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Conditional Gaussian Models of the Term Structure of Interest Rates

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CONDITIONAL GAUSSIAN MODELS OF THE TERM STRUCTURE OF INTEREST RATES[1]

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Abstract: We present a new family of yield curve models, termed "Conditional Gaussian". It provides both simplicity and extreme flexibility in constructing "market models". Any conditional co-variance structure - including features designed to capture volatility "skews" and/or GARCH effects - can be used, and the model can be embedded into a continuous-time whole yield curve model consistent with general equilibrium. Conditionally Gaussian increments in log one-plus-interest-rates enable "vanilla" and path-dependent derivatives to be valued easily by Monte Carlo without discretization error, whether or not their payoffs depend solely on the particular market rates being modelled directly.

1. INTRODUCTION

We present a new, "Conditional Gaussian" (CG), family of yield curve models, capable of being supported in general equilibrium. This family is significant in at least the following two respects, as well as opening up further possibilities to be discussed briefly in our concluding remarks.

Firstly, it enables us to construct multi-factor "market models" in a simple but extremely flexible manner. Moreover, "vanilla" and path-dependent interest-rate derivatives can be valued easily, whether or not their payoffs depend solely on the particular market rates being modelled directly. "Market models" carries here the general sense that the modelling focuses upon particular observable market interest rates; rather than the narrower sense of requiring lognormal simple rates. In many currencies, Black [1976] implied volatilities now exhibit "skews" with respect to strike; thus fitting market option prices closely requires rich alternatives to lognormality. Our models provide this through tremendous flexibility in the dependence of conditional covariances of term structure movements upon both rate levels and maturity.

Secondly, our models allow direct incorporation of GARCH dynamics into the whole term structure, within an arbitrage-free multi-factor model. This provides an alternative to existing models, in which heteroskedasticity (other than dependence on rate levels) enters only through stochastic volatility or GARCH dynamics of a single interest rate. Valuation remains easy under GARCH.

The key to our approach is that, if increments in log one-plus-interest rates are conditionally multivariate Normal under risk-adjusted probabilities, then a discrete-time no-arbitrage martingale condition reduces to a simple condition linking the means and co-variances. Any conditional co-

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[1] Views expressed in this paper are those of the author and not necessarily those of any organisation to which he is affiliated.

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variance structure - including GARCH - can be used, and the model can be embedded into a viable continuous-time model which, piecewise and conditionally, is multi-factor linear Gaussian. All kinds of volatility skews can be produced, and/or rates kept positive to an arbitrarily demanding confidence level, by appropriate combination of discrete time interval and co-variance structure. The linear Gaussian nature of both the discrete-time increments and the embedding model facilitates straightforward Monte Carlo valuations of derivatives.

The rest of this paper is organised as follows. Section 2 briefly reviews the existing "market models" and stochastic volatility/GARCH term structure literature, and sets our family of models in that context. Following this material, Sections 3-7 comprise the heart of the paper, dealing with Conditional Gaussian models in general.

Specifically, Section 3 begins by establishing a discrete-time martingale condition applicable to the dynamics of a sequence of forward simple interest rates, observed at a sequence of discrete observation dates. Reflection upon this condition motivates the construction of the discrete-time "kernel" of our family of models.

In Section 4, we construct a continuous-time whole yield curve model, consistent with general equilibrium, in which our discrete-time kernel is embedded, and in which spot rate changes over intervals within single discrete-time periods are conditionally Gaussian.

Econometric estimation and testing of any model operates under objective probabilities. To this end, Section 5 revisits the discrete time kernel, which we created in Section 3 under risk-adjusted probabilities, and re-casts it in a tractable way under the objective measure. To our knowledge, this is the first time that anything like such a rich class of term structure models has had available an exact arbitrage-free discrete-time representation for multiple market interest rates.

With the abstract theory thus complete, but still at the level of the full generality of the family of models, Section 6 addresses the valuation of what we term "kernel claims", ie those whose payoffs can be written as a function of the finite set of observations of forward simple interest rates contained in the discrete time kernel of the model. Standard interest rate caps and European-style swaptions, to which one would want to calibrate a model when carrying out derivatives pricing, are leading examples of such claims. We show that all kernel claims are directly amenable to Monte Carlo simulation, without discretization error.

At the same level of full generality, Section 7 shows that a broad class of "non-kernel" claims is equally amenable to Monte Carlo simulation. This is because we are able to show that the whole term structure at intermediate dates, like the particular rates and observations in the "kernel", can be simulated without discretization error using Normally distributed random numbers. The practical significance of this is that, once a Conditional Gaussian model has been calibrated to standard caps and swaptions, the resulting parameter values can be used directly in the valuation of other trades involving arbitrary dates.

Section 8 suggests various examples of specifications of the key model input – the covariance structure of discrete-time movements in the logarithms of forward one-plus-interest-rates. In particular, one example illustrates incorporation of GARCH effects.

A concluding section summarises the paper, discusses various implementation issues, and outlines some possible extensions. Proofs of any length are relegated to an appendix.

2. "MARKET MODELS" AND STOCHASTIC VOLATILITY / GARCH MODELS

2.1. *"MARKET MODELS"*

A significant branch of the term structure literature has recently developed, concerning so-called "market models" (Brace, Gatarek and Musiela [1997], Miltersen, Sandmann and Sondermann [1997], Musiela and Rutkowski [1997], Jamshidian [1997], and Andersen and Andreasen [1998]). Whereas the general continuous-time arbitrage-free framework developed independently by Babbs [1990] and Heath, Jarrow and Morton [1992] models the term structure across a continuum of maturities, without particular regard to any specific quoted interest rate, the "market models" assign a special role to forward simple interest rates of some particular tenor underlying commonly quoted interest rate cap or swaption contracts.[2] The main motivation for focusing on these rates is to seek a model in which theoretical prices of caps and/or swaption contracts will more closely mimic market prices. Considerable prominence is often given (eg Brace, Gatarek and Musiela [1997], Miltersen, Sandmann and Sondermann [1997]) to the production of cap and/or swaption pricing formulae which coincide with, or at least closely approximate, the adaptations of Black [1976] frequently used by traders.

Until very recently, this prominence could be supported by pointing out that - at least for the frequently traded contracts whose strike levels fell within about one percentage point of being at-the-money – the dependence of market price upon strike level was well approximated by Black's formula with no volatility "smile" or "skew". However, this approximation has now largely broken down in favour of noticeable volatility "skews" across all frequently traded strike levels.[3] Accommodating such a breakdown, Jamshidian [1997] and Andersen and Andreasen [1998], have relegated lognormal behaviour of simple interest rates[4] to a special case of their models.[5]

The family of models we shall present shares some features with that of Jamshidian. As in his framework, the dependence of volatility upon rate levels is at the user's discretion, as are the correlations between changes in different rates. Carefully engineered choices of his instantaneous volatility and correlation functions would even imply a discrete-time "kernel" identical with ours. However, there are also striking differences, stemming from different starting points. Unlike Jamshidian, we commence with the discrete-time analysis, and only afterwards extend to continuous-time, resulting in different continuous-time models. Our continuous-time model covers the whole continuum of maturities – which Jamshidian expressly eschews. Moreover, our

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[2] Earlier work by Goldys, Musiela and Sondermann [1994] and Sandmann and Sondermann [1993],[1994], using "effective" or other discretely compounded rates, contributed significantly to the creation of "market models".

[3] This was only to be expected in the light of empirical evidence (eg the application of the methodology of Nowman [1997] to more recent data) that, at least since the ending, in the mid-1980s, of aggressive targeting of monetary aggregates in the US, the amplitude of arithmetic changes in interest rates is not proportional to the level of rates, contrary to the assumption of the adaptations of Black [1976].

[4] Jamshidian develops separate models in which the simple rates concerned are, respectively, (single period) LIBOR rates, and (multi-period) swap rate. In general, two models using different tenors of LIBOR, or using LIBORs and swap rates, are mutually inconsistent.

[5] Similar relegation of lognormality is evident in Hunt, Kennedy and Pelsser's [1998] "Markov functional" models, which share many of the goals of market models while using only a low-dimensional process. It is currently unclear whether multi-factor versions of these models can be made tractable.

approach permits the valuation of contingent claims outside the scope of Jamshidian's model, such as those depending on rates outside the finite set upon which modelling is focused.

Further differences arise in numerical valuation. Both in our framework, and in Jamshidian's once lognormality is dropped, closed-form valuation formulae are lacking,[6] and Monte Carlo simulation is the obvious technique for contingent claims without early exercise or other discretionary features. In Jamshidian's LIBOR-based models, each forward LIBOR follows a continuous-time diffusion whose drift depends non-linearly on multiple LIBORs.[7] Hence, even to value payoffs depending only upon observations made at discrete dates, accurate Monte Carlo simulation will almost certainly require a finer mesh of timesteps, and it is non-trivial to decide how the stochastic processes should be discretized. (Although Glasserman and Zhao [1998] report significant work in this area, it remains not straightforward.) By contrast, Conditional Gaussian models provide exact multivariate Normal bases for Monte Carlo simulation of these contingent claims. This does not, of course, remove all performance and convergence issues; it does however completely eliminate discretization questions, and avoids the need for a finer time mesh with the higher dimensionality that entails.

2.2. STOCHASTIC VOLATILITY AND GARCH MODELS

Longstaff and Schwartz [1992] (LS) and Fong and Vasicek [1991] model the short (ie instantaneous spot) rate as having stochastic volatility imperfectly correlated with the rate itself. Equally validly, by an invertibility argument, the volatility can be written as a function of the short rate and some (arbitrarily chosen) other yield. Thus, when modelling interest rates, the distinction between stochastic volatility – and perhaps GARCH - models – and, by contrast, models in which volatility is a function of rate levels, becomes subtle and sometimes even meaningless. The distinction is perhaps best drawn between models in which short rate volatility can be written as a function of at most time and the current short rate, and models in which it cannot.[8] The vast majority of existing models fall into the former category. In doing so, they go against the conclusions of LS and Brenner, Harjes and Kroner [1996], that short rate volatility exhibits stochastic variation inadequately explained by that rate itself.

The stochastic volatility models, LS and Fong and Vasicek [1991], offer fragile tractability, in the sense that the availability of (a few) closed-form valuation formulae, and of an explicit transition density for the state variables, depend upon the mean-reversion and diffusion parameters being constant across time, and upon the overall diffusion coefficients being proportional to the square root – rather than any other non-zero power - of the respective variables.

Brenner, Harjes and Kroner [1996] explore econometrically several short-rate models, in most of which GARCH effects are present and appear to have significant explanatory power. They note that they "do not use the correct arbitrage-free drift" which would be "very difficult to do". We

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[6] Andersen and Andreasen [1998] report analytic caplet and European-style swaption approximations for the special case of a CEV formulation; these appear to work well except when Black [1976] implied volatilities are very high (as currently in Japan).

[7] See equation (13) in his Section 4, or equation (5) in his Section 6.

[8] We use "volatility" and "short rate" here in a widened sense to refer to the uncertainty attaching to instantaneous movements in the short rate in a continuous-time model, or to next period's shortest rate in a discrete-time model.

conjecture that such difficulties would prevent extending their particular approach to analyse multivariate time series of rates of various maturities. Duan [1996] constructs an equilibrium asymmetric GARCH model for the one-period rate in discrete time, thereby overcoming arbitrage problems; however, forward rates can only be solved for Monte Carlo valuation of bonds.

Against this background, the Conditional Gaussian framework offers attractive features. There are explicit transition densities regardless of the functional form of the dependence of volatility upon the current term structure, and despite possible GARCH effects. Moreover, arbitrage-free expected increments in multivariate forward simple interest rates are known exactly.

3. DISCRETE-TIME "KERNEL"

We commence by considering the implications of no-arbitrage martingale conditions for the dynamics of a sequence of forward simple interest rates, observed at a fixed sequence of discrete observation dates. These observation dates include, but need not be restricted to, the rate determination dates at which successive forward simple rates become spot rates.

We regard this "decoupling" of the observation frequency from the tenors of the forward rates as significant. Firstly, it will permit econometric use of, say, daily or weekly observations without the estimation of daily (weekly) forward rates and of their covariances. Secondly, it will permit the model to capture more accurately both the dependence of volatility upon rate levels, and also GARCH effects occurring on timescales shorter than the tenors of the interest rates in the "kernel". Thirdly, we conjecture that, by jointly varying the observation frequency, interest frequency and covariance structures, one could approximate any diffusion model of the term structure by a sequence of Conditional Gaussian models; each approximating model, of course, would enjoy the benefits of the exact arbitrage-free means given by (11), and the convenience of explicit multivariate Normal transition densities free of discretization error. We hope to confirm this conjecture in future work.

Formalizing the "decoupling" mathematically, we fix any sequence of dates $T_0 < T_1 < ... < T_N$, and denote the corresponding interest rate determination dates $T_0' < T_1' < ... < T_{N-1}'$.[9] We also fix a second sequence $0 = t_0 < t_1 < ... < t_J = T_{N-1}'$ of observation dates, such that

$$\forall n \in \{0,1,...,N-1\},\ \exists i(n) : t_{i(n)} = T_n' \tag{1}$$

Note that $i(n)$ is uniquely defined for each *n*, since each sequence of dates is strictly monotonic. It also has a converse in the form of a mapping:

$$k(i) \equiv \min\{k : T_k' \geq t_i\} \tag{2}$$

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[9] We distinguish with a prime the date on which rates are quoted, from the later "spot" date (in most currencies two business days later) on which any initial transfer of principal occurs. In currencies where such a "rate fixing lag" exists, "overnight" and "tom-next" interest rates spanning this interval are subject to different arrangements.

Since the final interest rate in our model kernel is determined at T_{N-1}', we have no interest in T_N'

Denoting the value at time t of unit pure discount bond maturing at M by $B(M,t)$, the value at any observation date t_i of a unit pure discount bond maturing at $T_n \geq T_{k(i)}$ can be expressed as:[10]

$$B(T_n,t_i) = B(T_{k(i)},t_i)\prod_{m=k(i)}^{n-1}\left(1+S_m(t_i)\delta_m\right)^{-1} \tag{3}$$

where: $B(T_{k(i)},t_i)$ reflects the discounting attributable to very near-term interest rates; $S_m(\bullet)$ is the stochastic process of the forward simple interest rate from T_m to T_{m+1}; and each δ_m is a strictly positive constant "accrual factor".

We normalise by dividing (3) through by the price of our numeraire, a discretely re-invested money market account[11] whose balance at any T_k is:[12]

$$\prod_{m=0}^{k-1}\left(1+S_m(T'_m)\delta_m\right)$$

and whose present value at t_i is therefore

$$A(t_i) = B(T_{k(i)},t_i)\prod_{m=0}^{k(i)-1}\left(1+S_m(T'_m)\delta_m\right) \tag{4}$$

and write the normalised pure discount bond price as

$$B^*(T_n,t_i) \equiv \frac{B(T_{k(i)},t_i)}{A(t_i)} = \prod_{m=1}^{n-1}\left(1+S_m(T'_m \wedge t_i)\delta_m\right)^{-1} \tag{5}$$

Taking pure discount bonds maturing at the dates $T_0 < T_1 <\ldots< T_N$ as the primitive securities of a frictionless securities market with trading opportunities at $0 = t_0 < t_1 <\ldots< t_J$, the model is arbitrage-free if and only if the change in the normalised price of every bond, over every single period has a conditional expectation of zero at the start of the period (ie t_i):

$$\mathrm{E}_i^*\left[B^*(T_n,t_{i+1})\right] = B^*(T_n,t_i); \qquad n = k(i+1),\ldots,N; \quad i = 0,1,\ldots,J-1 \tag{6}$$

where the conditional expectations are taken under any risk-adjusted probability measure, P^*, corresponding to our choice of numeraire.

Substituting (5) into (6) and simplifying, provides the martingale condition formula which must be satisfied:

$$\mathrm{E}_i^*\left[\prod_{m=k(i+1)}^{n-1}\frac{1+S_m(t_i)\delta_m}{1+S_m(t_{i+1})\delta_m}\right] = 1 \tag{7}$$

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[10] We apply the convention, throughout this paper, that a product (sum) – but not an integral - over a range of index values is unity (zero) if the starting index value strictly exceeds the finishing value.

[11] A very similar framework could be developed using a pure discount bond numeraire. However, this would result in details of the model depending on the maturity date of the bond in question, which we regard as undesirable.

[12] We assume that the account is initiated at date t_0 by purchasing a unit pure discount bond maturing at T_0; at each successive T'_m the shortly maturing balance (including interest) is re-invested in pure discount bonds due at T_{m+1}.

Reflecting on the form of the martingale condition formula, (7), leads readily to the thought that one distributional assumption under which the mathematics of enforcing the martingale condition would be especially tractable would be to suppose that the conditional joint distribution of the various terms in the product inside the conditional expectation operator is multivariate lognormal.[13] That is to say, we define the increments in the log "one-plus-interest-rates":

$$X_{m,i+1} \equiv \ln\left(1+S_m(t_{i+1})\delta_m\right) - \ln\left(1+S_m(t_i)\delta_m\right);\ m = k(i+1),\dots,N-1;\ i = 0,1,\dots,J-1 \tag{8}$$

and suppose that the distribution of $\{X_{m,i+1}: m = k(i+1),\dots,N-1\}$, conditional on information up to date t_i is multivariate Normal with:

$$\mathrm{E}_i^*\left[X_{m,i+1}\right] \equiv \mu_{m,i};\quad m = k(i+1),\dots,N-1 \tag{9}$$

$$\mathrm{cov}_i^*\left[X_{m,i+1}, X_{n,i+1}\right] \equiv c_{m,n,i};\quad m,n = k(i+1)\dots,N-1 \tag{10}$$

where cov_i^* signifies conditional covariance taken at time t_i with respect to the risk-adjusted probabilities.

This assumption is far less strong than might initially appear, since it is the date t_i <u>one-step conditional</u> distribution that is given an assumed form. We shall see that we have great freedom in specifying the covariance structure, including dependence on the date t_i term structure. Thus the <u>unconditional</u> distribution of interest rate changes can take a vast variety of forms.

Given the assumed multivariate distribution, substituting (8)-(10) into (7) reduces the latter to:

$$\exp\left\{-\sum_{m=k(i+1)}^{n-1}\left(\mu_{m,i} + \tfrac{1}{2}\sum_{j=k(i+1)}^{n-1} c_{n,j,i}\right)\right\} = 1$$

Taking logarithms of this equation, we see that the sum indexed by m is zero for any n. This will be so if and only if the marginal contribution for each successive value of n is zero. This determines the conditional means in terms of the conditional covariances:

$$\mu_{n,i} = \tfrac{1}{2}c_{n,n,i} + \sum_{m=k(i+1)}^{n-1} c_{m,n,i};\quad n = k(i+1),\dots,N-1 \tag{11}$$

We summarise the argument of this Section so far in our fundamental discrete-time result:

THEOREM. 3.1 *Suppose that the conditional distribution at each date $t_i : i = 0,1,\dots,J-1$, of the increments over the next period in the log "one-plus-interest rates" defined by (8), $\{X_{m,i+1}: m = k(i+1),\dots,N-1\}$, are multivariate Normal with conditional means and conditional (co-)variances defined by (9)-(10). Then the model satisfies the martingale condition formula, (7), if and only if the conditional moments are linked by (11).*

Proof Immediate by the arguments immediately before the statement of the theorem. ■

[13] We discuss briefly other possibilities, leading to Conditional Non-Gaussian models, in Section 9.

Summarising formally the ingredients used in our argument up to this point leads to the following definition of our concept of discrete-time kernel:

DEFINITION. 3.2 The discrete-time "kernel" of a Conditional Gaussian model consists of:

(i) a sequence, $T_0 < T_1 < \ldots < T_N$, of fixed dates, together with corresponding interest rate determination dates, $T_0' < T_1' < \ldots < T_{N-1}'$;

(ii) a sequence, $0 = t_0 < t_1 < \ldots < t_J = T_{N-1}'$, of fixed observation dates, including $T_0', T_1', \ldots, T_N'$;

(iii) an initial, ie as at date $t_0 = 0$, discrete term structure consisting of a discount factor for the earliest "spot" date, $B(T_0, 0)$, and forward simple interest rates, $\{S_m(0) : m = 0, \ldots, N-1\}$;[14]

(iv) for each $i = 0, 1, \ldots$, a mapping, C_i, from $\mathbb{R}^{d(i)}$ to the space of $(N - k(i+1)) \times (N - k(i+1))$ covariance matrices, where $d(i) = \sum_{j=0}^{i} N - k(j)$;

(v) the assumption that, for each $i = 0, 1, \ldots, J-1$, the conditional distribution under P^* of the increments, $\{X_{m,i+1} : m = k(i+1), \ldots, N-1\}$, in the log one-plus-simple-interest-rates, is multivariate Normal with moments given by (9)-(10) and linked by (11), where $c_{m,n,i}$ is the $(m - k(i+1) + 1, n - k(i+1) + 1)$th element of C_i, evaluated at the observation history so far, ie of $C_i\left(S_0(t_0), \ldots, S_{N-1}(t_0), S_{k(1)}(t_1), \ldots, S_{N-1}(t_1), \ldots, S_{k(i)}(t_i), \ldots, S_{N-1}(t_i)\right)$.

REMARK Note, from (3), that our "kernel" concept determines future discount factors $B(T_n, t_i)$, only up to scaling by the discount factor, $B\left(T_{k(i)}, t_i\right)$. This is perfectly adequate for pricing at t_i of payoffs due at some T_n and depending only on forward simple interest rates observed up to T_n'. We will amplify this point in Section 6, where we discuss pricing caps and European swaptions based on the forward rates in the "kernel".

COROLLARY. 3.3 *Any "kernel" satisfies the martingale condition formula, (7).*

Proof Immediate by Theorem. 3.1. ■

It will be evident to the reader that Definition. 3.2 and Corollary. 3.3 open up a tremendously flexible modelling framework. The "recipe" is simple: specify conditional covariances $\{c_{m,n,i} : m, n = k(i+1), \ldots, N-1; i \geq 0\}$ however you wish; then the assumption that the increments $\{X_{m,i+1} : m = k(i+1), \ldots, N-1; i \geq 0\}$ defined by (8) have the specified covariances, and have means determined by (11), generates a model satisfying the martingale condition formula.

Note, in particular, that this "recipe" allows us to specify whatever conditional correlation structure we want, and to have the conditional variances depend in whatever way we wish upon

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[14] Or equivalently, $B(T_0, 0)$ and further pure discount bond prices $\{B(T_n, 0) : n = 1, \ldots, N\}$.

the evolution of the term structure up to date t_i. Thus, for example, we could incorporate GARCH effects by making the conditional variances depend on the magnitude of past innovations. Moreover, whatever specification we adopt, the model will be directly amenable to Monte Carlo simulation since, for each *i*, the increments $\{X_{m,i+1}: m \geq k(i+1)\}$ required to generate forward rates at date t_{i+1} from that at t_i are conditionally multivariate Normally distributed.

4. CONTINUOUS-TIME WHOLE YIELD CURVE MODEL

It is certainly of theoretical interest to ask whether the conceptual framework set out above can be embedded in a continuous time whole yield curve model of the kind independently pioneered by Babbs [1990] and Heath, Jarrow and Morton [1992]. Moreover, embedding our discrete-time "kernel" in a continuous-time whole yield curve model will enable us - see Section 7 - to price consistently payoffs which depend upon the term structure at dates other than $t_0, t_1, \ldots, t_J$ and/or upon rates other than the finite set of forward LIBORs $\{S_m(t_i): m = k(i), \ldots, N-1; i = 0, 1, \ldots, J\}$.

In this section, we construct one way in which such an embedding can be achieved. In this construction, the conditional risk-adjusted distribution at any t_i of spot rates movements [15] occurring during the interval until t_{i+1}, is multivariate Normal. This reinforces the motivation for referring to our class of models as "Conditional Gaussian" (CG).

Before detailing our construction, we offer the following intuition.

Summing (8) over an appropriate range of values, and replacing simple interest rates by price ratios of relevant pure discount bonds we obtain:

$$\frac{B(T_{n+1}, t_k)}{B(T_n, t_k)} = \frac{B(T_{n+1}, t_j)}{B(T_n, t_j)} \exp\left\{ -\sum_{i=j}^{k-1} X_{n,i+1} \right\}$$

Observe that the first suffix on the *X* is *n*, throughout the summation on the RHS of the above equation. Moreover, *n* appears as that suffix only when it is the *n*th forward simple interest rate whose evolution we are considering. Observe also that, under our risk-adjusted probabilities, the conditional distribution of $X_{n,i+1}$ as at t_i is Normal, so that $X_{n,i+1}$ could be considered as the increment of an Ito process whose drift and diffusion coefficients are known at t_i.

These two observations prompt the conjecture that we are close to a multi-factor linear Gaussian model, in which the *q*th factor impacts instantaneous forward interest rates applying only to the period $(T_{q-1}, T_q]$; the only major point of difference being that the linear Gaussian behaviour holds piecewise – ie over intervals $(t_i, t_{i+1}]$ - with the coefficients being revised at each successive t_i.

We now provide a detailed construction along the lines of the above conjecture, representing innovations in each of the correlated factors by a linear combination of innovations in independent Brownian motions.

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[15] "Spot rates" here carries the academic meaning of continuously compounding yields on pure discount bonds.

The first task we undertake is to build a class of continuous-time models; our second task is to show that discrete time "kernels" can be embedded in models from this class. We commence the former task by adopting the following as primitive:

(i) sequences of dates as in items (i) and (ii) of Definition. 3.2

(ii) a filtered probability space $\left(\Omega,\mathcal{F},\left\{\mathcal{F}_t: t\in\left[0,T'_{N-1}\right]\right\},P\right)$ satisfying the "usual conditions"[16], and supporting N independent standard Brownian motions, $Z_1,\ldots,Z_N$;

(iii) a differentiable and non-vanishing initial term structure, $B(\bullet,0)$, consistent with the initial earliest spot discount factor $B(T_0,0)$ and forward simple interest rates, $S_1(0),\ldots,S_{N-1}(0)$;

(iv) for each $i=0,1,\ldots,J-1$, a mapping, $\hat{C}_i$, from $\mathbb{R}^{d(i)}$ to the space of $\left(N-k(i+1)+1\right)\times\left(N-k(i+1)+1\right)$ covariance matrices;[17]

(v) predictable "market price of risk" processes, $\theta_1,\ldots,\theta_N$.

Note that, whereas the discrete time framework set out in Section 3 requires covariances only for $m,n=k(i+1),\ldots,N-1$, the second "primitive" item above extends the covariance matrix to $m,n=k(i+1)-1,\ldots,N-1$. By this means, we provide for stochastic behaviour during $\left(t_i,t_{i+1}\right]$ of bonds maturing strictly before $T_{k(i+1)}$. For $m,n=k(i+1)-1,\ldots,N-1$, we define $\hat{c}_{m,n,i}$ as the $\left(m-k(i+1)+2,n-k(i+1)+2\right)$th element of $\hat{C}_i\left(S_0(t_0),\ldots,S_{N-1}(t_0),\ldots,S_{k(i)}(t_i),\ldots,S_{N-1}(t_i)\right)$

Since the range of each $\hat{C}_i$ mapping consists of covariance matrices, we can decompose the corresponding correlation matrices in terms of weightings to apply to uncorrelated increments. Choosing one such decomposition for each i, we obtain mappings

$$\alpha_{m,i,q}:\mathbb{R}^{d(i)}\to\mathbb{R};\qquad m=k(i+1)-1,\ldots,N-1;\quad q=1,\ldots,N$$

such that[18]

$$q<k(i+1)\Rightarrow\alpha_{m,i,q}\equiv 0;\qquad m=k(i+1)-1,\ldots,N-1$$

and, at $\left(S_0(t_0),\ldots,S_{N-1}(t_0),S_{k(1)}(t_1),\ldots,S_{N-1}(t_1),\ldots,S_{k(i)}(t_i),\ldots,S_{N-1}(t_i)\right)$,

$$\sum_{q=k(i+1)}^{N}\alpha_{m,i,q}\alpha_{n,i,q}=\frac{\hat{c}_{m,n,i}}{\sqrt{\hat{c}_{m,m,i}\hat{c}_{n,n,i}}};\qquad m,n=k(1+1)-1,\ldots,N-1 \tag{12}$$

We now proceed inductively. Trivially, our model is well-defined up to date t_0 with instantaneous forward rates:

$$f(M,t_0)=-\frac{\partial}{\partial m}\ln B(M,0)\ ;\ 0\le M\le T_N$$

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[16] see eg Karatzas and Shreve [1987] Definition 2.25 p10

[17] The notation $\hat{C}_i$ rather than C_i signifies the addition of an initial row and column to the latter.

[18] One effect of this is that interest rates applying to a date $t\in\left(T'_m,T_m\right]$ are deterministic after $t_{i(m)}=T'_m$. This is harmless for all obvious applications, but could be avoided at the cost of some rather fiddly extra complication.

Assuming our model is well-defined up to date t_i, with instantaneous forward rates $f(\bullet,t_i)$, implying forward simple interest rates, $S_{k(i)}(t_i),\ldots,S_{N-1}(t_i)$,[19] we extend our model up to date t_{i+1} by setting, for any $M \geq t \in (t_i,t_{i+1}]$,

$$f(M,t) = f(M,t_i) + \int_{t_i}^{t} a_i(s,M)ds + \sum_{q=1}^{N} \int_{t_i}^{t} b_{i,q}(M)dZ_q(s) \tag{13}$$

where

$$a_i(s,M) \equiv \sum_{q=1}^{N} b_{i,q}(M)\left\{\theta_q(s) + \int_s^M b_{i,q}(u)\,du\right\}$$

$$b_{i,q}(M) \begin{cases} \equiv 0 & ; \quad M \leq T_{k(i+1)-1} \\ \equiv \dfrac{1}{T_n - T_{n-1}} \sqrt{\dfrac{\hat{c}_{n-1,n-1,i}}{t_{i+1}-t_i}}\, \alpha_{n-1,i,q} & ; \quad M \in (T_{n-1},T_n];\, n = k(i+1),\ldots,N \end{cases} \tag{14}$$

Repeating this extension for successive values of i completes our construction.

We now establish the two main results of this section. The first of these, Theorem 4.1, confirms that the above construction delivers an arbitrage-based continuous-time whole yield curve model which is capable of being supported in general equilibrium:

THEOREM. 4.1 *Construct a term structure model as just described. Then the price dynamics of the pure discount bond maturing at any M>0 are given by:*

$$\frac{dB(M,t)}{B(M,t)} = \left\{r(t) + \sum_{q=1}^{N} \theta_q(t)\sigma_q(M,t)\right\}dt + \sum_{q=1}^{N} \sigma_q(M,t)dZ_q(t) \tag{15}$$

where

$$\sigma_q(M,t) \equiv -\int_t^M b_{i,q}(u)\,du; \qquad t \in (t_i,t_{i+1}];\, i = 0,1,\ldots,J-1 \tag{16}$$

and $r(\,)$ *is the spot instantaneous interest rate process:*

$$r(t) = \lim_{M \downarrow t} \frac{-\ln B(M,t)}{M-t}$$

Moreover, straightforward sufficient conditions for the model to be viable (ie capable of being supported in general equilibrium) are that each $\theta_q(\,)$ *and* $\hat{c}_{m,n,j}$ *be essentially bounded.*

Proof See Appendix. ■

REMARK Necessary and sufficient conditions for viability could be derived from the trivial multifactor extension of Theorem I on pp140-1 of Babbs [1990] (or equivalently, Theorem 3 on

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[19] The relationship, of course, is $1 + S_n(t_i)\delta_n = \dfrac{B(T_n,t_i)}{B(T_{n+1},t_i)} = \exp\left\{\int_{T_n}^{T_{n+1}} f(u,t_i)\,du\right\}$

pp262-3 of Babbs [1997]). However, essential boundedness can readily be met – or harmlessly imposed - in practice, and, as pointed out in the Theorem, is straightforward.

Our second main result shows that a discrete-time "kernel", as described by Definition. 3.2, can be embedded in a continuous-time model such as we have constructed above. Firstly, however, we make a convenient definition:

DEFINITION. 4.2 Say that the covariance mappings $\{\hat{C}_i : i = 0,1,\ldots,J-1\}$ "extend" $\{C_i : i = 0,1,\ldots,J-1\}$ if, for $m,n = k(i+1),\ldots,N-1$, the $(m-k(i+1)+2, n-k(i+1)+2)$th element of $\hat{C}_i$ coincides with the $(m-k(i+1)+1, n-k(i+1)+1)$th element of C_i.

Clearly, at least trivial extensions exist.

THEOREM. 4.3 *Take as given a discrete-time "kernel" as described in Definition. 3.2, and assume that the earliest "spot" discount factor and forward simple rates are taken from a differentiable and non-vanishing initial term structure, $B(\bullet,0)$. Let $\{\hat{C}_i : i = 0,1,\ldots,J-1\}$ be any extension of $\{C_i : i = 0,1,\ldots,J-1\}$, and let $\theta_1,\ldots,\theta_N$ be predictable processes on a filtered probability space satisfying the "usual conditions" and supporting N independent standard Brownian motions, $Z_1,\ldots,Z_N$. Then, taking these entities as primitive, Theorem. 4.1 holds.*

Under risk-adjusted probabilities corresponding to using as numeraire the same discretely re-invested money-market account as in Section 3, with continuous-time price process:

$$A(t)\begin{cases} = B(T_0,0) & ;\ t = t_0 = 0 \\ = B\left(T_{k(i+1)},t\right)\displaystyle\prod_{m=0}^{k(i+1)-1}\left(1+S_m(T'_m)\delta_m\right); & t_i < t \le t_{i+1}\ ; i = 0,\ldots,J-1 \end{cases} \tag{17}$$

the variables $\{X_{m,i+1} : m = k(i+1),\ldots,N-1\}$ defined for each j by (8) have a conditional multivariate Normal distribution at time t_i with means and (co-)variances given by (11) and (10) respectively.

Moreover, under those risk-adjusted probabilities, the conditional distribution at t_i of changes in spot rates over any sub-interval of $(t_i, t_{i+1}]$ is also Gaussian.

Proof See Appendix. ■

Note that while, for the purposes of Theorem. 4.3, it suffices that $\hat{C}_i$ be <u>any</u> extension of C_i, the following lemma establishes that the additional elements of $\hat{C}_i$ have a clearcut interpretation, which can be used to guide their specification in contexts – eg when constructing models under objective probabilities as in Section 5 below – where the additional elements are relevant to the analysis.

LEMMA. 4.4 *The additional variance element,* $\hat{c}_{k(i+1)-1,k(i+1)-1,i}$, *is proportional to the instantaneous variance of very short-term rates; thus setting* $n = k(i+1)-1$, *we have:*[20]

$$\hat{c}_{n,n,i} = (T_{n+1}-T_n)^2(t_{i+1}-t_i)\frac{\operatorname{var}_i[df(u,t)]}{dt}; \; u \in (t_i \vee T_n, T_{n+1}]; \; t \in [t_i, t_{i+1}) \tag{18}$$

and the additional implicit correlations equal the instantaneous correlations between forward rates for dates in the relevant time segments:

$$\frac{\hat{c}_{m,n,i}}{\sqrt{\hat{c}_{m,m,i}\hat{c}_{n,n,i}}} = \operatorname{corr}_i[df(s,t), df(u,t)]; \; s \in (T_m, T_{m+1}]; \; u \in (t_i \vee T_n, T_{n+1}]; \; t \in [t_i, t_{i+1}) \tag{19}$$

Proof A straightforward consequence of substituting (14) and (12) into (13). ■

5. THE DISCRETE-TIME KERNEL UNDER OBJECTIVE PROBABILITIES

If instances of our framework – with or without GARCH effects – are to be fully amenable to econometric testing, we must re-cast our discrete-time "kernel" (as defined in Definition. 3.2) in a tractable form, under objective, as opposed to risk-adjusted, probabilities.

To achieve this, we embed the kernel in a continuous-time model as in Section 4, under the restriction that the market prices of risk throughout each successive $(t_i, t_{i+1}]$ are known at t_i:

THEOREM. 5.1 *Under the conditions of Theorem. 4.3, and under the assumption that each* θ_i *is determined on each successive* $(t_i, t_{i+1}]$ *by the observation history to date,* $(S_m(t_j): m = k(j),\dots,N-1; j = 0,1,\dots,i)$, *the conditional distribution at* t_i *of* $\{X_{m,i+1}: m = k(i+1),\dots,N-1\}$, *under the objective probabilities, P, is multivariate Normal with:*

$$\mathrm{E}_i[X_{m,i+1}] = \mu_{m,i} + \left[1 - \frac{\left\{\frac{1}{2}[t_{i+1}+t_i^*] - T_{k(i+1)-1}\right\}\{t_{i+1}-t_i^*\}}{\left(T_{k(i+1)} - T_{k(i+1)-1}\right)(t_{i+1}-t_i)}\right]\hat{c}_{m,k(i+1)-1,i} + \tilde{\theta}_{m,i}\sqrt{(t_{i+1}-t_i)c_{m,m,i}} \tag{20}$$

$$\operatorname{cov}_i[X_{m,i+1}, X_{n,i+1}] = c_{m,n,i} \tag{21}$$

where $t_i^* \equiv t_i \vee T_{k(i+1)-1}$ *and*

$$\tilde{\theta}_{m,i} \equiv \sum_{q=1}^{N} \alpha_{m,i,q} \int_{t_i}^{t_{i+1}} \frac{\theta_q(t)}{t_{i+1}-t_i}\,dt$$

Proof See the Appendix. ■

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[20] Recall that forward instantaneous interest rates applying to dates in (T_n', T_n) are modelled as being non-stochastic after $T_n' = t_{i(n)}$ for any n; hence the $t_i \vee T_{k(i+1)-1}$ in (18) and (19).

A number of points need to be made about this result. Firstly, the apparent messiness of the coefficient of $\hat{c}_{m,k(i+1)-1,i}$ in (20) is merely a consequence of the fact that distance between the current observation date t_i and the next date, $T_{k(i)}$, in the payment date sequence $T_0 < T_1 <\ldots< T_N$ varies from one observation date to another; in the simplified case where the observation dates and payment dates coincide and are at perfectly regular intervals, the coefficient would reduce to $\frac{1}{2}$. Secondly, $\widetilde{\theta}_{m,i}$ can be interpreted as the market price of the risk of changes to instantaneous forward rates for dates in the interval $(T_m, T_{m+1}]$, averaged over the period of risk exposure from t_i to t_{i+1}. Thirdly, in specifying a model under the objective probabilities, we can work directly with the $\widetilde{\theta}_{m,i}$; this is most helpful, since it will be recalled that the orthogonalization of the covariance matrix in Section 4 was chosen arbitrarily. The final point to note is the appearance in (20) of $\hat{c}_{m,k(i+1)-1,i}$, representing the covariance between the forward rates in the kernel, and very short term rates – see Lemma. 4.4.

6. VALUING "KERNEL" CLAIMS

We define a "kernel" contingent claim as one having a single payoff, at a fixed date, T_M say, in the set $\{T_0,..,T_N\}$ forming part of the discrete-time "kernel" of the model, in an amount, Y say, which depends solely upon observations of the forward simple interest rate processes $\{S_m(\bullet): m = 0,\ldots,N-1\}$ at the observation dates $0 = t_0 < t_1 <\ldots< t_{i(M)} = T'_M$:

$$Y = y\left(S_0(t_0),\ldots,S_{N-1}(t_0),\ldots,S_{i(M)}(T'_M),\ldots,S_{N-1}(T'_M)\right) \tag{22}$$

We established, in Section 4, that discrete-time "kernels" of our Conditional Gaussian family of models can be embedded in viable continuous-time models of the whole yield curve. So why pay special attention to these "kernel" claims? Firstly, the interest rate caps and European swaptions to which practitioners would wish to calibrate a term structure model, are all "kernel" claims (or combinations thereof) if the dates $T_0,\ldots,T_N$ are chosen so as to include the cashflow dates of the caps and underlying forward starting swaps concerned – see Examples below.[21] Secondly, in some existing "market models" (eg Jamshidian [1997]), and in the "Markov functional model" of Hunt, Kennedy and Pelsser [1998], only such "kernel" claims (or combinations thereof) are modelled. For both these reasons, it is of especial interest to see what is involved in computing values of such claims.

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[21] We should concede at this point that we, in common with the entire "market models" literature, finesse the point that, under standard market conventions, the reference interest rate used to determine the caplet payoff at T_M say will not always relate to a period ending precisely at T_M. The practical significance of this for option modelling purposes is insignificant. This is fortunate, since picking at this seam would risk breaking the thread of abutting interest periods upon which market models focus, whereupon the whole fabric would unravel.

Adopting, as in Section 3, the discretely reinvested money market account as numeraire, it is immediate from that Section that the usual martingale valuation formula will take the form:

$$V(0) = A(0)\,\mathrm{E}^*\left[\frac{Y}{A(T_M)}\right]$$

$$= B(T_0,0)\,\mathrm{E}^*\left[y\left(S_0(0),\ldots,S_{N-1}(0),\ldots,S_{i(M)}(T'_M),\ldots,S_{N-1}(T'_M)\right)\prod_{m=1}^{M-1}\left(1+S_m(T'_m)\delta_m\right)^{-1}\right] \tag{23}$$

Thus, valuing the claim is precisely the task of computing the expectation of a function of the set of variables $\{S_m(t_i) : m = k(i),\ldots,N-1; i = 0,\ldots,i(M)\}$. But given (8),

$$S_m(t_i) = \frac{1}{\delta_m}\left[(1+S_m(0)\delta_m)\exp\left\{\sum_{j=0}^{i-1} X_{m,j+1}\right\} - 1\right]$$

so this computation is equivalent to computing the expectation of a function of (the initial forwards, $S_\bullet(0)$, and) the set of variables $\{X_{m,i+1} : m = k(i+1),\ldots,N-1; i = 0,\ldots,i(M)-1\}$.

As we noted at the end of Section 3, this is directly amenable to Monte Carlo simulation since, for each i, the increments required to generate forward rates at the next observation date, t_{i+1}, from the forward rates at t_i, are conditionally multivariate Normally distributed. The conditional moments pose us no problem, since, by Definition. 3.2, the moments during the ith time period are given by (9)-(11) as depending only on $\{X_{m,j+1} : m = k(j+1),\ldots,N-1; j = 0,\ldots,i-1\}$ through the covariance matrix $C_i\left(S_0(t_0),\ldots,S_{N-1}(t_0),S_{k(1)}(t_1),\ldots,S_{N-1}(t_1),\ldots,S_{k(i)}(t_i),\ldots,S_{N-1}(t_i)\right)$. The random drawings of the $X_{m,j+1}$ can be made, of course, using their exact multivariate Normal distribution; this contrasts with the situation in many market models (eg Jamshidian [1997]) where sampling such increments requires discretizing a vector SDE, thereby introducing discretization errors alongside statistical sampling issues.[22]

EXAMPLE. 6.1: CAP Consider a typical individual constituent caplet within a standard interest rate cap. The payoff per unit notional principal occurs at T_{L+1} say, in an amount $\max\{S_L(T'_L)-K,0\}\delta_L$ where K is the fixed strike level. This payoff is obviously of the form (22).

EXAMPLE. 6.2: EUROPEAN-STYLE SWAPTION A standard European-style receiver's swaption contract gives the option hold the right, at a given interest determination date, T'_L say, to enter an interest rate swap commencing at the corresponding "spot" date, T_L, and having "fixed leg" receipts at $T_{L+f} < T_{L+2f} \cdots < T_{L+nf}$ at a fixed interest rate K, against some sequence of "floating leg" payments at LIBOR. Applying the commonplace valuation assumption that this underlying swap can be valued as if decomposed into a notional fixed rate bond with coupon rate K and a floating rate note worth par, the value at T'_L of unit notional principal of this underlying swap is[23]

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[22] See Glasserman and Zhao [1998] for the complexities of addressing the discretization problem.

[23] We assume in (24) that fixed rate interest accrues in the swap on the same basis as floating interest. The modifications required for alternative interest rate conventions are obvious.

$$U=\sum_{m=1}^{n}\left(\delta_{L+(m-1)f}+\delta_{L+(m-1)f+1}\cdots+\delta_{L+mf-1}\right)KB\left(T_{L+mf},T_L'\right)+B\left(T_{L+nf},T_L'\right)-B\left(T_L,T_L'\right) \tag{24}$$

Hence, the payoff per unit notional principal of the swaption is $\max\{U,0\}$ at T_L', or equivalently, $Y=\max\{U,0\}/B(T_L,T_L')$ due at the "spot" date T_L. Substituting (24), we can re-write Y as:

$$Y=\max\left\{\sum_{m=1}^{n}\left(\delta_{L+(m-1)f}+\delta_{L+(m-1)f+1}\cdots+\delta_{L+mf-1}\right)K\frac{B\left(T_{L+mf},T_L'\right)}{B\left(T_L,T_L'\right)}+\frac{B\left(T_{L+nf},T_L'\right)}{B\left(T_L,T_L'\right)}-1,0\right\}$$

which takes the form (22) upon noting from (3) that

$$\frac{B\left(T_{L+mf},T_L'\right)}{B\left(T_L,T_L'\right)}=\prod_{m=L}^{L+mf-1}\left(1+S_m(T_L')\delta_m\right)^{-1}$$

7. VALUING "NON-KERNEL" CLAIMS

In Section 6, we showed that the valuation of what we termed "kernel" claims - ie whose payoffs occur at some T_M in an amount $y\left(S_0(0),\ldots,S_{N-1}(0),\ldots,S_{i(M)}(T_M'),\ldots,S_{N-1}(T_M')\right)$ - is directly amenable to Monte Carlo simulation, because the underlying variables requiring simulation are precisely the successive sets of multivariate Normal random variables, $\{X_{m,i+1}:m=k(i+1),\ldots,N-1\}$ for $i=0,\ldots,i(M)$. In the present section, we extend our analysis to show that the valuation of a broad class of other, "non-kernel", claims is equally directly amenable to Monte Carlo simulation.

The practical significance of this is that, once a Conditional Gaussian model has been "calibrated" to the prices of a set of "kernel" claims, the analysis in this Section will provide the means for the valuation of claims depending upon interest rates other than those in the kernel, or upon observations at other dates.

The class of "non-kernel" claims we consider consists of linear combinations of claims whose payoff structure can be expressed as an amount Y at a single fixed date, L_n say, where Y can be written as a function of the term structures at a sequence of fixed dates $L_1<\ldots<L_n$, ie: $Y=y\left(B(\bullet,L_1),\ldots,B(\bullet,L_n)\right)$. Obviously, interest rate caps and European-style swaptions are among the simpler instruments in this class.

To avoid lengthening this paper unduly, we will leave the reader to explore the payoff structures of individual instruments. The following results provide the keys to setting up the subsequent Monte Carlo valuation, by showing that the term structure at an arbitrary future date can be expressed in terms of the initial term structure and of discrete increments in processes that are standard Brownian motions under relevant risk-adjusted probabilities. The first of the two results is, in principle, quite sufficient. However, since it often turns out to be convenient to utilise a pure discount bond numeraire, rather than a money market one, when computing valuations, we offer the second result also, by way of alternative.

THEOREM. 7.1 *Given the covariance mappings, $\hat{C}_i : i = 0,\ldots,J-1$, the term structure at an arbitrary date, $t \in (t_i, t_{i+1}]$ say, is determined by the initial term structure, $B(\bullet,0)$, and the increments:*

$$Z_q^*(t_{j+1}) - Z_q^*(t_j) : q = k(j+1)-1,\ldots,N;\ j = 0,\ldots,i-1 \quad \textit{and} \quad Z_q^*(t) - Z_q^*(t_i) : q = k(i+1)-1,\ldots,N$$

Specifically,

$$\ln B(M,t) = \ln\frac{B(M,t_i)}{B(t,t_i)} - \tfrac{1}{2}(t-t_i) \sum_{q=k(i+1)-1}^{N} \left\{ \left\{ \int_{T_{k(i+1)}}^{M} b_{i,q}(u)\,du \right\}^2 - \left\{ \int_t^{T_{k(i+1)}} b_{i,q}(u)\,du \right\}^2 \right\}$$
$$- \sum_{q=k(i+1)-1}^{N} \int_t^M b_{i,q}(u)\,du \left\{ Z_q^*(t) - Z_q^*(t_i) \right\} \qquad (25)$$

Proof See the Appendix. ■

REMARK It can readily be shown that the distribution of the RHS of (25) is independent of the specific choice of orthogonalisation involved prior to the construction of the embedding continuous-time model. For computational purposes, we consider the result to be most conveniently expressed in the form stated. It is important to realise, moreover, that the functions $b_{i,q}(\bullet)$ are constant between successive interest payment dates T_n - see (14) – enabling the various integrals, both in (25) and in the corresponding equation in Theorem. 7.2 below, to be carried out analytically.

THEOREM. 7.2 *Suppose we adopt the pure discount bond maturing at L as numeraire, in place of the discretely re-invested money-market account, and that $P^{(L)}$ is a risk-adjusted probability measure corresponding to the new numeraire. Then, given the covariance mappings, $\hat{C}_i : i = 0,\ldots,J-1$, the term structure at an arbitrary date, $t \in (t_i, t_{i+1}]$ say, with $t \le L$, is determined by the initial term structure, $B(\bullet,0)$, and the increments:*

$$Z_q^{(L)}(t_{j+1}) - Z_q^{(L)}(t_j) : q = k(j+1)-1,..,N;\ j = 0,..,i-1 \quad \textit{and} \quad Z_q^{(L)}(t) - Z_q^{(L)}(t_i) : q = k(i+1)-1,..,N$$

where $Z_1^{(L)},\ldots,Z_N^{(L)}$ are independent standard Brownian motions under $P^{(L)}$, such that:

$$dZ_q^{(L)}(s) = \int_{T_{k(i+1)}}^{L} b_{i,q}(u)\,du\,ds + dZ_q^*(s) : \quad s \in (t_i, t_{i+1}];\ i = 0,\ldots,J-1;\ q = 1,\ldots,N$$

Specifically, we may re-write (25) as:

$$\ln B(M,t) = \ln\frac{B(M,t_i)}{B(t,t_i)} - \tfrac{1}{2}(t-t_i) \sum_{q=k(i+1)}^{N} \left\{ \left\{ \int_{T_{k(i+1)}}^{M} b_{i,q}(u)\,du \right\}^2 - \left\{ \int_t^{T_{k(i+1)}} b_{i,q}(u)\,du \right\}^2 \right\}$$
$$-(t-t_i) \sum_{q=k(i+1)}^{N} \int_t^M b_{i,q}(u)\,du \int_{T_{k(i+1)}}^{L} b_{i,q}(u)\,du \; - \sum_{q=k(i+1)}^{N} \int_t^M b_{i,q}(u)\,du \left\{ Z_q^{(L)}(t) - Z_q^{(L)}(t_i) \right\}$$

Proof Define the normalized bond price, with the new numeraire, as $B^{(L)}(M,t) \equiv B(M,t) \div B(L,t)$. Substituting on the RHS using (25), the result follows from the fact that $B^{(L)}(M,t)$ must be a $P^{(L)}$-martingale. ■

8. CHOOSING A SPECIFICATION

We provide here two sketches of how one might move from the general Conditional Gaussian framework set out in previous sections, towards a specific concrete model. Both examples could be made more complex, or varied in other ways, and their ideas could be combined – eg to produce a derivatives valuation framework with GARCH effects.

EXAMPLE. 8.1: DERIVATIVES VALUATION, WITHOUT GARCH EFFECTS

We can re-arrange (8) as:

$$X_{m,i+1} = \ln\left\{1+\frac{\left(S_m(t_{i+1})-S_m(t_i)\right)\delta_m}{1+S_m(t_i)\delta_m}\right\}$$

Now, except in extreme scenarios, the fraction inside the RHS of this equation will be small, so

$$X_{m,i+1} \cong \frac{\left(S_m(t_{i+1})-S_m(t_i)\right)\delta_m}{1+S_m(t_i)\delta_m};\ m,n=k(i+1),\ldots,N-1$$

and hence

$$c_{m,n,i} \cong \frac{\delta_m}{1+S_m(t_i)\delta_m}\frac{\delta_n}{1+S_n(t_i)\delta_n}\operatorname{cov}_i^*\left[S_m(t_{i+1})-S_m(t_i),\,S_n(t_{i+1})-S_n(t_i)\right]$$

A common assumption for the conditional standard deviation of interest rate changes (cf Chan, Karolyi, Longstaff and Saunders [1992] (CKLS) and Nowman [1997]) is that it is a power function of the rate level. As regards the behaviour of rates of different maturities, evidence (cf Litterman and Scheinkman [1991], Heath, Jarrow, Morton and Spindel [1992], Amin and Morton [1994], Bliss and Ritchken [1996] and Goncalves and Issler [1996]) suggests either monotonic decline of variance as maturity increases, or else a humped pattern. Correlations between movements between one forward rate and another tend to decrease as the periods to which the forwards relate are separated further, and to increase as the nearer period becomes increasingly distant. Finally, option prices invariably tend to suggest that prospective variability in rates is likely to be greater in some future periods than in others.

One formulation that captures all the above considerations, while not tying us to the power function of CKLS is:[24]

$$c_{m,n,i} = \frac{\delta_m f\left(S_m(t_i),T_m'-t_i\right)}{1+S_m(t_i)\delta_m}\frac{\delta_n f\left(S_n(t_i),T_n'-t_i\right)}{1+S_n(t_i)\delta_n}\rho\left(\left|T_m'-T_n'\right|,\min\{T_m',T_n'\}-t_i\right)\left(t_{i+1}-t_i\right)\sigma_i^2 \qquad (26)$$

In this formula, the function f captures the dependence of the conditional standard deviations of increments in rates upon the levels of rates and upon their tenors. For example, it might be supposed that f might be the product of a power function of rate with some monotonically declining or humped function of maturity.

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[24] An alternative attractive formulation might substitute the rate of a particular tenor, eg the current nearest one-period rate, $S_{k(i)}(t_i)$, for $S_m(t_i)$ and for $S_n(t_i)$ on the RHS of (26).

The function ρ captures the way in which the correlation between movements in forward rates for different periods depends upon the gap between the periods and upon the tenor of the nearer period. It might be set using values obtained from a principal components analysis of past correlations. Alternatively, it could be assumed to take some parameterized functional form designed to capture the general features of historic correlation patterns, and the parameter settings chosen as part of a calibration to current cap and swaption prices.

The sequence of values of σ reflects the anticipated pattern of the degree of interest rate variability in different future epochs. The length of the period $(t_i, t_{i+1}]$ permits the value of σ_i to be expressed in annualised terms, rather than depending on period length. Obvious simple ways of specifying $\sigma_1, \sigma_2, \ldots$ include "bucketing" various periods together and using a step function, or employing some simple functional form, $\sigma_i = \sigma(t_i)$.

EXAMPLE. 8.2: A SIMPLE GARCH MODEL UNDER OBJECTIVE PROBABILITIES

To model the covariance structure, we might proceed along the lines of Example. 8.1, adopting the functional form suggestion for ρ, and replacing the suggested schemes for σ_i by the following simple GARCH effects model, in which conditional variances depend on the most recent past innovation in the current nearest one-period rate:

$$\sigma_i^2 = \alpha_0 + \alpha_1 \left\{ S_{k(i)}(t_i) - \mathrm{E}_{i-1}\left[S_{k(i)}(t_i) \right] \right\}^2 \tag{27}$$

where α_0, α_1 are positive constants to be estimated and $\mathrm{E}_{i-1}\left[S_{k(i)}(t_i) \right]$ can be obtained from (20) and the relation:[25]

$$1 + \mathrm{E}_{i-1}\left[S_{k(i)}(t_i) \right] \delta_{k(i)} = \left(1 + S_{k(i)}(t_{i-1}) \delta_i \right) \exp\left\{ \mathrm{E}_{i-1}\left[X_{k(i),i} \right] + \tfrac{1}{2} c_{k(i),k(i),i-1} \right\} \tag{28}$$

It remains to specify the market prices of risk, $\tilde{\theta}_{m,i}$, whose stationary form is $\tilde{\theta}_{m,i} = \tilde{\theta}(T'_m - t_i)$. Setting $\tilde{\theta}(s) = h_0 + h_1 e^{-h_2 s}$ for some constants h_0, h_1, h_2 with $h_2 > 0$ would seem a reasonably flexible but simple choice.

9. CONCLUDING REMARKS

The key to this paper was the observation that the discrete-time martingale condition formula, (7), could be enforced very straightforwardly if we imposed the Conditional Gaussian assumption, ie that increments in log one-plus-interest-rates are conditionally multivariate Normal. Indeed, as Corollary. 3.3 showed, we are free to select <u>any</u> covariance structure, while Theorem. 4.3 showed that any choice could be embedded in a continuous-time whole yield curve model consistent with general equilibrium. At the same level of generality, Theorem. 5.1 showed how to recast the discrete-time "kernel" of a Conditional Gaussian model under the objective probabilities, for econometric purposes; Sections 6-7 demonstrated how Conditional Gaussian models lend

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[25] To establish (28), apply (8) to get $1 + S_{k(i)}(t_i)\delta_{k(i)} = \left(1 + S_{k(i)}(t_{i-1})\delta_{k(i)}\right)\exp\left\{X_{k(i),i}\right\}$ and take conditional expectations.

themselves to Monte Carlo valuation of contingent claims without the issues of discretization error arising in other "market models". Finally, Section 8 sketched some possible ways in which one might move towards specific concrete models within the Conditional Gaussian framework.

We acknowledge that we have not discussed calibration of Conditional Gaussian models to market prices of caps and swaptions, nor the valuation of instruments with early exercise or other discretionary features (eg Bermudan swaptions). We have nothing to report on these topics, save that, like Andersen and Andreasen [1998] we regard calibration of models relying on Monte Carlo valuation as "computationally feasible, albeit slow".

We intend to extend the framework developed in this paper. While the Conditional Gaussian assumption is a convenient way of enforcing the martingale condition formula, (7), it is certainly not the only way. We are beginning to explore alternative assumptions, leading to "Conditional Non-Gaussian" models.

APPENDIX

Proof of Theorem. 4.1 Define

$$a(t,M) \equiv a_i(t,M),\ b_q(t,M) \equiv b_{i,q}(M); \qquad t \in (t_i, t_{i+1}];\ i = 0,1,\ldots,J-1$$

Then the versions of (13) for successive values of j can be chained together to give

$$f(M,t) = f(M,0) + \int_0^t a(s,M)ds + \sum_{q=1}^{N} \int_0^t b_q(s,M)dZ_q(s)$$

It is trivial to verify that $a()$ and each $b_q()$ satisfy the regularity conditions of the Theorem on pp115-6 of Babbs [1990] (equivalently the Proposition and Theorem 1 on pp257-8 of Babbs [1997])[26]. The first part of the present theorem follows immediately.

Viability flows from the results in Section 3 of Chapter 5 in Babbs [1990] (pp140-144), (equivalently pp259-263 of Babbs [1997]), [27] since essential boundedness of each $\theta_q()$ and each $\hat{c}_{m,n,i}$ ensures[28] that (35) and (37) in Babbs'[1990] (equivalently (21) and (23) in Babbs [1997]) define square integrable martingales under P and P^* respectively. ■

Proof of Theorem. 4.3 The first part of the theorem is trivial.

For the second part, commence by noting that it is trivial to verify that (17) is indeed the continuous-time price process of the same numeraire as that used in Section 3. We now define the normalised pure discount bond price process $B^*(M,t) \equiv B(M,t) \div A(t)$, apply Ito's lemma, and substitute using (15) to obtain

$$\frac{dB^*(M,t)}{B^*(M,t)} = \sum_{q=1}^{N} \left\{\sigma_q(M,t) - \sigma_q\left(T_{k(i+1)},t\right)\right\}\left[\left\{\theta_q(t) - \sigma_q\left(T_{k(i+1)},t\right)\right\}dt + dZ_q(t)\right]$$

which (invoking Girsanov's theorem and the fact that normalised security prices must be martingales under risk adjusted probabilities) must be equivalent to

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[26] The cited result extends trivially to the multifactor case by summing over i in the obvious places.

[27] Again extended trivially to the multifactor case.

[28] To see this, simply apply the Novikov condition (see eg Karatzas and Shreve [1987] Corollary 3.5.13 p199).

$$\frac{dB^*(M,t)}{B^*(M,t)} = \sum_{q=1}^{N}\left\{\sigma_q(M,t) - \sigma_q\left(T_{k(i+1)},t\right)\right\} dZ_q^*(t) \tag{A1}$$

where

$$dZ_q^* = \left\{\theta_q(t) - \sigma_q\left(T_{k(i+1)},t\right)\right\}dt + dZ_q(t); \quad q = 1,\ldots,N \tag{A2}$$

defines a set of independent standard Brownian motions under risk-adjusted probabilities corresponding to our discretely-reinvested money market account as numeraire.

(A1) implies

$$\begin{aligned}\ln\frac{B^*(M,t)}{B^*(M,t_i)} = &-\tfrac{1}{2}\sum_{q=1}^{N}\int_{t_i}^{t}\left\{\sigma_q(M,u) - \sigma_q\left(T_{k(i+1)},u\right)\right\}^2 du \\ &+ \sum_{q=1}^{N}\int_{t_i}^{t}\left\{\sigma_q(M,u) - \sigma_q\left(T_{k(i+1)},u\right)\right\}dZ_q^*\end{aligned} \tag{A3}$$

Setting $M = T_m, t = t_{i+1}$ in this equation, and subtracting the corresponding equation with $M = T_{m+1}, t = t_{i+1}$,

$$\ln\frac{B(T_m,t_{i+1})}{B(T_{m+1},t_{i+1})} - \ln\frac{B(T_m,t_i)}{B(T_{m+1},t_i)} = \hat{\mu}_{m,i} + \sum_{q=1}^{N}\int_{t_i}^{t_{i+1}}\left\{\sigma_q(T_m,t) - \sigma_q(T_{m+1},t)\right\}dZ_q^* \tag{A4}$$

where

$$\hat{\mu}_{m,i} = \tfrac{1}{2}\sum_{q=1}^{N}\int_{t_i}^{t_{i+1}}\left\{\sigma_q(T_{m+1},t) - \sigma_q\left(T_{k(i+1)},t\right)\right\}^2 - \left\{\sigma_q(T_m,t) - \sigma_q\left(T_{k(i+1)},t\right)\right\}^2 dt$$

Now, the LHS of (A4) is simply $X_{m,j+1}$, while the fact that each $\sigma_q()$ is conditionally (ie at date t_i) deterministic over each $(t_i,t_{i+1}]$ implies that $\hat{\mu}_{m,i}$ is known at t_i and that the RHS of (A4) is conditionally Normally distributed; indeed, it tells us, moreover, that

$$\mathrm{cov}_i^*\left[X_{m,i+1}, X_{n,i+1}\right] = \sum_{q=1}^{N}\int_{t_i}^{t_i}\left\{\sigma_q(T_m,t) - \sigma_q(T_{m+1},t)\right\}\left\{\sigma_q(T_n,t) - \sigma_q(T_{n+1},t)\right\}dt \tag{A5}$$

If we can show that the RHS of (A5) equals $c_{m,n,i}$, we shall be done, since the knowledge that the model is consistent with general equilibrium – and thus must satisfy the martingale condition (7), as per Theorem. 3.1– will then tell us that $\hat{\mu}_{m,i} = \mu_{m,i}$. We proceed as follows.

Noting that the range of integration in (A5) implies $t \in (t_i,t_{i+1}]$, the use of (16) followed by (14) gives us

$$\sigma_q(T_m,t) - \sigma_q(T_{m+1},t) = \int_{T_m}^{T_{m+1}} b_{i,q}(u)du = \sqrt{\frac{c_{m,m,i}}{t_{i+1}-t_i}}\,\alpha_{m,i,q} \tag{A6}$$

Substituting this into the RHS of (A5) and then applying (12) yields the required relation:

$$\mathrm{cov}_i^*\left[X_{m,i+1}, X_{n,i+1}\right] = \sqrt{c_{m,m,i}c_{n,n,i}}\sum_{q=1}^{N}\alpha_{m,i,q}\alpha_{n,i,q} = c_{m,n,i}$$

For the final part of the theorem, subtract from (A3) the corresponding equation with $M = t \in (t_i,t_{i+1}]$, and exploit the fact that $B(M,t) = B(M,t) \div B(t,t) = B^*(M,t) \div B^*(t,t)$ to obtain

$$\ln B(M,t) = -\tfrac{1}{2}\sum_{q=1}^{N}\int_{t_i}^{t}\left\{\sigma_q(M,s)-\sigma_q\left(T_{k(i+1)},s\right)\right\}^2 - \left\{\sigma_q(t,s)-\sigma_q\left(T_{k(i+1)},s\right)\right\}^2 ds$$
$$+\ln\frac{B(M,t_i)}{B(t,t_i)} + \sum_{q=1}^{N}\int_{t_i}^{t}\left\{\sigma_q(M,s)-\sigma_q(t,s)\right\}dZ_q^*(s) \tag{A7}$$

whence the result follows from the fact that the functions in the integrands are known at t_i. ■

Proof of Theorem. 5.1 We substitute (A2) into (A4), noting from the proof of Theorem. 4.3 that the LHS of (A4) is simply $X_{m,i+1}$ and that $\hat{\mu}_{m,i} = \mu_{m,i}$, and the take conditional expectations to obtain:

$$\mathrm{E}_i\left[X_{m,i+1}\right] = \mu_{m,i} + \sum_{q=1}^{N}\int_{t_i}^{t_{i+1}}\left\{\sigma_q(T_m,t)-\sigma_q(T_{m+1},t)\right\}\left\{\theta_q(t)-\sigma_q\left(T_{k(i+1)},t\right)\right\}dt$$

Now, for $t\in(t_i,t_{i+1}]$, (16) and (14) give us not only (A6), but also:

$$\sigma_q\left(T_{k(i+1)},t\right) = -\int_t^{T_{k(i+1)}} b_{i,q}(u)\,du = -\int_{t\vee T_{k(i+1)-1}}^{T_{k(i+1)}}\frac{1}{T_{k(i+1)}-T_{k(i+1)-1}}\sqrt{\frac{c_{k(i+1)-1,k(i+1)-1,i}}{t_{i+1}-t_i}}\,\alpha_{k(i+1)-1,i,q}\,du$$
$$= -\frac{T_{k(i+1)}-\left(t\vee T_{k(i+1)-1}\right)}{T_{k(i+1)}-T_{k(i+1)-1}}\sqrt{\frac{c_{k(i+1)-1,k(i+1)-1,i}}{t_{i+1}-t_i}}\,\alpha_{k(i+1)-1,i,q}$$

whence

$$\int_{t_i}^{t_{i+1}}\left\{\sigma_q(T_m,t)-\sigma_q(T_{m+1},t)\right\}\theta_q(t)\,dt = \sqrt{\frac{c_{m,m,i}}{t_{i+1}-t_i}}\,\alpha_{m,i,q}\int_{t_i}^{t_{i+1}}\theta_q(t)\,dt$$

and, using (12) also, and setting $t_i^*\equiv t_i\vee T_{k(i+1)-1}$,

$$-\sum_{q=1}^{N}\int_{t_i}^{t_{i+1}}\left\{\sigma_q(T_m,t)-\sigma_q(T_{m+1},t)\right\}\sigma_q\left(T_{k(i+1)},t\right)dt = \int_{t_i}^{t_{i+1}}\frac{T_{k(i+1)}-\left(t\vee T_{k(i+1)-1}\right)}{T_{k(i+1)}-T_{k(i+1)-1}}dt\,\frac{\hat{c}_{m,k(i+1)-1,i}}{t_{i+1}-t_i}$$
$$=\left[t_i^*-t_i-\frac{\left\{T_{k(i+1)}-t_{i+1}\right\}^2-\left\{T_{k(i+1)}-t_i^*\right\}^2}{2\left(T_{k(i+1)}-T_{k(i+1)-1}\right)}\right]\frac{\hat{c}_{m,k(i+1)-1,i}}{t_{i+1}-t_i}$$
$$=\left[t_i^*-t_i+\frac{\left\{T_{k(i+1)}-\tfrac{1}{2}\left[t_{i+1}+t_i^*\right]\right\}\left\{t_{i+1}-t_i^*\right\}}{T_{k(i+1)}-T_{k(i+1)-1}}\right]\frac{\hat{c}_{m,k(i+1)-1,i}}{t_{i+1}-t_i}$$
$$=\left[1-\frac{\left\{\tfrac{1}{2}\left[t_{i+1}+t_i^*\right]-T_{k(i+1)-1}\right\}\left\{t_{i+1}-t_i^*\right\}}{\left(T_{k(i+1)}-T_{k(i+1)-1}\right)\left(t_{i+1}-t_i\right)}\right]\hat{c}_{m,k(i+1)-1,i}$$

The result follows. ■

Proof of Theorem. 7.1 We proceed inductively. The result holds trivially for $t=0=t_0$. Now suppose that the result holds $\forall t\le t_i$, and consider any $t\in(t_i,t_{i+1}]$. Now, $\ln B(M,t)$ is given by (A7). From (16), we have:

$$\sigma_q(M,s)-\sigma_q\left(T_{k(i+1)},s\right) = -\int_{T_{k(i+1)}}^{M} b_{i,q}(u)\,du;\quad \sigma_q(t,s)-\sigma_q\left(T_{k(i+1)},s\right) = \int_t^{T_{k(i+1)}} b_{i,q}(u)\,du$$

both of which are determined by the evolution of the term structure up to t_i and independent of s. Hence the integrands in (A7) are independent of s. The result follows. ■

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