	0.417	0.625	0.833
No Missing Factors	Value of Noncentrality Parameter		0	0.205	1.281	5.123	11.526	20.490	0	0.205	1.281	5.123	11.526	20.490
(K=3)	Test Level	1%	0.01	0.01	0.02	0.09	0.31	0.67	0.01	0.01	0.03	0.12	0.39	0.75
		5%	0.05	0.05	0.08	0.24	0.56	0.86	0.05	0.06	0.10	0.28	0.61	0.90
		10%	0.10	0.11	0.16	0.37	0.69	0.92	0.10	0.11	0.17	0.39	0.72	0.94
Size FMP	Value of Noncentrality Parameter		0.00	0.20	1.28	5.12	11.51	20.46	0.00	0.20	1.28	5.12	11.51	20.46
(K=2)	Test Level	1%	0.01	0.01	0.02	0.09	0.31	0.66	0.01	0.01	0.02	0.10	0.35	0.72
		5%	0.04	0.05	0.08	0.23	0.55	0.85	0.04	0.05	0.08	0.25	0.58	0.88
		10%	0.09	0.10	0.15	0.36	0.68	0.92	0.09	0.10	0.15	0.37	0.71	0.94
Growth FMP	Value of Noncentrality Parameter		0.00	0.20	1.28	5.12	11.52	20.48	0.00	0.20	1.28	5.12	11.52	20.48
(K=2)	Test Level	1%	0.01	0.01	0.02	0.09	0.31	0.66	0.01	0.01	0.02	0.10	0.35	0.72
		5%	0.04	0.05	0.08	0.24	0.55	0.85	0.04	0.05	0.08	0.25	0.58	0.88
		10%	0.09	0.10	0.15	0.36	0.69	0.92	0.09	0.10	0.15	0.37	0.71	0.94
Excess Market	Value of Noncentrality Parameter		0.00	0.05	0.30	1.18	2.66	4.74	0.00	0.05	0.30	1.18	2.66	4.74
(K=2)	Test Level	1%	0.01	0.01	0.01	0.01	0.02	0.05	0.00	0.00	0.00	0.02	0.12	0.40
		5%	0.03	0.03	0.04	0.05	0.10	0.20	0.01	0.01	0.02	0.08	0.30	0.66
		10%	0.07	0.07	0.08	0.11	0.19	0.33	0.03	0.03	0.05	0.16	0.44	0.79
Growth FMP	Value of Noncentrality Parameter		0.00	0.20	1.28	5.12	11.51	20.46	0.00	0.20	1.28	5.12	11.51	20.46
Size FMP	Test Level	1%	0.01	0.01	0.02	0.08	0.30	0.66	0.01	0.01	0.02	0.09	0.32	0.69
(K=1)		5%	0.04	0.05	0.08	0.23	0.55	0.85	0.04	0.05	0.08	0.24	0.57	0.87
		10%	0.09	0.10	0.14	0.36	0.68	0.92	0.09	0.10	0.14	0.36	0.70	0.93
Excess Market	Value of Noncentrality Parameter		0.00	0.05	0.30	1.18	2.66	4.72	0.00	0.05	0.30	1.18	2.66	4.72
Size FMP	Test Level	1%	0.01	0.01	0.01	0.01	0.02	0.05	0.00	0.00	0.00	0.02	0.10	0.37
(K=1)		5%	0.03	0.03	0.03	0.05	0.09	0.19	0.01	0.01	0.02	0.08	0.30	0.65
		10%	0.07	0.07	0.07	0.11	0.18	0.32	0.02	0.03	0.04	0.15	0.43	0.78
Excess Market	Value of Noncentrality Parameter		0.00	0.05	0.31	1.26	2.83	5.02	0.00	0.05	0.31	1.26	2.83	5.02
Growth FMP	Test Level	1%	0.01	0.01	0.01	0.01	0.02	0.04	0.00	0.00	0.00	0.01	0.07	0.31
(K=1)		5%	0.03	0.03	0.03	0.05	0.09	0.17	0.01	0.01	0.01	0.06	0.24	0.59
		10%	0.07	0.07	0.07	0.10	0.17	0.29	0.02	0.02	0.03	0.12	0.37	0.73

Notes: Randomly selected N excess equity returns included in S&P500 are estimated with three factors in the linear regression model; excess market return, size and growth FMP returns. Then the coefficients and the variance of disturbance terms are used to generate N excess equity returns, which are used to test the power of the F test and our new test in the presence of missing factors.
See section 3.2 for a detailed explanation on the simulation proceddure. The numbers in 'Elements of the (*Nx1*) Vector ***a***' are monthly returns equivalent to 0%, 1%, 2.5%, 5%, 7.5%, and 10% in annual term. T=120 represents 120 monthly returns from April 1989 to March 1999.
K is the number of factors used for the power test. The results on the power of tests are obtained with 10,000 replications.

D. T=120, N=110

Missing Factors			Power of Conventional F Test						Power of New Test					
			Elements of the (*Nx1*) Vector ***a***						Elements of the (*Nx1*) Vector ***a***					
			0	0.083	0.208	0.417	0.625	0.833	0	0.083	0.208	0.417	0.625	0.833
	Value of Noncentrality Parameter		0	2.303	14.395	57.578	129.551	230.314	0	2.303	14.395	57.578	129.551	230.314
No Missing Factors		1%	0.01	0.01	0.01	0.03	0.08	0.18	0.01	0.02	0.10	0.83	1.00	1.00
(K=3)	Test Level	5%	0.05	0.05	0.07	0.14	0.29	0.52	0.05	0.07	0.25	0.94	1.00	1.00
		10%	0.10	0.11	0.13	0.25	0.46	0.71	0.10	0.13	0.37	0.97	1.00	1.00
	Value of Noncentrality Parameter		0.00	2.17	13.53	54.14	121.81	216.55	0.00	2.17	13.53	54.14	121.81	216.55
Size FMP		1%	0.01	0.01	0.01	0.03	0.09	0.21	0.01	0.01	0.06	0.78	1.00	1.00
(K=2)	Test Level	5%	0.05	0.05	0.06	0.14	0.31	0.55	0.03	0.05	0.20	0.92	1.00	1.00
		10%	0.09	0.10	0.13	0.25	0.48	0.73	0.08	0.10	0.32	0.96	1.00	1.00
	Value of Noncentrality Parameter		0.00	2.30	14.38	57.52	129.41	230.07	0.00	2.30	14.38	57.52	129.41	230.07
Growth FMP		1%	0.01	0.01	0.01	0.03	0.10	0.23	0.01	0.01	0.08	0.80	1.00	1.00
(K=2)	Test Level	5%	0.05	0.05	0.07	0.15	0.33	0.58	0.04	0.06	0.23	0.93	1.00	1.00
		10%	0.09	0.10	0.13	0.27	0.50	0.76	0.09	0.12	0.35	0.97	1.00	1.00
	Value of Noncentrality Parameter		0.00	0.30	1.90	7.61	17.13	30.45	0.00	0.30	1.90	7.61	17.13	30.45
Excess Market		1%	0.01	0.01	0.01	0.01	0.02	0.02	0.00	0.00	0.00	0.13	0.96	1.00
(K=2)	Test Level	5%	0.05	0.05	0.05	0.06	0.07	0.10	0.00	0.00	0.00	0.33	0.99	1.00
		10%	0.09	0.10	0.10	0.12	0.14	0.18	0.00	0.00	0.01	0.48	1.00	1.00
	Value of Noncentrality Parameter		0.00	2.16	13.49	53.94	121.37	215.77	0.00	2.16	13.49	53.94	121.37	215.77
Growth FMP		1%	0.01	0.01	0.01	0.03	0.10	0.25	0.01	0.01	0.06	0.77	1.00	1.00
Size FMP	Test Level	5%	0.04	0.05	0.07	0.15	0.34	0.60	0.03	0.05	0.19	0.91	1.00	1.00
(K=1)		10%	0.09	0.10	0.13	0.27	0.52	0.78	0.07	0.09	0.30	0.96	1.00	1.00
	Value of Noncentrality Parameter		0.00	0.28	1.76	7.06	15.88	28.22	0.00	0.28	1.76	7.06	15.88	28.22
Excess Market		1%	0.01	0.01	0.01	0.01	0.01	0.02	0.00	0.00	0.00	0.11	0.95	1.00
Size FMP	Test Level	5%	0.04	0.04	0.05	0.06	0.07	0.09	0.00	0.00	0.00	0.30	0.99	1.00
(K=1)		10%	0.09	0.09	0.10	0.11	0.14	0.18	0.00	0.00	0.01	0.44	1.00	1.00
	Value of Noncentrality Parameter		0.00	0.31	1.93	7.73	17.40	30.93	0.00	0.31	1.93	7.73	17.40	30.93
Excess Market		1%	0.01	0.01	0.01	0.01	0.02	0.02	0.00	0.00	0.00	0.06	0.89	1.00
Growth FMP	Test Level	5%	0.04	0.05	0.05	0.06	0.07	0.10	0.00	0.00	0.00	0.19	0.98	1.00
(K=1)		10%	0.09	0.09	0.09	0.11	0.14	0.18	0.00	0.00	0.00	0.30	0.99	1.00

Notes: Randomly selected N excess equity returns included in S&P500 are estimated with three factors in the linear regression model; excess market return, size and growth FMP returns. Then the coefficients and the variance of disturbance terms are used to generate N excess equity returns, which are used to test the power of the F test and our new test in the presence of missing factors.
See section 3.2 for a detailed explanation on the simulation proceddure. The numbers in 'Elements of the (*Nx1*) Vector ***a***' are monthly returns equivalent to 0%, 1%, 2.5%, 5%, 7.5%, and 10% in annual term. T=120 represents 120 monthly returns from April 1989 to March 1999.
K is the number of factors used for the power test. The results on the power of tests are obtained with 10,000 replications.

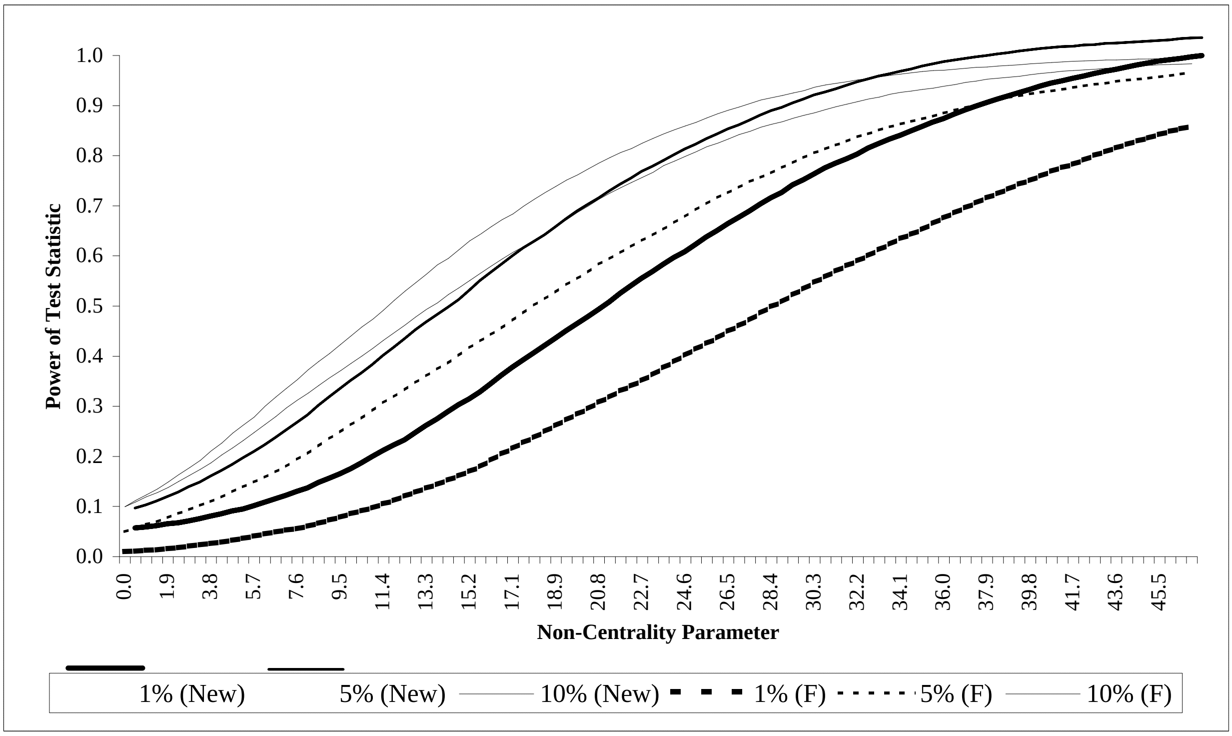
E. T=60, N=110

Missing Factors			Power of New Test					
			Elements of the (*Nx1*) Vector ***a***					
			0	0.083	0.208	0.417	0.625	0.833
No Missing Factors *(K=3)*	Value of Noncentrality Parameter		0	1.195	7.469	29.876	67.221	119.503
		1%	0.01	0.01	0.04	0.32	0.90	1.00
	Test Level	5%	0.05	0.06	0.13	0.56	0.97	1.00
		10%	0.10	0.12	0.22	0.69	0.99	1.00
Size FMP *(K=2)*	Value of Noncentrality Parameter		0.00	1.10	6.88	27.53	61.93	110.10
		1%	0.01	0.01	0.03	0.27	0.88	1.00
	Test Level	5%	0.04	0.05	0.10	0.50	0.96	1.00
		10%	0.08	0.09	0.17	0.64	0.98	1.00
Growth FMP *(K=2)*	Value of Noncentrality Parameter		0.00	1.18	7.37	29.49	66.34	117.94
		1%	0.01	0.01	0.03	0.29	0.89	1.00
	Test Level	5%	0.04	0.05	0.11	0.53	0.97	1.00
		10%	0.09	0.10	0.19	0.66	0.99	1.00
Excess Market *(K=2)*	Value of Noncentrality Parameter		0.00	0.24	1.50	6.00	13.50	24.00
		1%	0.00	0.00	0.00	0.01	0.32	0.94
	Test Level	5%	0.00	0.00	0.00	0.05	0.56	0.99
		10%	0.00	0.00	0.00	0.11	0.70	1.00
Growth FMP Size FMP *(K=1)*	Value of Noncentrality Parameter		0.00	1.09	6.82	27.29	61.40	109.16
		1%	0.00	0.00	0.02	0.20	0.83	1.00
	Test Level	5%	0.03	0.03	0.08	0.45	0.95	1.00
		10%	0.06	0.07	0.14	0.58	0.98	1.00
Excess Market Size FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.24	1.50	5.99	13.47	23.94
		1%	0.00	0.00	0.00	0.01	0.23	0.91
	Test Level	5%	0.00	0.00	0.00	0.04	0.50	0.98
		10%	0.00	0.00	0.00	0.08	0.64	0.99
Excess Market Growth FMP *(K=1)*	Value of Noncentrality Parameter		0.00	0.22	1.39	5.56	12.50	22.23
		1%	0.00	0.00	0.00	0.00	0.09	0.75
	Test Level	5%	0.00	0.00	0.00	0.01	0.27	0.92
		10%	0.00	0.00	0.00	0.02	0.41	0.96

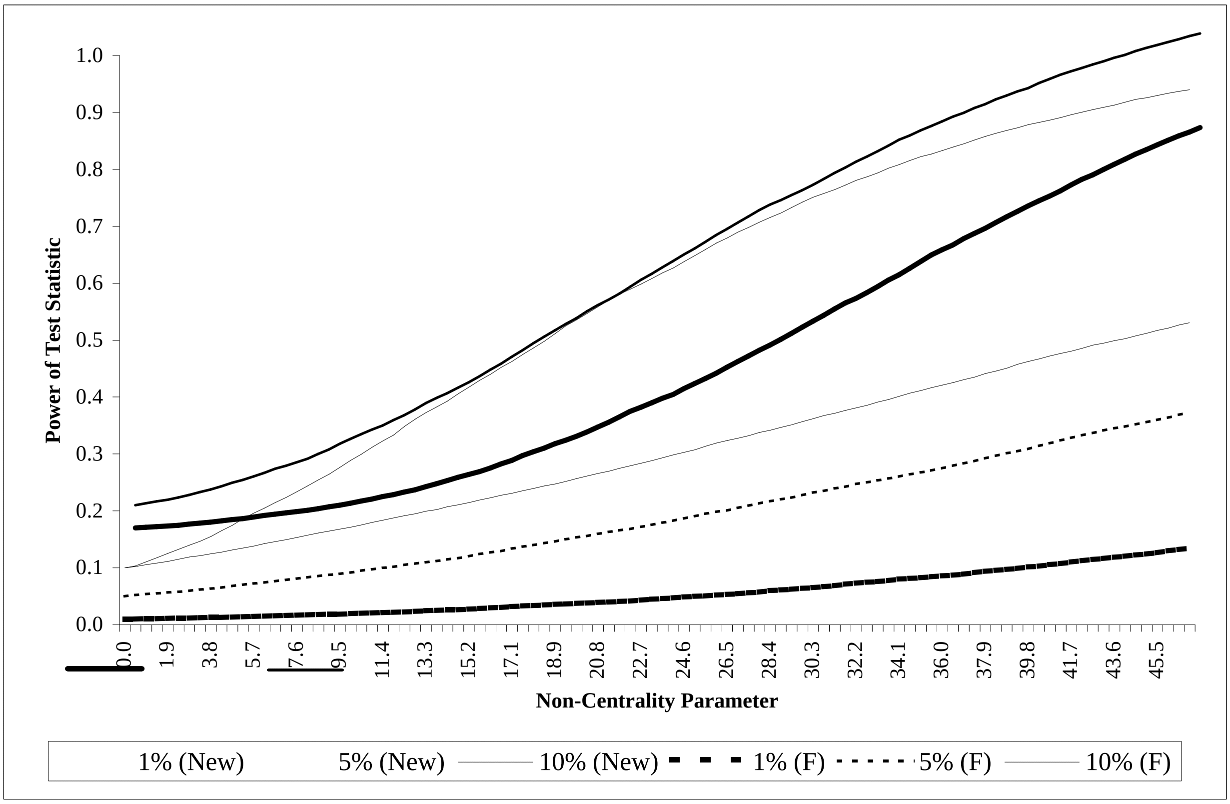
Notes: Randomly selected N excess equity returns included in S&P500 are estimated with three factors in the linear regression model; excess market return, size and growth FMP returns. Then the coefficients and the variance of disturbance terms are used to generate *N* excess equity returns, which are used to test the power of the new test in the presence of missing factors.
See section 3.2 for a detailed explanation on the simulation proceddure. The numbers in 'Elements of the (*Nx1*) Vector ***a*** ' are monthly returns equivalent to 0%, 1%, 2.5%, 5%, 7.5%, and 10% in annual term. T=60 represents 60 monthly returns from April 1994 to March 1999.
Note that the conventional F test is not available in this case. K is the number of factors used for the power test.
The results on the power of tests are obtained with 10,000 replications.

Figure 1 Power of the New and F (N, T-N-K) Test Statistics for the Null Hypothesis $H_0: a_j = 0$ for all j against the Alternative Hypothesis $H_1: a_1 = m$ and $a_j = 0$ for $j > 1$:

A. Case 1: T=120, N=30, K=1



B. Case 2: T=120, N=90, K=1



Notes: The figures A and B show the simulation results on case of the null hypothesis $H_0: a_j = 0, j = 1, \ldots, N$ against the alternative hypothesis $H_1: a_j = m$, for j=1, and $a_j = 0$, for $j = 2, \ldots, N$. Simulation procedures are explained in section 3.2. The non-central variables for the alternative hypothesis are generated with equations (24) and (25) for the conventional F test and the new test, respectively.