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Abstract

Simple analytical pricing formulae have been derived, by different authors and for several derivatives, under the Gaussian Langetieg (1980) model. The purpose of this paper is to use such exact Gaussian solutions in order to obtain approximate analytical pricing formulas under the most general stochastic volatility specification of the Duffie and Kan (1996) model. Using Gaussian Arrow-Debreu state prices, first order stochastic volatility approximate pricing solutions will be derived only involving one integral with respect to the time-to-maturity of the contingent claim under valuation. Such approximations will be shown to be much faster than the existing exact numerical solutions, as well as accurate.

Key words: Exponential-affine models, Stochastic volatility, Arrow-Debreu prices, Bonds, Interest rate futures, European path-independent interest rate options

JEL Classification: G13

1 Introduction

The Duffie and Kan (1996) model can be considered as the most general multifactor time-homogeneous affine term structure framework, including as special cases several well known models such as Fong and Vasicek (1991), Longstaff and Schwartz (1992), Langetieg (1980), and Chen and Scott (1995). As additional appealing features, it incorporates *mean reversion*, and accommodates both deterministic and stochastic volatility specifications.

Under its most general stochastic volatility specification, Duffie and Kan (1996) derived a quasi-closed form pricing formula for default-free pure discount bonds (involving the numerical solution of Riccati differential equations), and priced path-independent interest rate options, in a two-factor model, through an alternating directions implicit (ADI) finite-difference method. Unfortunately, such algorithm can not be easily extended to higher dimensions, for which, and accordingly to Duffie and Kan (1994), Monte Carlo simulation appears to be the best pricing methodology available. Consequently, the *expedite* and *accurate* analytic approximate pricing solutions that will be proposed in this paper are intended to provide more efficient pricing and calibration alternative tools for this general affine class of term structure models, specially for high dimensional formulations (e.g. three-factor models).

Recently, Duffie, Pan and Singleton (1998) proposed exact Fourier transform pricing solutions for an affine jump-diffusion model that nests, as a special case, the Duffie and Kan (1996) framework under analysis.[1] Although such exact formulae are also applicable to the Duffie and Kan (1996) model, the advantage of the approximate pricing solutions derived in this paper is the fact that they are, in general, much faster to implement than the corresponding exact ones obtainable from Duffie, Pan and Singleton (1998). In fact, if the functional form of the relevant characteristic function -Duffie, Pan and Singleton (1998, equation B.2)- is known, then the exact Fourier transform pricing formulae are "explicit" or closed-form solutions (in the sense that only one-dimensional Fourier inverse integrals are involved). However, in general the characteristic function does not possess an explicit solution and must be numerically obtained from a complex-valued system consisting of one unidimensional ODE and another n-dimensional Riccati differential equation, where n denotes the number of state variables. Because the computation, for instance, of the exact price of an European option on a pure discount bond requires two inverse Fourier transforms (and thus two one-dimensional integrals; one for each exercise probability), and since the characteristic function must be evaluated numerically at each integration point, then, for n state variables, the Fourier transform exact solution involves a $2m(n+1)$ integration problem,[2] where m is the chosen number of steps in the numerical integration, whereas the corresponding first order approximate formulae proposed in this paper will only include one time-integral (no matter the order of n). For the valuation of a cap (or a floor), the difference between the two (exact and approximate) solutions, in terms of computational effort, is even multiplied by the number of component caplets (or floorlets). Consequently, in the general case where no closed-form solution exists for the characteristic function, the proposed approximate pricing formulae will be shown to be much faster to implement than the exact Fourier transform ones. Moreover, when the characteristic function is not known in closed-form, the optimization of both the grid size and the upper bound of integration for the computation of the inverse Fourier transforms becomes also very time-demanding, since it requires the numerical evaluation of the characteristic function.

By imposing a deterministic volatility specification to the Duffie and Kan (1996) formulation, the Langetieg (1980) *multivariate elastic random walk model* is obtained. This type of Gaussian multifactor affine models has received an extensive treatment in the literature, and exact closed-form pricing formulas have been derived for several interest rate contingent claims, among others, by El Karoui et al. (1991), Jamshidian (1993), Brace and Musiela (1994), and Nunes (1998). The purpose of this paper is to use such Gaussian solutions in order to obtain approximate closed-form pricing formulas, under the stochastic volatility specification of the Duffie and Kan (1996) model, for several European-style interest rate contingent claims[3], namely for: default-free bonds, FRAs, IRSs, short-term and long-term interest rate futures, European spot and futures options on zero-coupon bonds, interest rate caps and floors, European (*conventional* and *pure*) futures options on short-term interest rates, and even for European swaptions.[4]

[1] As already suggested in Chen and Scott (1995, page 58).

[2] In other words, the computational burden grows linearly with the number of model' factors.

[3] That is derivatives with only one future admissible payoff.

[4] The valuation of LIBOR-rate derivatives will be based on the Duffie and Singleton (1997) assumption of symmetric coun-

In order to derive the above mentioned stochastic volatility approximate pricing solutions, first, the functional form of an Arrow-Debreu price, for the Gaussian specification of the Duffie and Kan (1996) model, will be obtained in a slightly more general form than the one given by Beaglehole and Tenney (1991, page 73). Then, each stochastic volatility approximate analytic solution will be expressed in terms of the previously derived Gaussian Arrow-Debreu state price, in terms of the corresponding Gaussian exact pricing formula, and in terms of the model' parameters imposing stochastic volatility. The resulting first order approximate pricing formula will include one integral with respect to each one of the model' state variables, and another integral with respect to the time-to-maturity of the contingent claim under valuation. Hence, the methodology employed in this paper follows, up to this point, the work of Chen (1996), although his "general" and "special" three-factor model specifications are different from the ones used here.

However, this type of multidimensional integral solutions would have to be computed numerically through repeated one-dimensional integration or by using Monte Carlo integration, which does not necessarily represent any improvement in terms of efficiency when compared with the existing exact numerical solutions. Consequently, because the practical usefulness of these multidimensional integral approximations may be questionable, a different approach is pursued: to reduce the dimensionality of the problem analytically. Unlike in Chen (1996) and as the main contribution of the present work, all the stochastic volatility first order approximate closed-form solutions will be simplified into equivalent pricing formulas that do not involve any integration with respect to the model' factors. Such simplification will be allowed by the tractability of the chosen nested Gaussian specification, and will increase enormously the numerical efficiency of the stochastic volatility pricing approximations: only one time-integral is involved, irrespective of the model dimension. Therefore, such first order analytic approximations will be shown to be extremely fast, as well as accurate.

To the authors' knowledge, although the use of approximations involving Arrow-Debreu securities is common in Finance, the derivation of factor-integral independent pricing solutions (in the context of the most general multifactor affine term structure model) represents an original result. In addition, exact pricing formulas (involving the numerical solution of Riccati differential equations) are also found for long-term and short-term interest rate futures, under the stochastic volatility specification of the Duffie and Kan (1996) model.

Next sections are organized as follows. Section 2 provides a summary of both (deterministic and stochastic volatility) specifications of the Duffie and Kan (1996) model. Section 3 derives an analytical solution for Arrow-Debreu state prices under the deterministic volatility specification. Then, section 4 provides a series expansion pricing equation for a generic interest rate derivative, under the stochastic volatility specification, and based on the previously derived Gaussian Arrow-Debreu state-prices. Section 5 simplifies the previous pricing solution for any "exponential-affine" interest rate contingent claim, and yields a general first order explicit approximation only involving one time-integral. Such explicit approximate stochastic volatility pricing formula is then applied to different contracts: bonds, FRAs, IRSs, bond futures, and short-term interest rate futures. Similarly, in section 6 the global series expansion pricing equation is converted into an explicit first order approximation (only involving one time-integral) for a generic European and path-independent interest rate option. This explicit generic solution is then specialized to options on pure discount bonds, caps and floors, swaptions, yield options, futures options on zero-coupon bonds, and options on short-term interest rate futures. Finally, section 7 summarizes the main conclusions. All accessory proves are relegated to the appendix, while the more illustrative ones are kept in the text.

2 Description of the Model

The Duffie and Kan (1996) model imposes an exponential-affine form for the price of a riskless (unit face value) pure discount bond:

$$P(t,T) = \exp\left[A(\tau) + \underline{B}'(\tau) \cdot \underline{X}(t)\right], \tag{1}$$

where $P(t,T)$ represents the time-t price of a default-free pure discount bond expiring at time T, $\tau = T - t$ is the time-to-maturity of the zero-coupon bond, $\underline{X}(t) \in \Re^n$ is the time-t vector of state variables[5], and

terparty credit risk.

[5]Unlike in the *yield-factor model* proposed by Duffie and Kan (1996), in this paper the state variables will be assumed to be unobservable, instead of being obtained from a selected basis of fixed maturity spot interest rates. In fact, the existence of

$\cdot$ denotes the inner product in $\Re^n$. In order to respect the boundary condition $P(T,T) = 1$, the time-homogeneous functions $A(\tau) \in \Re$ and $\underline{B}(\tau) \in \Re^n$ must be such that $A(0) = 0$ and $\underline{B}(0) = \underline{0}$. Therefore, the short-term interest rate $r(t)$ is an affine function of the n factors:

$$
\begin{aligned}
r(t) &= \lim_{\tau \to 0} \left[-\frac{\ln P(t,T)}{\tau} \right] \\
&= f + \underline{G}' \cdot \underline{X}(t),
\end{aligned} \tag{2}
$$

where $f = -\left.\frac{\partial A(\tau)}{\partial \tau}\right|_{\tau=0}$, and the i^{th} element of vector $\underline{G} \in \Re^n$ is defined as $g_i = -\left.\frac{\partial B_i(\tau)}{\partial \tau}\right|_{\tau=0}$, being $B_i(\tau)$ the i^{th} element of vector $\underline{B}(\tau)$.

Duffie and Kan (1996) also assume that the n state variables follow, under a martingale measure Q, a parametric Markov diffusion process, where the drift and the variance of these risk-adjusted stochastic processes also have an affine form, in order to support the exponential-affine specification of equation (1):

$$
d\underline{X}(t) = [a \cdot \underline{X}(t) + \underline{b}]\, dt + \Sigma \cdot \sqrt{V^D(t)} \cdot d\underline{W}^Q(t), \underline{X}(t) \in \mathbf{D}, \tag{3}
$$

where $a, \Sigma \in \Re^{n \times n}$, $\underline{b} \in \Re^n$,

$$
\sqrt{V^D(t)} = diag\left\{ \sqrt{v_1(t)}, \ldots, \sqrt{v_n(t)} \right\},
$$

$$
v_i(t) = \alpha_i + \underline{\beta_i}' \cdot \underline{X}(t), \quad for\ i = 1, ..., n,
$$

$\alpha_i \in \Re$, $\underline{\beta_i} \in \Re^n$, $d\underline{W}^Q(t) \in \Re^n$ is a vector of n independent Brownian motion increments under measure Q, and

$$
\mathbf{D} = \left\{ \underline{X} \in \Re^n : \alpha_i + \underline{\beta_i}' \cdot \underline{X} \geq 0, i = 1, \ldots, n \right\} \tag{4}
$$

is the admissible domain of the model' state variables.

Hereafter, the martingale measure Q will denote the probability measure obtained when the "money market account" is taken as the numeraire of the economy underlying the model under analysis. And, a stochastic intertemporal economy will be considered with a finite time horizon $\mathcal{T} = [0, T]$, where uncertainty is represented by a probability space $(\Omega, \mathcal{F}, Q)$, and where all the information accruing to all the agents in the economy is described by a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ satisfying the usual conditions: namely, $\mathcal{F}_0$ is assumed to be almost trivial, and $\mathcal{F}_T = \mathcal{F}$.

Equations (1) and (3) summarize the stochastic volatility specification of the Duffie and Kan (1996) model (since $\underline{\beta_i}$ is not constrained to be equal to $\underline{0}$). Under this general framework, Duffie and Kan (1994) point out that path-independent interest rate contingent claims can only be valued by a finite-difference method or, for large n, by Monte Carlo simulation. The only exception seems to be the valuation of default-free pure discount bonds, for which an exact quasi-closed form solution is provided by Duffie and Kan (1996). Using equations (3.9) and (3.10) of Duffie and Kan (1996), first the *duration* vector $\underline{B}'(\tau)$ must be found through the solution of a system of n Riccati differential equations (for instance, by using a fifth order Runge-Kutta method),

$$
\frac{\partial}{\partial \tau} \underline{B}'(\tau) = -\underline{G}' + \underline{B}'(\tau) \cdot a + \frac{1}{2} \sum_{k=1}^{n} \left[\sum_{j=1}^{n} B_j(\tau) \varepsilon_{jk} \right]^2 \underline{\beta_k}', \tag{5}
$$

subject to the initial condition $\underline{B}(0) = \underline{0}$, where A' denotes the transpose of A, and ε_{jk} is the j^{th}-row k^{th}-column element of matrix Σ. Then, $A(\tau)$ is obtained through the solution of a first order ordinary differential equation (for instance, by using Romberg's integration method),

$$
\frac{\partial}{\partial \tau} A(\tau) = -f + \underline{B}'(\tau) \cdot \underline{b} + \frac{1}{2} \sum_{k=1}^{n} \left[\sum_{j=1}^{n} B_j(\tau) \varepsilon_{jk} \right]^2 \alpha_k, \tag{6}
$$

market imperfections (e.g. bid-ask spreads) does not allow, in practice, all the (factor) yields to be always observed without error. This assumption greatly generalizes the scope of the interest rate model under consideration, but at the cost of additional difficulties in terms of model estimation: filtering methods must be used to estimate the model' parameters and to recover the latent state variables.

subject to the initial condition $A(0) = 0$. Finally, $P(t,T)$ is given by equation (1). However, under this general specification of the Duffie and Kan (1996) model, even the above ODEs must be solved numerically.

In order to obtain closed form solutions for a wide range of European interest rate options, Nunes (1998) used a deterministic volatility specification for the Duffie and Kan (1996) model by imposing $\underline{\beta_i} = \underline{0}$ for $i = 1,\ldots,n$. This Gaussian specification of the Duffie and Kan (1996) model is given by equations (1) and (7):

$$d\underline{X}(t) = \left[a \cdot \underline{X}(t) + \underline{b}\right] dt + S \cdot d\underline{W}^Q(t), \underline{X}(t) \in \Re^n, \tag{7}$$

where $S = \Sigma \cdot \sqrt{U^D}$, and $\sqrt{U^D} = diag\left\{\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right\}$. In essence, this formulation corresponds to the Langetieg (1980) *multivariate elastic random walk model*, and thus an analytic formula exists for default-free pure discount bonds.

Proposition 1 *Under the deterministic volatility specification of the Duffie and Kan (1996) model and assuming that matrix a is nonsingular, the price of a riskless zero-coupon bond is given by equation (1), where*

$$\underline{B}'(\tau) = \underline{G}' \cdot a^{-1} \cdot \left(I_n - e^{a\tau}\right), \tag{8}$$

$$\begin{aligned} A(\tau) \quad = \quad & \tau\left(\underline{G}' \cdot a^{-1} \cdot \underline{b} - f\right) + \underline{B}'(\tau) \cdot a^{-1} \cdot \underline{b} + \frac{\tau}{2}\underline{G}' \cdot a^{-1} \cdot \Theta \cdot \left(a^{-1}\right)' \cdot \underline{G} \\ & + \underline{G}' \cdot a^{-1} \cdot \left(I_n - e^{a\tau}\right) \cdot a^{-1} \cdot \Theta \cdot \left(a^{-1}\right)' \cdot \underline{G} \\ & + \frac{1}{2}\underline{G}' \cdot a^{-1} \cdot \Delta(\tau) \cdot \left(a^{-1}\right)' \cdot \underline{G}, \end{aligned} \tag{9}$$

and

$$\Delta(\tau) = \int_t^T e^{a(T-s)} \cdot \Theta \cdot e^{a'(T-s)} ds, \tag{10}$$

with $\Theta = S \cdot S'$, *and* $I_n \in \Re^{n \times n}$ *denoting an identity matrix.*

Proof. See equations (5), (30), (32), and (33) of Langetieg (1980). Alternatively, equations (5) and (6) can also be solved explicitly with $\underline{\beta_k} = \underline{0}$ for $k = 1,\ldots,n$. ■

Remark 1 *As noticed in Langetieg (1980, footnote 20), matrix a will be singular only if one of the state variables follows a random walk. If this is the case, equations (8) to (10) can always be replaced by the more general solutions described in Lund (1994, appendix A).*

Proposition 2 *Under the deterministic volatility specification of the Duffie and Kan (1996) model and assuming that matrix a is diagonalizable, the function* $\Delta(\tau)$ *possesses the following explicit solution:*

$$\Delta(\tau) = e^{a(T-t)} \cdot Y \cdot e^{a'(T-t)} - Y, \tag{11}$$

where $\Theta^* = Q^{-1} \cdot \Theta \cdot \left(Q^{-1}\right)' \equiv \left\{\sigma_{ij}^*\right\}_{i,j=1,\ldots,n}$, $\Theta^{**} = \left\{\frac{\sigma_{ij}^*}{\lambda_i + \lambda_j}\right\}_{i,j=1,\ldots,n}$, $Y = Q \cdot \Theta^{**} \cdot Q'$, λ_i $(i = 1,\ldots,n)$ *is the* i^{th} *eigenvalue of matrix a, and Q is a* $n \times n$ *matrix whose columns correspond to the eigenvectors of matrix a.*

Proof. See Langetieg (1980, footnote 23) or Nunes (1998, appendix B). ■

Remark 2 *As argued by Duan and Simonato (1995, page 26), this "assumption of diagonability does not involve an appreciable loss of generality" because the eigenvalues of a matrix are continuous functions of its elements (and thus multiple roots of the characteristic equation can be avoided by a small adjustment in the original matrix). Nevertheless, for the numerical experiments presented in this paper all matrix exponentials are computed using Padé approximations with scaling and squaring. For details, see Van Loan (1978).*

3 Arrow-Debreu Prices under the Gaussian Specification

In this section a closed form solution will be derived for an Arrow-Debreu state price, under the Gaussian specification corresponding to equations (1) and (7). The formula that will be obtained is equivalent to the one given by Beaglehole and Tenney (1991, page 73), with just two differences: the short-term interest rate is not constrained to be one of the model' factors; and, proposition 2 ensures that no single integral is involved when all the eigenvalues of matrix a are assumed to be distinct.

Proposition 3 *Let $G\left[\underline{X}(T), T; \underline{X}(t), t\right]$ represent the value, at time t (and in state $\underline{X}(t)$), of a unit payoff occurring at time T ($\geq t$) and in state $\underline{X}(T)$. Under the deterministic volatility specification of the Duffie and Kan (1996) model, the Arrow-Debreu price $G\left[\underline{X}(T), T; \underline{X}(t), t\right]$ possesses the following analytical form:*

$$\begin{aligned} & G\left[\underline{X}(T), T; \underline{X}(t), t\right] \\ = \; & P_G(t,T) \frac{\exp\left\{-\frac{1}{2}\left[\underline{X}(T) - \underline{M}(\tau)\right]' \cdot \Delta^{-1}(\tau) \cdot \left[\underline{X}(T) - \underline{M}(\tau)\right]\right\}}{\sqrt{(2\pi)^n \left|\Delta(\tau)\right|}}, \end{aligned} \tag{12}$$

where

$$\underline{M}(\tau) = a^{-1} \cdot \left(e^{a\tau} - I_n\right) \cdot \left[\underline{b} + \Theta \cdot \left(a^{-1}\right)' \cdot \underline{G}\right] + e^{a\tau} \cdot \underline{X}(t) - \Delta(\tau) \cdot \left(a^{-1}\right)' \cdot \underline{G}$$

and $P_G(t,T)$ denotes a pure discount bond price computed under proposition 1.

Proof. See appendix A. ■

Remark 3 *The fundamental solution (12) corresponds simply to the product between the time-t price of a pure discount bond with maturity at time T, and the probability density function of $\underline{X}(T)$, conditional on $\underline{X}(t)$, under the equivalent martingale probability measure obtained when such zero-coupon bond is taken as the numeraire. This result is in line with corollary 2 of Jamshidian (1991), which was obtained in the context of a one-factor Gaussian term structure model.*

Remark 4 *The fundamental solution (12) corresponds to an Arrow-Debreu state price and not precisely to a Green's function, in the mathematical sense of the term. Nevertheless, both terms are often used interchangeably in the Finance literature.*

4 Series Expansion Solution for the Stochastic Volatility Specification

4.1 A general result

This section provides the theoretical background needed to produce approximate pricing formulas under the stochastic volatility specification, from the Gaussian "Green's function" derived earlier, and using the corresponding exact solutions already known for the deterministic volatility version of the Duffie and Kan (1996) model. The result obtained is a very general one in the sense that it can be applied to any interest rate contingent claim.

Theorem 1 *Let $V_G\left[\underline{X}(t), t\right]$ and $V_S\left[\underline{X}(t), t\right]$ be the time-t prices, for the same contingent claim with maturity at time T ($\geq t$), computed under the Gaussian and the stochastic volatility specifications of the Duffie and Kan (1996) model, respectively. Assuming that the terminal payoff function and the dividend yield process are of the same form for both $V_S\left[\underline{X}(t), t\right]$ and $V_G\left[\underline{X}(t), t\right]$, when $\underline{X} \in \mathbf{D}$, but identically zero if $\underline{X} \notin \mathbf{D}$, then:*

$$V_S\left[\underline{X}(t), t\right] = \sum_{p \geq 0} \frac{1}{2^p} V_p\left[\underline{X}(t), t\right], \tag{13}$$

where $V_0[\underline{X}(t),t] \equiv V_G[\underline{X}(t),t]$, *and*[6]

$$V_{p+1}[\underline{X}(t),t] = \int_t^T dl \int_{\underline{X}(l)\in\mathbf{D}} d\underline{X}(l)\, G[\underline{X}(l),l;\underline{X}(t),t] \\ tr\left\{\frac{\partial^2 V_p[\underline{X}(l),l]}{\partial \underline{X}(l)\,\partial \underline{X}'(l)} \cdot \Sigma \cdot W^D(l) \cdot \Sigma'\right\}, \quad (14)$$

for $p \geq 0$, *with* $W^D(t) = diag\{\underline{\beta_1}' \cdot \underline{X}(t), \ldots, \underline{\beta_n}' \cdot \underline{X}(t)\}$.

Proof. See appendix B. ■

Remark 5 *The series expansion pricing formula (13) is similar to equation (1.21) of Chen (1996). As Chen (1996, page 19) notices, all the terms* $\frac{1}{2}V_1, \frac{1}{4}V_2, \ldots$ *are strictly decreasing in magnitude, and therefore a good approximation should be obtained by only retaining the first few terms in the expansion.*

Remark 6 *The series expansion pricing formula (13) only depends on the Gaussian Arrow-Debreu state-price G, on the corresponding exact pricing formula under the deterministic volatility specification* V_G, *and on the "stochastic volatility parameters"* $\underline{\beta_i}$ $(i = 1, \ldots, n)$, *through matrix* W^D.

Remark 7 *Intuitively, equation (13) arises essentially because the Gaussian specification is nested into the more general stochastic volatility one. More formally, because the stochastic volatility and Gaussian instantaneous variances of the model' factors are related through the identity (77).*

Remark 8 *The recursive relation (14) shows that the* p^{th} *order approximating term,* $V_p[\underline{X}(t),t]$, *involves p time-integrals and p factor-integrals (on* **D***), and therefore its numerical computation would require the use of repeated one-dimensional or Monte Carlo integration. Next sections will simplify such general result by extending the integration with respect to the state variables to the all n-dimensional Euclidean space.*

4.2 Asymptotic properties

Next two propositions describe the limiting behavior of the general pricing solution (13) as the stochastic volatility model tends to its nested Gaussian specification, and when the series expansion (13) is truncated, while the domain of integration, in (14), is expanded from **D** to $\Re^n$.

Proposition 4 *The limit of the series expansion (13), as the perturbed parameters tend to zero, exists and is well defined:*

$$\lim_{\beta \to O_n} \sum_{p\geq 0} \frac{1}{2^p} V_p[\underline{X}(t),t] = V_G[\underline{X}(t),t], \quad (15)$$

where $O_n \in \Re^{n\times n}$ *is a null matrix, and* $\beta \in \Re^{n\times n}$ *is a matrix whose* i^{th}*-column is given by vector* $\underline{\beta_i}$.

Proof. Because $W^D(l) \to O_n$ as $\beta \to O_n$, then $V_p[\underline{X}(t),t] \to 0$ as $\beta \to O_n$, for $p \geq 1$. ■

Remark 9 *The limit (15) is well behaved in the sense that* $V_G[\underline{X}(t),t]$ *is the solution of the initial value problem (74)-(75) when* $\beta = O_n$.

In practice, it is usually impossible to obtain analytically series terms of order higher than the first, and the series expansion (13) must be truncated, which induces a "truncation" error. Moreover, even for the first order term to be computed explicitly (that is without involving any factor-integral) it is almost always necessary to extend the integration bounds from **D** to $\Re^n$, introducing an "integration" error. The following proposition shows that the "integration" and "truncation" errors involved in the first order explicit solutions (with extended integration bounds) proposed hereafter are of order *strictly smaller* than the perturbed parameters.[7]

[6] The authors wish to thank Qiang Dai for deriving the elegant recursive relation (14).

[7] The authors wish to thank Qiang Dai for showing how to generalize proposition 5 from the more restrictive $A_n(n)$ specification to the more general $A_m(n)$ canonical form.

Proposition 5 *Under the $A_m(n)$ canonical formulation of Dai and Singleton (1998, definition III.1), let $\beta = \lambda\bar{\beta}$, where $\lambda \in \Re^+$ is a common scale for the perturbed parameters,*

$$\bar{\beta} = \begin{bmatrix} I_{m\times m} & \bar{\beta}^{DB}_{m\times(n-m)} \\ O_{(n-m)\times m} & O_{(n-m)\times(n-m)} \end{bmatrix}$$

and $\bar{\beta}^{DB}$ is a matrix of positive constants.[8] *Let also the series $\left\{\tilde{U}_p\left[\underline{X}(t),t\right], p \geq 0\right\}$ be defined by $\tilde{U}_0\left[\underline{X}(t),t\right] = V_G\left[\underline{X}(t),t\right]$ and, for $p \geq 1$,*

$$\begin{aligned}\tilde{U}_p\left[\underline{X}(t),t\right] &= \int_t^T dl \int_{\underline{X}(l)\in\Re^n} d\underline{X}(l)\, G\left[\underline{X}(l),l;\underline{X}(t),t\right] \\ & \quad tr\left\{\frac{\partial^2 \tilde{U}_{p-1}\left[\underline{X}(l),l\right]}{\partial \underline{X}(l)\,\partial \underline{X}'(l)} \cdot \Sigma \cdot \bar{W}^D(l) \cdot \Sigma'\right\},\end{aligned}$$

with $\bar{W}^D(l) = diag\left\{\underline{\bar{\beta}_1}' \cdot \underline{X}(l), \ldots, \underline{\bar{\beta}_n}' \cdot \underline{X}(l)\right\}$, and where $\underline{\bar{\beta}_i}$ denotes the i^{th}-column of matrix $\bar{\beta}$.

If $\left|\tilde{U}_p\left[\underline{X}(t),t\right]\right| < \infty$ for $p > 1$, then for every $c_0 \in \Re^+$ there exists a $\lambda_0 \in \Re^+$ such that

$$\left|V_S\left[\underline{X}(t),t\right] - V_G\left[\underline{X}(t),t\right] - \frac{1}{2}\tilde{V}_1\left[\underline{X}(t),t\right]\right| \leq c_0|\lambda| \qquad for\ |\lambda| \leq \lambda_0, \tag{16}$$

where

$$\begin{aligned}\tilde{V}_1\left[\underline{X}(t),t\right] &= \int_t^T dl \int_{\underline{X}(l)\in\Re^n} d\underline{X}(l)\, G\left[\underline{X}(l),l;\underline{X}(t),t\right] \\ & \quad tr\left\{\frac{\partial^2 V_G\left[\underline{X}(l),l\right]}{\partial \underline{X}(l)\,\partial \underline{X}'(l)} \cdot \Sigma \cdot W^D(l) \cdot \Sigma'\right\}.\end{aligned}$$

Proof. See appendix C. ■

Remark 10 *The cast of proposition 5 under the Dai and Singleton (1998) canonical form does not represent any loss of generality because any exponential-affine model already proposed in the literature can always be nested under the $A_m(n)$ specification, through an appropriate invariant transformation.*

Remark 11 *Along the same lines, it can also be shown that, without approximating the integration domain, the asymptotic "truncation" error for a fixed partial sum of the series (13) would be of the order of the first omitted term, that is:*

$$V_S\left[\underline{X}(t),t\right] - \sum_{p=0}^{k} \frac{1}{2^p} V_p\left[\underline{X}(t),t\right] = O\left(\lambda^{k+1}\right).$$

However, if one is to go beyond the first order approximation term, attention should also be paid for the corresponding "integration" error.

4.3 Invariant affine transformations and nested models

Before actually applying Theorem 1, it is usually necessary to perform an affine invariant transformation (along the lines of Dai and Singleton (1998)), in order to ensure: *i*) the existence of solution for the stochastic differential equation (7), satisfied by the state vector when zeroing off the parameters $\underline{\beta_i}$ $(i = 1, \ldots, n)$;[9] and *ii*) that the nested Gaussian specification is close enough to the general stochastic volatility one.[10]

[8] Dai and Singleton (1998, definition III.1) normalize λ to unity.

[9] For instance, it is impossible, *a priori*, to nest a Gaussian specification into a multifactor CIR model, and thus it would seem impossible to apply the Theorem 1 to such stochastic volatility formulation. It will be shown shortly that this is not the case.

[10] The closer is $V_G\left[\underline{X}(t),t\right]$ to $V_S\left[\underline{X}(t),t\right]$, the less important should be the neglected approximating terms $\frac{1}{2^p}V_p\left[\underline{X}(t),t\right]$, $p > k$, where k is the order of a truncated series (13). In this paper $k = 1$.

In order to illustrate the analysis, let us consider the stochastic volatility specification defined by equations (2) and (3). The problem is that if one tries to apply directly Theorem 1 to such stochastic volatility specification, by simply imposing that $\underline{\beta_i} = \underline{0}$ $(for\ i = 1, \ldots, n)$, in some cases, the resulting Gaussian nested formulation that is obtained is too far apart from the original general stochastic volatility model. Moreover, if $\alpha_i \leq 0$ for some i, then Theorem 1 can not even be used.

However, by redefining the vector of state variables through an invariant affine transformation

$$\underline{\tilde{X}}(t) = \underline{X}(t) - \underline{u}, \tag{17}$$

where $\underline{\tilde{X}}(t), \underline{u} \in \Re^n$, and applying Itô's lemma, an exactly equivalent[11] stochastic volatility formulation follows:

$$r(t) = \tilde{f} + \underline{G}' \cdot \underline{\tilde{X}}(t), \tag{18}$$

with

$$\tilde{f} = f + \underline{G}' \cdot \underline{u}, \tag{19}$$

and

$$d\underline{\tilde{X}}(t) = \left[a \cdot \underline{\tilde{X}}(t) + \underline{\tilde{b}}\right] dt + \Sigma \cdot \sqrt{\tilde{V}^D(t)} \cdot d\underline{W}^Q(t), \tag{20}$$

where

$$\underline{\tilde{b}} = a \cdot \underline{u} + \underline{b}, \tag{21}$$

$$\sqrt{\tilde{V}^D(t)} = diag\left\{\sqrt{\tilde{v}_1(t)}, \ldots, \sqrt{\tilde{v}_n(t)}\right\}, \tag{22}$$

$$\tilde{v}_i(t) = \tilde{\alpha}_i + \underline{\beta_i}' \cdot \underline{\tilde{X}}(t), \tag{23}$$

and

$$\tilde{\alpha}_i = \alpha_i + \underline{\beta_i}' \cdot \underline{u}, \quad i = 1, ..., n. \tag{24}$$

The advantage of this transformed stochastic volatility specification is that $\underline{u}$ can be defined in such a way that Theorem 1 is applicable (i.e. $\tilde{\alpha}_i > 0$ for all i), and that the Gaussian nested specification, obtained with $\underline{\beta_i} = \underline{0}$ $(for\ i = 1, \ldots, n)$, is close enough to the more general stochastic volatility one.

Alternative transformations, distinguished by different definitions of $\underline{u} \in \Re^n$, will be used for the numerical examples presented in this paper.[12] The authors' preferred transformation consists in matching the first two time-t conditional moments of the new state vector (evaluated at the maturity date of the derivative under valuation, which is denominated by $T\ (\geq t)$, where t is the current pricing date), between the nested and the general specifications of the Duffie and Kan (1996) model. In appendix D it is shown that, no matter how $\underline{u}$ is defined, the conditional mean of $\underline{\tilde{X}}(T)$ is always the same for both Gaussian and stochastic volatility specifications. Furthermore, it is also shown that the transformation

$$\underline{u} = \underline{X}(t) \tag{25}$$

approximates the conditional Gaussian and stochastic volatility covariance matrices of $\underline{\tilde{X}}(T)$, at least for short maturity derivatives. This is precisely the same type of transformation as taken by Leblanc and Scaillet (1998, page 360) in order to ensure that the stationary distributions (under Q) of the state variables, for both general and nested models, have the same first two moments. Additional transformations of the form

$$\underline{u} = \eta \underline{X}(t), \tag{26}$$

where $\eta \in \Re$ but $\eta \neq 1$, and

$$\underline{u} = -a^{-1} \cdot \underline{b}, \tag{27}$$

will be also considered. In the last case (equation (27)), the unconditional mean of the state vector is used to minimize the stochastic volatility effects that can arise from the drift of the state process. On average, the numerical experiments presented in this paper will suggest that transformation (25) yields the lowest pricing errors for the proposed first order approximations.

[11] In the sense that all interest rate contingent claims' prices and price probability distributions remain unchanged.

[12] Although all pricing formulae are stated under the stochastic volatility specification (2)-(3), if an affine invariant transformation is used, it is understood that $\underline{X}(t)$, f, $\underline{b}$, and α_i are implicitly replaced by $\underline{\tilde{X}}(t)$, $\tilde{f}$, $\underline{\tilde{b}}$, and $\tilde{\alpha}_i$, respectively.

Although other candidates for nested specifications exist (such as the multifactor CIR model or the three-factor Chen (1996) benchmark model), the Gaussian Langetieg (1980) specification was selected as the *bare* model from which each pricing solution for the *full* Duffie and Kan (1996) model is expanded, for two reasons:

i) Firstly, because the chosen nested specification must possess analytically tractable closed-form pricing solutions in order to yield explicit first order approximating terms. In other words, the chosen nested Gaussian specification provides an analytical solution for Arrow-Debreu state prices, which will allow all factor-integrals to be transformed into expectations with respect to a Gaussian kernel. It is exactly this feature that enables the first order approximating term $V_1[\underline{X}(t), t]$ just to involve one time-integral (no matter the dimension of the interest rate model under consideration).[13]

ii) Secondly, and as shown in appendix D, the selected nested model possesses the advantage that its first conditional moment for the state vector is automatically equal to the one given by the general stochastic volatility model.

5 Pricing of Exponential-Affine Derivatives

5.1 Explicit stochastic volatility approximation

When the Gaussian price of the interest rate contingent claim under valuation can be expressed as an exponential-affine function of the vector of state variables, the general stochastic volatility valuation equation (13) can be easily converted into a first order approximation that is "explicit" in the sense that it does not involve any factor-integral. Corollary 1 proposes a first order approximate and analytical pricing solution (only involving one time-integral) for exponential-affine derivatives, by extending the bounds of integration, in equation (14) and for $p = 0$, to the n-dimensional Euclidean space. It also provides bounds for the approximation error involved in extending the domain of integration, and contains the exact analytical solution of the first order approximating term for the univariate case ($n = 1$).

Corollary 1 *Under the assumptions of Theorem 1, let*

$$V_G[\underline{X}(t), t] = \exp\left[\varphi(t,T) + \underline{\psi}'(t,T) \cdot \underline{X}(t)\right], \tag{28}$$

with $\varphi(t,T) \in \Re$ and $\underline{\psi}(t,T) \in \Re^n$, denote the time-t price of a contingent claim computed under the Gaussian version of the Duffie and Kan (1996) model.

1. *Approximating* **D** *by $\Re^n$, a first order analytical stochastic volatility approximate solution is obtained from (13), with*[14]

$$\begin{aligned} V_1[\underline{X}(t), t] \cong & \int_t^T dlP_G(t,l) \exp\left[\varphi(l,T) + \frac{1}{2}\underline{\psi}'(l,T) \cdot \Delta(l-t)\right. \\ & \left.\cdot \underline{\psi}(l,T) + \underline{\psi}'(l,T) \cdot \underline{M}(l-t)\right] \\ & \left[\sum_{k=1}^{n} \left(\underline{\psi}'(l,T) \cdot \underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\right] \cdot \left[\Delta(l-t) \cdot \underline{\psi}(l,T) + \underline{M}(l-t)\right] \end{aligned} \tag{29}$$

and where $\underline{\varepsilon_k}$ is the k^{th} column of matrix Σ.

[13]The price to pay for such simplicity is that perhaps another *bare* model could provide a better zeroth order approximation. However, the computation of the corresponding first order approximating term would be too time-consuming for practical purposes.

[14]It can be easily checked that the p^{th} order ($p > 1$) approximating term is still exponential-affine, modulo a p^{th}-degree polynomial pre-factor. For higher accuracy, it can be computed analytically (up to a numerical p-dimensional integration over time). Formulae available upon request. However, the examples presented in this paper suggest that a first order "explicit" approximation should be enough for the valuation of simple "exponential-affine" derivatives.

2. *For the unidimensional case ($n = 1$), the integration over* $\mathbf{D}$ *can be solved analytically, and the following solution becomes exact:*

$$V_1\left[\underline{X}(t), t\right] = \int_t^T dlP_G(t,l)\exp\left[\varphi(l,T) + \frac{1}{2}\underline{\psi}'(l,T)\cdot\Delta(l-t) \cdot\underline{\psi}(l,T) + \underline{\psi}'(l,T)\cdot\underline{M}(l-t)\right]\sum_{k=1}^{n}\left[\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right]^2 \left\{\sqrt{\frac{\underline{\beta_k}'\cdot\Delta(l-t)\cdot\underline{\beta_k}}{2\pi}}e^{-\frac{1}{2}\frac{\left(\alpha_k+\underline{\beta_k}'\cdot\Delta(l-t)\cdot\underline{\mu}(l)\right)^2}{\underline{\beta_k}'\cdot\Delta(l-t)\cdot\underline{\beta_k}}} + \left[\underline{\beta_k}'\cdot\Delta(l-t)\cdot\underline{\mu}(l)\right]\Phi\left[\frac{\alpha_k+\underline{\beta_k}'\cdot\Delta(l-t)\cdot\underline{\mu}(l)}{\sqrt{\underline{\beta_k}'\cdot\Delta(l-t)\cdot\underline{\beta_k}}}\right]\right\}, \tag{30}$$

with

$$\underline{\mu}(l) = \underline{\psi}(l,T) + \Delta^{-1}(l-t)\cdot\underline{M}(l-t), \tag{31}$$

and where Φ *represents the cumulative density function of the univariate standard normal distribution.*

3. *The exact value of the first order approximation term can be bounded from above and from bellow, using the following inequalities:*[15]

$$V_1\left[\underline{X}(t), t\right] \leq \int_t^T dlP_G(t,l)\exp\left[\varphi(l,T) + \frac{1}{2}\underline{\psi}'(l,T)\cdot\Delta(l-t) \cdot\underline{\psi}(l,T) + \underline{\psi}'(l,T)\cdot\underline{M}(l-t)\right]\sum_{k=1}^{n}\left[\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right]^2 \sqrt{\underline{\beta_k}'\cdot\left[\Delta(l-t)+\Delta(l-t)\cdot\underline{\mu}(l)\cdot\underline{\mu}'(l)\cdot\Delta(l-t)\right]\cdot\underline{\beta_k}} \prod_{j=1}^{n}\left\{\Phi\left[\frac{\alpha_j+\underline{\beta_j}'\cdot\Delta(l-t)\cdot\underline{\mu}(l)}{\sqrt{\underline{\beta_j}'\cdot\Delta(l-t)\cdot\underline{\beta_j}}}\right]\right\}^{\frac{1}{2j+1}}, \tag{32}$$

$$V_1\left[\underline{X}(t), t\right] \geq -\int_t^T dlP_G(t,l)\exp\left[\varphi(l,T) + \frac{1}{2}\underline{\psi}'(l,T)\cdot\Delta(l-t) \cdot\underline{\psi}(l,T) + \underline{\psi}'(l,T)\cdot\underline{M}(l-t)\right]\sum_{k=1}^{n}\left[\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right]^2\cdot\alpha_k \prod_{j=1}^{n}\Phi\left[\frac{\alpha_j+\underline{\beta_j}'\cdot\Delta(l-t)\cdot\underline{\mu}(l)}{\sqrt{\underline{\beta_j}'\cdot\Delta(l-t)\cdot\underline{\beta_j}}}\right]. \tag{33}$$

Proof. In order to eliminate the factor-integral from equation (14) for $p = 0$, the first order approximating term will be represented as an expectation with respect to a Gaussian kernel, and then such expectation will be computed explicitly. Because equation (28) implies that

$$tr\left\{\frac{\partial^2 V_G\left[\underline{X}(l), l\right]}{\partial\underline{X}(l)\,\partial\underline{X}'(l)}\cdot\Sigma\cdot W^D(l)\cdot\Sigma'\right\} = V_G\left[\underline{X}(l), l\right]\left[\sum_{k=1}^{n}\left(\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right)^2\underline{\beta_k}'\right]\cdot\underline{X}(l),$$

[15] These loose bounds are used in the examples presented in this section simply to emphasize that the approximation error involved in assuming $\mathbf{D} = \Re^n$ is negligeable. In proposition 5, such error has already been shown to be of smaller (asymptotic) order then the perturbed parameters.

equation (14) yields, for $p = 0$, the following functional form for the first order approximating term:

$$V_1\left[\underline{X}(t), t\right] = \int_t^T dl \int_{\underline{X}(l)\in\mathbf{D}} d\underline{X}(l)\, G\left[\underline{X}(l), l; \underline{X}(t), t\right] V_G\left[\underline{X}(l), l\right] \sum_{k=1}^{n} \left(\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\cdot\underline{X}(l). \tag{34}$$

Approximating $\mathbf{D}$ by $\Re^n$, using the analytical solution (12) for the Gaussian Arrow-Debreu prices, and rearranging terms:

$$\begin{aligned} V_1\left[\underline{X}(t), t\right] \cong & \int_t^T dl P_G(t,l) \exp\left[\varphi(l,T) - \frac{1}{2}\underline{M}'(l-t)\cdot\Delta^{-1}(l-t) \cdot\underline{M}(l-t) + \frac{1}{2}\underline{\mu}'(l)\cdot\Delta(l-t)\cdot\underline{\mu}(l)\right] \\ & \int_{\underline{X}(l)\in\Re^n} d\underline{X}(l) \frac{\left[\sum_{k=1}^{n}\left(\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\right]\cdot\underline{X}(l)}{\sqrt{(2\pi)^n}\left|\Delta(l-t)\right|} \\ & \exp\left\{-\frac{1}{2}\left[\underline{X}(l) - \Delta(l-t)\cdot\underline{\mu}(l)\right]'\cdot\Delta^{-1}(l-t) \cdot\left[\underline{X}(l) - \Delta(l-t)\cdot\underline{\mu}(l)\right]\right\}. \end{aligned}$$

The factor-integral contained in the last expression for the first order approximating term can be interpreted as the expectation of the random variable $\left[\sum_{k=1}^{n}\left(\underline{\psi}'(l,T)\ \cdot\underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\right]\cdot\underline{X}(l)$, conditional on $\underline{X}(t)$, under some equivalent probability measure with respect to which $\underline{X}(l)$ is normally distributed with mean $\Delta(l-t)\cdot\underline{\mu}(l)$ and covariance $\Delta(l-t)$, i.e.[16] $\underline{X}(l) \cap N^n\left(\Delta(l-t)\cdot\underline{\mu}(l), \Delta(l-t)\right)$. Computing the expectation explicitly,

$$\begin{aligned} V_1\left[\underline{X}(t), t\right] \cong & \int_t^T dl P_G(t,l) \exp\left[\varphi(l,T) - \frac{1}{2}\underline{M}'(l-t)\cdot\Delta^{-1}(l-t) \cdot\underline{M}(l-t) + \frac{1}{2}\cdot\underline{\mu}'(l)\cdot\Delta(l-t)\cdot\underline{\mu}(l)\right] \\ & \left[\sum_{k=1}^{n}\left(\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\right]\cdot\Delta(l-t)\cdot\underline{\mu}(l), \end{aligned}$$

and simplifying terms, the factor-integral independent solution (29) arises.

Items 2 and 3 of Corollary 1 are derived in appendix E. ■

Remark 12 *Because the exact analytical solution (30) is only valid for one-factor models and the focus of this paper is on multifactor frameworks, the approximate solution (29) will be used hereafter. In fact, the numerical experiments implemented in this section show that the pricing errors resulting from extending* $\mathbf{D}$ *to* $\Re^n$ *in computing* $V_1\left[\underline{X}(t), t\right]$ *are small enough to be neglected.*

Remark 13 *The "explicit" approximation (29) is very fast to implement since it only involves one time-integral, and can be easily computed using, for example, Romberg's integration method (on a closed interval).*

For the remaining of this section Corollary 1 will be specialized for different types of "exponential-affine" interest rate contingent claims, by nesting each Gaussian price into formula (28).

[16] The notation $\underline{Y} \cap N^d\left(\underline{\mu}, C\right)$ will be used to state that the random variable $\underline{Y} \in \Re^d$ possesses a d-dimensional normal distribution, with mean $\underline{\mu} \in \Re^d$, and variance $C \in \Re^{d\times d}$.

5.2 Bonds, FRAs and IRSs

5.2.1 A first order explicit approximation

The following proposition offers a first order approximate explicit solution for the price of a zero-coupon bond.

Proposition 6 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t price* $P_S(t,T)$ *of a default-free pure discount bond with maturity at time* T $(T \geq t)$ *can be approximated by the following first order solution:*

$$P_S(t,T) \cong P_G(t,T) + \frac{1}{2} V_1 \left[\underline{X}(t), t \right], \tag{35}$$

where $P_G(t,T)$ *is the corresponding exact Gaussian bond price computed under proposition 1, and* $V_1 [\underline{X}(t), t]$ *is given by equation (29) with* $\varphi(l,T) = A(T-l)$*, and* $\underline{\psi}(l,T) = \underline{B}(T-l)$*.*

Proof. This result simply follows from Corollary 1, by comparing equations (1) and (28). ■

The analytical results obtained so far in this section can be further used to value all interest rate contingent claims whose price can be decomposed into a portfolio of pure discount bonds (as it is the case, for instance, of a coupon-bearing bond).

Moreover, under the Duffie and Singleton (1997) assumption of symmetric counterparty credit risk, proposition 6 can also be used to value forward rate agreements and interest rate swaps. In fact, Duffie and Singleton (1997) have shown that, as long as the counterparties have symmetric probabilities of default, any term structure model previously formulated for government yield curves can also be used to price defaultable interest rate contingent claims, after the short-term interest rate process is adjusted for default and liquidity factors. Therefore, the symmetric credit risk assumption as well as the other four implicit hypothesis described in Duffie and Singleton (1997, section 1) will be adopted in this paper whenever the pricing of LIBOR-rate derivatives is dealt with. Note however that, since the risk-free short-term interest rate must be replaced by a risk- and liquidity-adjusted instantaneous interest rate process when the term structure model is applied to LIBOR-rate derivatives, it is not possible to price simultaneously riskless and defaultable interest rate contingent claims.

Following, for instance, Baxter and Rennie (1996, section 5.6), the time-t fixed rate corresponding to a zero FRA value, under the stochastic volatility Duffie and Kan (1996) model, is equal to the forward interest rate

$$\frac{1}{t_2 - t_1} \left[\frac{P_S(t,t_1)}{P_S(t,t_2)} - 1 \right],$$

where t_1 and t_2 are the maturity dates of the FRA contract and of its underlying borrowing/lending operation, respectively $(t \leq t_1 \leq t_2)$. Similarly, the time-t fixed rate corresponding to a zero present value for a forward-start "plain-vanilla" IRS, under the stochastic volatility Duffie and Kan (1996) model, is equal to the forward swap rate

$$\frac{P_S(t,t_0) - P_S(t,t_m)}{\sum_{k=1}^{m} (t_k - t_{k-1}) P_S(t,t_k)},$$

where the swap starts at time t_0, and generates m cash flows at times t_k $(k = 1, \ldots, m)$, with $t_k \geq t_0 \geq t$. All the risky zero-coupon bond prices involved in the last two formulas can be quickly computed using proposition 6.

Now, the relevant (empirical) question is to verify the accuracy of the proposed first order approximation. That is to test whether the approximating terms of order higher then the first are small enough to be neglected, as predicted before.

5.2.2 Examples

Table 1 prices (unit face value) pure discount bonds and a 20-year swap rate (with semi-annually compounding), for different affine invariant transformations, using the three-factor CIR model of Schlogl and Sommer

(1998, Figure 5) where:

$$\begin{aligned} f &= 0, \underline{G} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}', \underline{X}(t) = 0.02 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}', a = diag\{-0.1, -0.15, -0.2\} \\ \underline{b} &= \begin{bmatrix} 0.002607 & 0.003 & 0.003426 \end{bmatrix}', \Sigma = diag\{0.03, 0.04, 0.05\}, \underline{\alpha} = \underline{0}, \beta = I_3 \end{aligned}$$

being $\underline{\alpha} \in \Re^n$ a vector with α_i as its i^{th} element. Exact stochastic volatility zero-coupon bond prices are computed using the exact numerical solution of equations (5) and (6), through an adaptive stepsize fifth-order Runge-Kutta method (for $\underline{B}(\tau)$), and Romberg's integration (for $A(\tau)$).[17] Approximate stochastic volatility prices are given by the "explicit" first order formula (35), where its second term is implemented using Romberg's integration method on a closed interval, and the associated Gaussian pure discount bond prices are computed from proposition 1. For each affine transformation, instead of the zero and first order approximate prices, the corresponding percentage pricing errors are presented. Throughout this paper, the CPU time is always shown in seconds (except if stated otherwise), and all computations are made running Pascal programs on a Pentium 233MHz with 32MB of RAM memory.

For the transformation (25), the lower and upper bounds of the first order approximating term are computed accordingly to equations (33) and (32), respectively. Based on these, the maximum absolute percentage error arising from assuming that $\mathbf{D} = \Re^n$ in computing $V_1[\underline{X}(t), t]$ is presented.

The overall conclusion is that the proposed approximation is very accurate: all invariant transformations produce pricing errors for the IRS smaller than a tenth of a basis point. In other words, the neglected approximating terms (of order higher then the first) seem to constitute an irrelevant part of the (stochastic volatility) pure discount bond price. Moreover, the use of the approximate formula is also faster since it avoids the solution of the Riccati equations (5) through Runge-Kutta methods: the swap rate was forty times faster to compute using the explicit first order approximation! Notice also that the first order approximation is always more accurate than the zeroth order one.

Table 2 presents the same empirical analysis as before, but using the following $A_2(3)$ model:[18]

$$\begin{aligned} f &= -0.00394, \underline{G} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}', a = \begin{bmatrix} -2.78 & -0.41238 & 1386.106 \\ 0 & 0.02138 & 39.9 \\ 0 & 0.000741 & -2.2328 \end{bmatrix}, \underline{\alpha} = \underline{0}, \\ \underline{b} &= \begin{bmatrix} -6.1e-18 \\ 0.002445 \\ 9.49e-05 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & -1 & -252 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.00237 & 0 \\ 1 & 0 & 6.45e-05 \end{bmatrix}, \end{aligned}$$

where the state variables' values, $\underline{X}(t) = \begin{bmatrix} 0.01 & 0.03 & 0.0001 \end{bmatrix}'$, were defined in order to have an upward slopping yield curve (the spot rates with continuous compounding vary from 4.564%, for three months, to 10.285%, for 20 years). The first order explicit approximation is still fast to implement and accurate (although the pricing errors are higher for longer maturities). As before, the pricing errors resulting from extending $\mathbf{D}$ to $\Re^n$ in computing $V_1[\underline{X}(t), t]$ are small (at least for short maturities).

5.3 Bond futures

5.3.1 Exact pricing solutions

In order to obtain a stochastic volatility approximate pricing formula, zero-coupon bond futures (e.g. futures on Treasury Bills) must be first priced, in exact terms, under the Gaussian Duffie and Kan (1996) specification. Hereafter the hypothesis of continuous marking to market will be assumed, whenever futures contracts are involved.

Proposition 7 *Under the deterministic volatility specification of the Duffie and Kan (1996) model, the time-t price, $FP_G(t, T_f, T_1)$, of a futures contract for delivery at time T_f and on a pure discount bond with*

[17] Although the exact analytical solution of Chen and Scott (1995, page 54) is also available, in general, the stochastic volatility Duffie and Kan (1996) model does not produce exact closed-form solutions. Therefore, the efficiency of the explicit first order approximations shall be compared against the exact numerical solutions.

[18] This model specification was borrowed from a previous version (Table IV) of the Dai and Singleton (1998) paper.

maturity at time T_1 $(t \leq T_f \leq T_1)$ *is equal to*

$$FP_G(t, T_f, T_1) = \frac{P_G(t, T_1)}{P_G(t, T_f)} \exp[-J(t)], \tag{36}$$

where

$$\begin{aligned} J(t) &= \underline{G}' \cdot a^{-1} \cdot \left\{ \Theta \cdot \left(a^{-1}\right)' \cdot \left[e^{a'(T_1 - T_f)} + e^{a'(T_f - t)} - e^{a'(T_1 - t)} - I_n \right] \right. \\ &\quad \left. + \Delta (T_f - t) \cdot \left[e^{a'(T_1 - T_f)} - I_n \right] \right\} \cdot \left(a^{-1}\right)' \cdot \underline{G}. \end{aligned}$$

Proof. See, for instance, Nunes (1998, subsection 5.1). ■

Under the stochastic volatility specification, and as Duffie and Kan (1996) did for pure discount bonds, it is also possible to find an exponential-affine exact pricing formula for futures on zero-coupon bonds that involves maturity-dependent functions satisfying Riccati differential equations.

Proposition 8 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t price,* $FP_S(t, T_f, T_1)$*, of a futures contract for delivery at time* T_f *and on a pure discount bond with maturity at time* T_1 $(t \leq T_f \leq T_1)$ *is equal to*

$$FP_S(t, T_f, T_1) = \frac{P_S(t, T_1)}{P_S(t, T_f)} \exp\left[C(t, T_f, T_1) + \underline{D}'(t, T_f, T_1) \cdot \underline{X}(t)\right], \tag{37}$$

where $\underline{D}(t, T_f, T_1) \in \Re^n$ *is the solution of*

$$\begin{aligned} \frac{\partial \underline{D}'(t, T_f, T_1)}{\partial t} &= -\underline{D}'(t, T_f, T_1) \cdot a \\ &\quad - \sum_{k=1}^{n} \underline{B}'(T_f - t) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \cdot \left[\underline{B}(T_f - t) - \underline{B}(T_1 - t)\right] \underline{\beta_k}' \\ &\quad - \frac{1}{2} \sum_{k=1}^{n} \underline{D}'(t, T_f, T_1) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \\ &\quad \cdot \left[2\underline{B}(T_1 - t) - 2\underline{B}(T_f - t) + \underline{D}(t, T_f, T_1)\right] \underline{\beta_k}' \end{aligned} \tag{38}$$

subject to $\underline{D}(T_f, T_f, T_1) = \underline{0}$*, and* $C(t, T_f, T_1) \in \Re$ *is obtained from*

$$\begin{aligned} \frac{\partial C(t, T_f, T_1)}{\partial t} &= -\underline{D}'(t, T_f, T_1) \cdot \underline{b} \\ &\quad - \sum_{k=1}^{n} \underline{B}'(T_f - t) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \cdot \left[\underline{B}(T_f - t) - \underline{B}(T_1 - t)\right] \alpha_k \\ &\quad - \frac{1}{2} \sum_{k=1}^{n} \underline{D}'(t, T_f, T_1) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \\ &\quad \cdot \left[2\underline{B}(T_1 - t) - 2\underline{B}(T_f - t) + \underline{D}(t, T_f, T_1)\right] \alpha_k \end{aligned} \tag{39}$$

subject to $C(T_f, T_f, T_1) = 0.$

Proof. See appendix F. ■

Remark 14 *Equation (38) can be solved numerically through Runge-Kutta methods, while equation (39) seems to only require univariate integration algorithms. However, both (38) and (39) involve the solution of Riccati equations similar to (5) at each evaluation point, since they are both functions of the duration vectors* $\underline{B}(T_f - t)$ *and* $\underline{B}(T_1 - t)$*. Therefore, the following explicit approximation should provide significant efficiency gains.*

5.3.2 A first order explicit approximation

Next proposition proposes an approximate stochastic volatility pricing solution that is easier to implement than the exact numerical one offered by proposition 8.

Proposition 9 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t price, $FP_S(t,T_f,T_1)$, of a futures contract for delivery at time T_f and on a pure discount bond with maturity at time T_1 $(t \leq T_f \leq T_1)$ can be approximated by*

$$FP_S(t,T_f,T_1) \cong FP_G(t,T_f,T_1) + \frac{1}{2}V_1[\underline{X}(t),t], \tag{40}$$

where $FP_G(t,T_f,T_1)$ is computed from proposition 7, and $V_1[\underline{X}(t),t]$ has the "explicit" solution given by equation (29) but with $T = T_f$, $\varphi(l,T) = A(T_1 - l) - A(T_f - l) - J(l)$, and $\underline{\psi}(l,T) = \underline{B}(T_1 - l) - \underline{B}(T_f - l)$.

Proof. This result follows from Corollary 1, by comparing equations (36) and (28). ■

To value futures on coupon-bearing bonds, and following Nunes (1998, equation (55)), it is just necessary to consider the summation of the prices of futures on zero-coupon bonds with maturity dates corresponding to the moments where cash flows are paid by the coupon bond, and with contract sizes equal to the value of such cash flows. That is[19]

$$F_S(t,T_f) = \sum_{i=1}^{N_f} k_i FP_S(t,T_f,T_i),$$

where $T_f < T_i$ $(\forall i)$, $F_S(t,T_f)$ represents the stochastic volatility time-t price of a futures contract for delivery at time T_f, on a coupon-bearing bond paying N_f cash flows k_i $(i = 1,\ldots,N_f)$ from the futures' expiry date and until the bond's maturity date (T_{N_f}), and $FP_S(t,T_f,T_i)$ is computed under proposition 9.

5.3.3 Example

Table 3 values futures with a maturity of 6 months on (unit face value) zero-coupon bonds with maturities ranging from 1 year to 20.5 years, using the three-factor CIR model of Schlogl and Sommer (1998, Figure 5), as well as a futures contract with a maturity of 6 months, on a theoretical coupon-bearing bond with a maturity of 20 years at the futures expiry date, with a semi-annual coupon rate of 8% per annum, and with a face value of 100. No provision is made for the existence of delivery options. Exact stochastic volatility futures prices were obtained from proposition 8, by using an adaptive stepsize fifth-order Runge-Kutta method for equation (38), and Romberg's integration method for equation (39). Approximate stochastic volatility futures prices were computed through the first order "explicit" solution obtained in proposition 9, and the corresponding Gaussian futures price resulted from proposition 7.

A new transformation is also tested where $\underline{u}$ is defined in order for the variance of the state variables, at the futures' expiry date and conditional on the current value of the state vector, to be equal between the nested Gaussian and the multifactor CIR models.[20] As before, different affine invariant transformations produce similar results: the proposed first order explicit stochastic volatility approximation is still accurate and extremely fast to implement. Moreover, the pricing errors arising from approximating the integration domain in computing $V_1[\underline{X}(t),t]$ are again negligeable.

[19] Ignoring the existence of quality and/or timing options.

[20] Writing the multifactor CIR model, under measure Q, as $r(t) = \sum_{j=1}^{n} X_j(t)$ with

$$dX_j(t) = [k_j\theta_j - (k_j + \lambda_j)X_j(t)]\,dt + \sigma_j\sqrt{X_j(t)}dW_j^Q(t), j = 1,\ldots,n,$$

and applying the invariant transformation (17), it can be shown that the matching of the factor Gaussian and multifactor CIR variances for maturity $T(\geq t)$ is obtained if

$$u_j = \frac{\theta_j + [2X_j(t) - \theta_j]\,e^{-(k_j+\lambda_j)(T-t)}}{e^{-(k_j+\lambda_j)(T-t)} + 1}, j = 1,\ldots,n,$$

where u_j is the j^{th} element of vector $\underline{u}$.

5.4 Short-term interest rate futures

This subsection considers the valuation of futures on short-term *nominal*[21] money-market forward interest rates. This is the case, for instance, of the widely traded Eurodollar futures contract, where the underlying nominal interest rate is the LIBOR of the USD for a three months period. In what follows, all interest rates and all bond prices are assumed to be risk-adjusted along the lines of Duffie and Singleton (1997).

5.4.1 Exact pricing solutions

Proposition 10 *Under the deterministic volatility specification of the Duffie and Kan (1996) model, the time-t price, $FR_G(t, T_f, T_1)$, of a futures contract with maturity at time T_f and on the nominal interest rate for the period $(T_1 - T_f)$, with $T_1 \geq T_f \geq t$, is equal to*

$$FR_G(t, T_f, T_1) = 100\left\{1 - \frac{1}{T_1 - T_f}\left[\frac{P_G(t, T_f)}{P_G(t, T_1)} e^{L(t)} - 1\right]\right\}, \tag{41}$$

where

$$\begin{aligned} L(t) &= \underline{G}' \cdot a^{-1} \cdot \left\{\Theta \cdot \left(a^{-1}\right)' \cdot \left[e^{a'(T_1 - T_f)} + e^{a'(T_f - t)} - e^{a'(T_1 - t)} - I_n\right]\right. \\ &\quad \left. + e^{a(T_1 - T_f)} \cdot \Delta(T_f - t) \cdot \left[e^{a'(T_1 - T_f)} - I_n\right]\right\} \cdot \left(a^{-1}\right)' \cdot \underline{G}. \end{aligned}$$

Proof. See appendix G. ■

Proposition 11 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t price, $FR_S(t, T_f, T_1)$, of a futures contract with maturity at time T_f and on the nominal interest rate for the period $(T_1 - T_f)$, with $T_1 \geq T_f \geq t$, is equal to*

$$\begin{aligned} FR_S(t, T_f, T_1) &= 100\left\{1 - \frac{1}{T_1 - T_f}\left[\frac{P_S(t, T_f)}{P_S(t, T_1)}\right.\right. \\ &\quad \left.\left. \exp\left(E(t, T_f, T_1) + \underline{F}'(t, T_f, T_1) \cdot \underline{X}(t)\right) - 1\right]\right\}, \end{aligned} \tag{42}$$

where $\underline{F}(t, T_f, T_1) \in \Re^n$ is the solution of

$$\begin{aligned} \frac{\partial \underline{F}'(t, T_f, T_1)}{\partial t} &= -\underline{F}'(t, T_f, T_1) \cdot a \\ &\quad - \sum_{k=1}^{n} \underline{B}'(T_1 - t) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \cdot \left[\underline{B}(T_1 - t) - \underline{B}(T_f - t)\right] \underline{\beta_k}' \\ &\quad - \frac{1}{2} \sum_{k=1}^{n} \underline{F}'(t, T_f, T_1) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \\ &\quad \cdot \left[2\underline{B}(T_f - t) - 2\underline{B}(T_1 - t) + \underline{F}(t, T_f, T_1)\right] \underline{\beta_k}' \end{aligned} \tag{43}$$

subject to $\underline{F}(T_f, T_f, T_1) = \underline{0}$, and $E(t, T_f, T_1) \in \Re$ is obtained from

$$\begin{aligned} \frac{\partial E(t, T_f, T_1)}{\partial t} &= -\underline{F}'(t, T_f, T_1) \cdot \underline{b} \\ &\quad - \sum_{k=1}^{n} \underline{B}'(T_1 - t) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \cdot \left[\underline{B}(T_1 - t) - \underline{B}(T_f - t)\right] \alpha_k \\ &\quad - \frac{1}{2} \sum_{k=1}^{n} \underline{F}'(t, T_f, T_1) \cdot \underline{\varepsilon_k} \cdot \underline{\varepsilon_k}' \\ &\quad \cdot \left[2\underline{B}(T_f - t) - 2\underline{B}(T_1 - t) + \underline{F}(t, T_f, T_1)\right] \alpha_k \end{aligned} \tag{44}$$

subject to $E(T_f, T_f, T_1) = 0$.

Proof. The derivation of the above exact numerical result is similar to the proof of proposition 8, and can be obtained upon request. ■

[21] In the present context, the term *nominal* means simple (as opposed to continuous) compounding, and is not used to distinguish from *real* interest rates.

5.4.2 A first order explicit approximation

Proposition 10 allows the next result to be extracted from Corollary 1.

Proposition 12 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t price, $FR_S(t,T_f,T_1)$, of a futures contract with maturity at time T_f and on the nominal interest rate for the period $(T_1 - T_f)$, with $T_1 \geq T_f \geq t$, can be approximated by*

$$FR_S(t,T_f,T_1) \cong FR_G(t,T_f,T_1) - \frac{50}{T_1 - T_f} V_1[\underline{X}(t), t], \tag{45}$$

where $FR_G(t,T_f,T_1)$ is computed from proposition 10, and $V_1[\underline{X}(t), t]$ has the "explicit" solution given by equation (29) but with $T = T_f$, $\varphi(l,T) = A(T_f - l) - A(T_1 - l) + L(l)$, and $\underline{\psi}(l,T) = \underline{B}(T_f - l) - \underline{B}(T_1 - l)$.

Proof. This result follows from Corollary 1, by comparing equation (28) with $\left[1 + (T_1 - T_f)\frac{100 - FR_G(t,T_f,T_1)}{100}\right]$, where $FR_G(t,T_f,T_1)$ is given by equation (41). ■

5.4.3 Example

Table 4 prices three-month Eurodollar futures contracts, with maturities varying from one month to 9 years, and based on the $A_1(3)_{DS}$ model of Dai and Singleton (1998, Table II), where

$$f = 0, \underline{G} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}', a = \begin{bmatrix} -0.33458 & 0 & 0 \\ 0.878876 & -0.226 & 0 \\ -9.190106 & 17.4 & -17.4 \end{bmatrix}, \underline{\alpha} = \underline{0},$$

$$\underline{b} = \begin{bmatrix} 0.005475 \\ 0.012350 \\ 0.021683 \end{bmatrix}, \Sigma = \begin{bmatrix} 0.088431 & 0 & 0 \\ 0 & 1 & -0.0943 \\ 0.377599 & -3.42 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the factor' values, $\underline{X}(t) = \begin{bmatrix} 0.01 & 0.12 & 0.11 \end{bmatrix}'$, were defined in order to have a downward slopping yield curve (the spot rates with continuous compounding vary from 11.396%, for six months, to 10.392%, for 20 years). The Gaussian prices are computed under proposition 10, while the approximate stochastic volatility ones are obtained from proposition 12. Exact numerical stochastic volatility futures prices result from proposition 11 (using a fifth-order Runge-Kutta method for equation (43), and Romberg's integration for expression (44)).

For all the invariant transformations tested, the pricing errors are almost inexistent. Again, the error induced by the extension of the integration domain from **D** to $\Re^n$ is very small, and the proposed first order approximation is much faster than the exact numerical solution as well as always more accurate than the zeroth order one.

6 Pricing of European Interest Rate Options

6.1 Explicit stochastic volatility approximation

Besides the already considered exponential-affine derivatives, it is also possible to obtain explicit first order pricing solutions for several European interest rate options, such as: options on pure discount bonds, caps and floors, yield options, and (*conventional* or *pure*) futures options on zero-coupon bonds and on short-term interest rates. Next Corollary establish the general first order explicit solution which can be applied (specialized) to any of the specific option contracts described before.

Corollary 2 *Let the time-t price of an European option, with maturity at date T_0 $(\geq t)$, and computed under the Gaussian specification of the Duffie and Kan (1996) model, be represented by:*

$$\begin{aligned} V_G[\underline{X}(t), t] &= \theta q\{\exp[U(t,\cdot) + \underline{Q}'(t,\cdot) \cdot \underline{X}(t)]\, \Phi[\theta d_1(t)] \\ &\quad - K \exp[S(t,T_0) + \underline{T}'(t,T_0) \cdot \underline{X}(t)]\, \Phi[\theta d_0(t)]\}, \end{aligned} \tag{46}$$

with

$$d_1(t) = \frac{\ln\left\{\frac{\exp\left[U(t,\cdot)+\underline{Q}'(t,\cdot)\cdot\underline{X}(t)\right]}{K\exp\left[S(t,T_0)+\underline{T}'(t,T_0)\cdot\underline{X}(t)\right]}\right\}+\frac{\sigma^2(t)}{2}}{\sigma(t)}, \tag{47}$$

$$d_0(t) = d_1(t) - \sigma(t), \tag{48}$$

$$\sigma^2(t) = \underline{Q}'(T_0,\cdot)\cdot\Delta(T_0-t)\cdot\underline{Q}(T_0,\cdot), \tag{49}$$

and where $\theta \in \{-1,1\}$, $q, K, U(t,\cdot) \in \Re$, $S(t,T_0) \in \Re$ *is such that* $S(T_0,T_0) = 0$, $\underline{Q}(t,\cdot) \in \Re^n$, *and* $\underline{T}(t,T_0) \in \Re^n$ *satisfies* $\underline{T}(T_0,T_0) = \underline{0}$.[22]

Under the assumptions of Theorem 1, and approximating **D** *by* $\Re^n$, *the corresponding price of the same option contract but for the stochastic volatility version of the Duffie and Kan (1996) model can be approximated by the first order analytical solution obtained from (13) with:*

$$V_1[\underline{X}(t),t] \cong q\int_t^{T_0} dl\,[V_{11}(l)+V_{12}(l)+V_{13}(l)]. \tag{50}$$

For $i = 1,2$*:*

$$\begin{aligned} V_{1i}(l) \;=\; & \theta[(2-i)-K(i-1)]P_G(t,l)\sqrt{\frac{|\Omega^{-1}(l)\cdot\Psi^{-1}(l)|}{|\Delta(l-t)\cdot\Delta(T_0-l)|}} \\ & \exp\left[F_i(l)-\frac{1}{2}\underline{M}'(l-t)\cdot\Delta^{-1}(l-t)\cdot\underline{M}(l-t)+\frac{1}{2}\underline{\mu c_i}'(l)\right. \\ & \cdot\Omega^{-1}(l)\cdot\underline{\mu c_i}(l)-\frac{1}{2}\underline{MC_i}'(T_0-l)\cdot\Delta^{-1}(T_0-l)\cdot\underline{MC_i}(T_0-l) \\ & \left.+\frac{1}{2}\underline{N_i}'(l)\cdot\Psi^{-1}(l)\cdot\underline{N_i}(l)\right]\left\{\left[\sum_{k=1}^{n}\left(\underline{D_i}'(l)\cdot\underline{\varepsilon_k}\right)^2\underline{\beta_k}'\right]\right. \\ & \cdot\Omega^{-1}(l)\cdot\underline{\mu c_i}(l)\,\Phi\left[\theta\frac{\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{N_i}(l)-K^*}{\sqrt{\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{H}(T_0)}}\right] \\ & +\frac{\lambda_i(l)}{\theta}\sqrt{\frac{\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{H}(T_0)}{2\pi}} \\ & \exp\left[-\frac{1}{2}\frac{\left(K^*-\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{N_i}(l)\right)^2}{\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{H}(T_0)}\right]+\lambda_i(l) \\ & \left.\left[\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{N_i}(l)\right]\Phi\left[\theta\frac{\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{N_i}(l)-K^*}{\sqrt{\underline{H}'(T_0)\cdot\Psi^{-1}(l)\cdot\underline{H}(T_0)}}\right]\right\}, \end{aligned} \tag{51}$$

where

$$F_i(l) = (2-i)U(l,\cdot)+(i-1)S(l,T_0),$$

$$\underline{D_i}(l) = (2-i)\underline{Q}(l,\cdot)+(i-1)\underline{T}(l,T_0),$$

$$\underline{H}(l) = \underline{Q}(l,\cdot)-\underline{T}(l,T_0),$$

$$K^* = \ln(K)-U(T_0,\cdot),$$

$$\begin{aligned} \underline{MC_i}(T_0-l) \;=\; & a^{-1}\cdot\left[e^{a(T_0-l)}-I_n\right]\cdot\left[\underline{b}+\Theta\cdot\left(a^{-1}\right)'\cdot\underline{G}\right] \\ & -\Delta(T_0-l)\cdot\left[\left(a^{-1}\right)'\cdot\underline{G}-(2-i)\cdot\underline{Q}(T_0,\cdot)\right], \end{aligned}$$

$$\Omega(l) = \Delta^{-1}(l-t)+e^{a'(T_0-l)}\cdot\Delta^{-1}(T_0-l)\cdot e^{a(T_0-l)},$$

[22]The functions $U(t,\cdot)$ and $\underline{Q}(t,\cdot)$ may involve other maturities than just the current time, depending on the contract specifications.

$$\underline{\mu c_i}(l) = \Delta^{-1}(l-t)\cdot \underline{M}(l-t) + \underline{D_i}(l) \\ -e^{a'(T_0-l)}\cdot \Delta^{-1}(T_0-l)\cdot \underline{MC_i}(T_0-l),$$

$$\Psi(l) = \Delta^{-1}(T_0-l)\cdot\left[I_n - e^{a(T_0-l)}\cdot \Omega^{-1}(l)\cdot e^{a'(T_0-l)}\cdot \Delta^{-1}(T_0-l)\right],$$

$$\underline{N_i}(l) = \Delta^{-1}(T_0-l)\cdot\left[\underline{MC_i}(T_0-l) + e^{a(T_0-l)}\cdot \Omega^{-1}(l)\cdot \underline{\mu c_i}(l)\right],$$

$$\lambda_i(l) = \frac{\underline{C_i}'(l)\cdot \underline{1}}{\underline{H}'(T_0)\cdot \underline{1}},$$

and

$$\underline{C_i}'(l) = \left[\sum_{k=1}^{n}\left(\underline{D_i}'(l)\cdot\underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\right]\cdot \Omega^{-1}(l)\cdot e^{a'(T_0-l)}\cdot \Delta^{-1}(T_0-l).$$

For $i = 3$:

$$V_{13}(l) = \frac{KP_G(t,l)}{\sqrt{2\pi\sigma^2(l)}}\sqrt{\frac{|\varphi^{-1}(l)|}{|\Delta(l-t)|}}\exp\left\{S(l,T_0) - \frac{[d_0^*(l)]^2}{2\sigma^2(l)}\right. \\ \left. -\frac{1}{2}\underline{M}'(l-t)\cdot\Delta^{-1}(l-t)\cdot\underline{M}(l-t) + \frac{1}{2}\underline{m}'(l)\cdot\varphi^{-1}(l)\cdot\underline{m}(l)\right\} \\ \left\{\sum_{k=1}^{n}\left[\underline{H}'(l)\cdot\underline{\varepsilon_k}\right]^2\underline{\beta_k}'\right\}\cdot\varphi^{-1}(l)\cdot\underline{m}(l), \tag{52}$$

where

$$d_0^*(l) = U(l,\cdot) - S(l,T_0) - \ln(K) - \frac{\sigma^2(l)}{2},$$

$$\underline{m}(l) = \Delta^{-1}(l-t)\cdot\underline{M}(l-t) + \underline{T}(l,T_0) - \frac{d_0^*(l)}{\sigma^2(l)}\cdot\underline{H}(l),$$

and

$$\varphi(l) = \Delta^{-1}(l-t) + \frac{1}{\sigma^2(l)}\underline{H}(l)\cdot\underline{H}'(l).$$

Proof. See appendix H. ■

6.2 Options on pure discount bonds

6.2.1 A first order explicit approximation

In order to apply Corollary 2 to the valuation of European options on zero-coupon bonds under the stochastic volatility specification of the Duffie and Kan (1996) model, an exact pricing formula for these contingent claims but under the Langetieg (1980) model specification must be used. Such closed form solution is provided by the following proposition.

Proposition 13 *Under the Gaussian specification of the Duffie and Kan (1996) model, the time-t price of an European call on the riskless pure discount bond $P_G(t,T_1)$, with a strike price equal to K, and with maturity at time T_0 (such that $t \le T_0 \le T_1$) is equal to*

$$c_t^G[P_G(t,T_1);K;T_0] = P_G(t,T_1)\,\Phi[d_1(t)] - KP_G(t,T_0)\,\Phi[d_0(t)], \tag{53}$$

with

$$d_1(t) = \frac{\ln\left[\frac{P_G(t,T_1)}{KP_G(t,T_0)}\right] + \frac{V_1(\tau_0)}{2}}{\sqrt{V_1(\tau_0)}},$$

$$d_0(t) = d_1(t) - \sqrt{V_1(\tau_0)},$$

$$V_1(\tau_0) = \underline{B}'(T_1 - T_0) \cdot \Delta(T_0 - t) \cdot \underline{B}(T_1 - T_0),$$

and where $\tau_0 = T_0 - t$. *The corresponding put price is*

$$p_t^G[P_G(t, T_1); K; T_0] = -P_G(t, T_1)\,\Phi[-d_1(t)] + K P_G(t, T_0)\,\Phi[-d_0(t)]. \tag{54}$$

Proof. See, for instance, Nunes (1998, subsections 4.1 and 4.2). ■

Using the last proposition, Corollary 2 can now be specialized for European options on pure discount bonds.

Proposition 14 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t price of an European call on the riskless pure discount bond* $P_S(t, T_1)$, *with a strike price equal to* K, *and with maturity at time* T_0 *(such that* $t \leq T_0 \leq T_1$*) can be approximated by*

$$c_t^S[P_S(t, T_1); K; T_0] \cong c_t^G[P_G(t, T_1); K; T_0] + \frac{1}{2} V_1[\underline{X}(t), t], \tag{55}$$

where $c_t^G[P_G(t, T_1); K; T_0]$ *is computed from equation (53), and* $V_1[\underline{X}(t), t]$ *has the "explicit" solution given by equation (50) but with* $q = 1$, $U(t, \cdot) = A(T_1 - t)$, $\underline{Q}(t, \cdot) = \underline{B}(T_1 - t)$, $S(t, T_0) = A(T_0 - t)$, $\underline{T}(t, T_0) = \underline{B}(T_0 - t)$, *and* $\theta = 1$. *The corresponding stochastic volatility put price can be approximated by*

$$p_t^S[P_S(t, T_1); K; T_0] \cong p_t^G[P_G(t, T_1); K; T_0] + \frac{1}{2} V_1[\underline{X}(t), t], \tag{56}$$

where $p_t^G[P_G(t, T_1); K; T_0]$ *is obtained from equation (54), and* $V_1[\underline{X}(t), t]$ *is similarly computed but with* $\theta = -1$.

Proof. Comparing equations (53) and (54) with the general Gaussian option price (46), proposition 14 follows immediately. ■

6.2.2 Caps, floors, yield options, and swaptions

The results obtained so far for European options on default-free pure discount bonds can be easily generalized for European options on *nominal* "money-market" forward interest rates, under the Duffie and Singleton (1997) assumption of symmetric counterparty credit risk.

For instance, the value of an interest rate cap can be decomposed into a portfolio of caplets. The terminal payoff of a standard caplet for the compounding period $(t_{i+1} - t_i)$, with $t_{i+1} > t_i$, occurs at time t_{i+1}, and is equal to:

$$[R(t_i, t_{i+1}) - k]^+ (t_{i+1} - t_i),$$

where $R(t_i, t_{i+1})$ is the time-t_i spot interest rate (with a compounding period of $(t_{i+1} - t_i)$ years) for the period $(t_{i+1} - t_i)$, k is the cap rate, and the cap is assumed to have a unit contract size. Therefore, the time-t value of the caplet, with $t \leq t_i$, is equal to the price of an European call on the time-t forward rate for the period $(t_{i+1} - t_i)$, with a strike equal to k, with maturity at time t_{i+1}, and with a contract size of $(t_{i+1} - t_i)$. However, it is well known -see, for instance, Baxter and Rennie (1996, page 171)- that the same caplet can be valued as an European put with maturity at time t_i, with a contract size of $[1 + (t_{i+1} - t_i)k]$, with a strike price of $\frac{1}{1+(t_{i+1}-t_i)k}$, and on a pure discount bond with maturity at time t_{i+1}. That is the time-t value of the caplet corresponds to

$$[1 + (t_{i+1} - t_i)k]\, p_t^S\left[P_S(t, t_{i+1}); \frac{1}{1 + (t_{i+1} - t_i)k}; t_i\right],$$

which can be computed, for the stochastic volatility specification of the Duffie and Kan (1996) model, using the first order approximation proposed in proposition 14.

Similarly, the time-t value of a floorlet for the compounding period $(t_{i+1} - t_i)$, with $t_{i+1} > t_i$, can be shown to be equal to the price of an European call with maturity at time t_i, with a contract size of $[1 + (t_{i+1} - t_i)k]$, with a strike price of $\frac{1}{1+(t_{i+1}-t_i)k}$, and on a pure discount bond with maturity at time t_{i+1}:

$$[1 + (t_{i+1} - t_i)k]\, c_t^S\left[P_S(t, t_{i+1}); \frac{1}{1 + (t_{i+1} - t_i)k}; t_i\right],$$

where k is now a floor rate. Consequently, an interest rate floor (i.e. a portfolio of floorlets) can also be valued using proposition 14. The same can be said about the valuation of interest rate borrowing/lending collars, since their value is decomposable into a long/short cap and a short/long floor, respectively.

In order to value European yield call and put options, with settlement in arrears (i.e. with payoff generated at time t_{i+1}), on the time-t nominal forward rate for the period $(t_{i+1} - t_i)$, with a strike equal to k, with maturity at time t_i, and with a unit contract size, it is simply necessary to divide the valuation formulas previously given for caplets and floorlets by the compounding period $(t_{i+1} - t_i)$.

Unfortunately, although an approximate solution can also be derived for European options on coupon-bearing bonds (and thus for European swaptions as well), based on the Gaussian *rank 1* Brace and Musiela (1994, equation (6.1)) formula, such first order stochastic volatility approximation can not be made explicit (i.e. the integration with respect to the model' state variables can not be eliminated from the final solution). However, because an analytical stochastic volatility first order solution exists for European options on pure discount bonds, it is always possible to price European swaptions using the *stochastic duration* approximation suggested by Wei (1997) and Munk (1998).[23]

In summary, the first order stochastic volatility explicit approximation derived for European options on pure discount bonds can be applied to a wide variety of effectively traded interest rate options.

6.2.3 Examples

Tables 5 to 7 price a five-year interest rate floor (with quarterly compounding), for different strikes and different invariant transformations, using the three-factor CIR model of Schlogl and Sommer (1998, Figure 5). The floor value is divided into 19 European calls:

$$Floor_0 = (1 + 0.25k) \sum_{i=1}^{19} c_0 \left[P(0, 0.25(i+1)); (1 + 0.25k)^{-1}; 0.25i \right],$$

where k is the floor rate and $c_0 [S; X; T]$ denotes the time-0 price of an European call on the asset S, with a strike X, and with maturity at time T. The exact multifactor CIR call prices are computed using the analytical Fourier transforms' approach of Chen and Scott (1995). The Duffie, Pan and Singleton (1998) pricing methodology is also implemented by computing the characteristic function not analytically but rather numerically,[24] using a 10-point Gaussian quadrature to invert each Fourier transform. Gaussian call prices are obtained from proposition 13, and first order approximate stochastic volatility prices are computed using proposition 14 (where equation (50) was implemented using Romberg's integration on an open interval).

Table 8 values an at-the-money five-year interest rate floor (with quarterly compounding), using the same $A_2(3)$ specification as in table 2. Because, in this case, no closed-form solution exists for European options on pure discount bonds, the exact price of each call was estimated through standard Monte Carlo simulation, using the usual Euler discretization of equation (3) with 1,000 time steps per year, independent normal variates generated through the Box-Muller algorithm, 200,000 simulations, and the numerical solution of equations (5) and (6) in order to compute the option' terminal payoff. Besides the Monte Carlo price estimate, the percentage of its standard error on the mean price is also shown.

In general terms, all the previous examples show that: *i)* the first order stochastic volatility approximation is still accurate and fast to implement for interest rate options; *ii)* the pricing errors increase with the maturity of the contingent claim and are higher for out-of-the-money options; *iii)* the first order approximating term improves significantly the zeroth order approximation; and *iv)* the proposed transformation (25) yields, on average, the best results.

Finally, using again the same $A_2(3)$ specification, Table 9 prices a 6-month European call on a 5-year coupon-bearing bond (with a 6% annual coupon and a face value of 100), for different strikes, through the approximation of Wei (1997) and Munk (1998). Once more, the proposed first order stochastic volatility explicit approximation is fast and accurate.

[23] In essence, an European call on a coupon-bearing bond, with strike X and maturity T, is approximated by ξ times an European call, with strike $\frac{X}{\xi}$ and maturity T, on a pure discount bond with expiry equal to the *stochastic duration* of the coupon bond. The constant ξ is the forward price of the coupon-bearing bond for its *stochastic duration*.

[24] Because, in general, the analytical form of the relevant characteristic function is unknown, this procedure enables the assessment of the computational time involved in this pricing methodology.

6.3 Futures options on pure discount bonds

This subsection only considers options with stock-style margining, also known as *conventional* futures options (using the terminology of Duffie (1989)): that is contracts with premium paid at the beginning of the option's life.

6.3.1 A first order explicit approximation

Starting with the Gaussian exact analytical solution,

Proposition 15 *Under the deterministic volatility specification of the Duffie and Kan (1996) model, the time-t premium of an European conventional call on the asset $FP_G(t,T_f,T_1)$, with a strike price of K_f, and expiry date at date T_0 (such that $t \le T_0 \le T_f \le T_1$), is equal to*

$$c_t^G[FP_G(t,T_f,T_1);K_f;T_0] = h(t,T_f,T_1)\,\Phi\left[d_1^f(t)\right] - P_G(t,T_0)\,K_f\Phi\left[d_0^f(t)\right] \tag{57}$$

where

$$d_1^f(t) = \frac{\ln\left[\frac{h(t,T_f,T_1)}{P_G(t,T_0)K_f}\right] + \frac{\sigma_h^2(t)}{2}}{\sigma_h(t)},$$

$$d_0^f(t) = d_1^f(t) - \sigma_h(t),$$

$$h(t,T_f,T_1) = P_G(t,T_0)\,FP_G(t,T_f,T_1)\exp[I(t)],$$

$$\begin{aligned} I(t) &= \underline{G}'\cdot a^{-1}\cdot\Theta\cdot(a')^{-1}\cdot[\underline{B}(T_1-t)+\underline{B}(T_f-T_0)-\underline{B}(T_1-T_0) \\ &\quad -\underline{B}(T_f-t)] + \underline{G}'\cdot a^{-1}\cdot\Delta(T_0-t)\cdot[\underline{B}(T_f-T_0)-\underline{B}(T_1-T_0)], \end{aligned}$$

and

$$\begin{aligned} \sigma_h^2(t) &= [\underline{B}'(T_1-T_0)-\underline{B}'(T_f-T_0)]\cdot\Delta(T_0-t) \\ &\quad \cdot[\underline{B}(T_1-T_0)-\underline{B}(T_f-T_0)]. \end{aligned}$$

The corresponding put price is

$$\begin{aligned} p_t^G[FP_G(t,T_f,T_1);K_f;T_0] &= -h(t,T_f,T_1)\,\Phi\left[-d_1^f(t)\right] \\ &\quad + P_G(t,T_0)\,K_f\Phi\left[-d_0^f(t)\right]. \end{aligned} \tag{58}$$

Proof. See, for instance, Nunes (1998, subsections 6.1 and 6.2). ■

Next proposition applies Corollary 2 to the valuation of European options on pure discount bond futures.

Proposition 16 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t premium of an European conventional call on the asset $FP_S(t,T_f,T_1)$, with a strike price of K_f, and expiry date at time T_0 (such that $t \le T_0 \le T_f \le T_1$), can be approximated by*

$$c_t^S[FP_S(t,T_f,T_1);K_f;T_0] \cong c_t^G[FP_G(t,T_f,T_1);K_f;T_0] + \frac{1}{2}V_1[\underline{X}(t),t], \tag{59}$$

where $c_t^G[FP_G(t,T_f,T_1);K_f;T_0]$ is computed from equation (57), and $V_1[\underline{X}(t),t]$ has the "explicit" solution given by equation (50) but with $q=1$, $U(t,\cdot) = A(T_0-t)+A(T_1-t)-A(T_f-t)-J(t)+I(t)$, $\underline{Q}(t,\cdot) = \underline{B}(T_0-t)+\underline{B}(T_1-t)-\underline{B}(T_f-t)$, $S(t,T_0) = A(T_0-t)$, $\underline{T}(t,T_0) = \underline{B}(T_0-t)$, $K = K_f$, and $\theta = 1$. The corresponding stochastic volatility put price can be approximated by

$$p_t^S[FP_S(t,T_f,T_1);K_f;T_0] \cong p_t^G[FP_G(t,T_f,T_1);K_f;T_0] + \frac{1}{2}V_1[\underline{X}(t),t], \tag{60}$$

where $p_t^G[FP_G(t,T_f,T_1);K_f;T_0]$ is obtained from equation (58), and $V_1[\underline{X}(t),t]$ is similarly computed but with $\theta = -1$.

Proof. Comparing equations (57) and (58) with the general Gaussian option price (46), proposition 16 is obtained. ■

6.3.2 Example

Table 10 prices European futures calls on pure discount bonds, for different strikes, with $(T_0 - t, T_f - t, T_1 - t) = (0.25, 0.5, 2.5)$, and using the three-factor CIR model of Schlogl and Sommer (1998, Figure 5). Exact stochastic volatility prices are obtained through standard Monte Carlo simulation, as described in 6.2.3 (although now the terminal option payoff is computed from proposition 8). The Gaussian or zeroth order price is given by equation (57), and the first order approximation is obtained from proposition 16.

Again, the accuracy of the first order explicit stochastic volatility solution is acceptable (pricing errors of about one standard error of the Monte Carlo estimate), while its computational time is significantly lower than the one taken by the exact numerical result.

6.4 Options on short-term interest rate futures

This subsection is devoted to the valuation of European futures options on short-term *nominal* "money-market" forward interest rates, and makes use of the symmetric credit risk assumption of Duffie and Singleton (1997). First, exact Gaussian pricing formulas will be presented both for *conventional* and *pure* futures options. Then, such analytical Gaussian solutions will be generalized for the stochastic volatility specification of the Duffie and Kan (1996) model.

6.4.1 A first order explicit approximation

Proposition 17 *Under the deterministic volatility specification of the Duffie and Kan (1996) model, the time-t premium of an European conventional call on the futures contract $FR_G(t, T_f, T_1)$, with a strike price equal to K_R, and expiring at time T_0 (such that $t \leq T_0 \leq T_f \leq T_1$), is equal to*

$$\begin{aligned} & c_t^G \left[FR_G(t, T_f, T_1); K_R; T_0 \right] \\ = & \frac{100 P_G(t, T_0)}{T_1 - T_f} \left\{ -\frac{P_G(t, T_f)}{P_G(t, T_1)} \exp\left[L(T_0) + \rho(t) \right] \Phi\left[-d_1^R(t) \right] \right. \\ & \left. + \left[1 + (T_1 - T_f) \frac{100 - K_R}{100} \right] \Phi\left[-d_0^R(t) \right] \right\}, \end{aligned} \tag{61}$$

where

$$d_0^R(t) = \frac{\ln\left[\frac{\frac{P_G(t, T_f)}{P_G(t, T_1)}}{1 + (T_1 - T_f)\frac{100 - K_R}{100}} \right] + L(T_0) + \rho(t) - \frac{\sigma_R^2(t)}{2}}{\sigma_R(t)},$$

$$d_1^R(t) = d_0^R(t) + \sigma_R(t),$$

$$\sigma_R^2(t) = \left[\underline{B}'(T_1 - T_0) - \underline{B}'(T_f - T_0) \right] \cdot \Delta(\tau_0) \cdot \left[\underline{B}(T_1 - T_0) - \underline{B}(T_f - T_0) \right],$$

and

$$\rho(t) = \left[\underline{B}'(T_1 - T_0) - \underline{B}'(T_f - T_0) \right] \cdot \Delta(\tau_0) \cdot \underline{B}(T_1 - T_0).$$

The time-t premium of the corresponding European conventional put option is given by

$$\begin{aligned} & p_t^G \left[FR_G(t, T_f, T_1); K_R; T_0 \right] \\ = & \frac{100 P_G(t, T_0)}{T_1 - T_f} \left\{ \frac{P_G(t, T_f)}{P_G(t, T_1)} \exp\left[L(T_0) + \rho(t) \right] \Phi\left[d_1^R(t) \right] \right. \\ & \left. - \left[1 + (T_1 - T_f) \frac{100 - K_R}{100} \right] \Phi\left[d_0^R(t) \right] \right\}. \end{aligned} \tag{62}$$

Proof. See appendix I. ■

Remark 15 *If the maturity date of the futures option is the same as the delivery date of the underlying futures contract (as is the case, for instance, of the Quarterly Eurodollar futures options traded at the International Money Market Division of the Chicago Mercantile Exchange), equations (61) and (62) are still applicable but with T_0 replaced by T_f, i.e. with $L(T_0) + \rho(t) = \sigma_R^2(t) = V_1(T_f - t)$.*

Remark 16 *If $T_0 = T_f$, then equation (107) can be rewritten as*

$$
\begin{aligned}
& c_{T_f}^{G}\left[FR_G\left(T_f,T_f,T_1\right);K_R;T_f\right] \\
= & \ 100\left\{\frac{100-K_R}{100}-\frac{1}{T_1-T_f}\left[\frac{P_G\left(T_f,T_f\right)}{P_G\left(T_f,T_1\right)}-1\right]\right\}^{+},
\end{aligned}
$$

and

$$
p_{T_f}^{G}\left[FR_G\left(T_f,T_f,T_1\right);K_R;T_f\right] = -c_{T_f}^{G}\left[FR_G\left(T_f,T_f,T_1\right);K_R;T_f\right],
$$

where $R_G\left(t,T_f,T_1\right) = \frac{1}{T_1-T_f}\left[\frac{P_G(t,T_f)}{P_G(t,T_1)}-1\right]$ is the Gaussian time-t nominal forward rate for the time period $(T_1 - T_f)$. Therefore, equations (61) and (62), when $T_0 = T_f$, are also the pricing solutions for European puts and calls, respectively, on the nominal forward interest rate $R_G\left(t,T_f,T_1\right)$, with a strike equal to $\frac{100-K_R}{100}$, with a contract size of 100*, and with settlement at the option's expiry date (instead of settlement in arrears, as was the case in 6.2.2).*

All the valuation formulas derived so far in this subsection are only valid for futures options with stock-style margining. However, the short-term interest rate futures options traded at the London International Financial Futures Exchange (LIFFE) have futures-style margining requirements, that is are *pure* futures options accordingly to Duffie (1989). This means that the option premium is not paid at the time of purchase, but only when the contract is exercised. Moreover, option positions are marked-to-market daily, in exactly the same way as the underlying futures contract. Next proposition takes these features into account.

Proposition 18 *Under the deterministic volatility specification of the Duffie and Kan (1996) model, the time-t premium of a pure European futures call on the futures contract $FR_G\left(t,\ T_f,T_1\right)$, with a strike price equal to K_R, and maturity at date T_0 (such that $t \leq T_0 \leq T_f \leq T_1$), is equal to*

$$
\begin{aligned}
& Fc_t^G\left[FR_G\left(t,T_f,T_1\right);K_R;T_0\right] \qquad (63)\\
= & \ \frac{100}{T_1-T_f}\left\{-\frac{P_G\left(t,T_f\right)}{P_G\left(t,T_1\right)}\exp\left[L\left(t\right)\right]\Phi\left[-d_1^{FR}\left(t\right)\right]\right. \\
& \left.+\left[1+\left(T_1-T_f\right)\frac{100-K_R}{100}\right]\Phi\left[-d_0^{FR}\left(t\right)\right]\right\},
\end{aligned}
$$

where

$$
d_0^{FR}\left(t\right) = \frac{\ln\left[\frac{\frac{P_G\left(t,T_f\right)}{P_G\left(t,T_1\right)}}{1+\left(T_1-T_f\right)\frac{100-K_R}{100}}\right]+L\left(t\right)-\frac{\sigma_R^2(t)}{2}}{\sigma_R\left(t\right)},
$$

and

$$
d_1^{FR}\left(t\right) = d_0^{FR}\left(t\right)+\sigma_R\left(t\right).
$$

The time-t premium of the corresponding pure European futures put is given by

$$
\begin{aligned}
& Fp_t^G\left[FR_G\left(t,T_f,T_1\right);K_R;T_0\right] \qquad (64)\\
= & \ \frac{100}{T_1-T_f}\left\{\frac{P_G\left(t,T_f\right)}{P_G\left(t,T_1\right)}\exp\left[L\left(t\right)\right]\Phi\left[d_1^{FR}\left(t\right)\right]\right. \\
& \left.-\left[1+\left(T_1-T_f\right)\frac{100-K_R}{100}\right]\Phi\left[d_0^{FR}\left(t\right)\right]\right\}.
\end{aligned}
$$

Proof. The derivation of equations (63) and (64) is similar to the proof presented for proposition 17, and can be obtained upon request. ■

Remark 17 *Equations (63) and (64) can also be applied to value pure American futures options, because, and as shown by Chen and Scott (1993), the price of a pure American futures option before expiration will always exceed its intrinsic value, and therefore early exercise should not occur.*

The following proposition generalizes all the above results to the stochastic volatility specification of the Duffie and Kan (1996) model.

Proposition 19 *Under the stochastic volatility specification of the Duffie and Kan (1996) model, the time-t premium of an European option on the futures contract* $FR_S(t, T_f, T_1)$*, with a strike price equal to* K_R*, and expiring at time* T_0 *(such that* $t \leq T_0 \leq T_f \leq T_1$*), can be approximated by a first order solution where* $V_G[\underline{X}(t), t]$ *is computed under propositions 17 or 18, and* $V_1[\underline{X}(t), t]$ *has the "explicit" solution given by equation (50) but with* $q = \frac{100}{T_1 - T_f}$*,* $U(t, \cdot) = A(T_f - t) - A(T_1 - t) + \phi[A(T_0 - t) + L(T_0) + \rho(t)] + (1 - \phi) L(t)$*,* $\underline{Q}(t, \cdot) = \underline{B}(T_f - t) - \underline{B}(T_1 - t) + \phi \underline{B}(T_0 - t)$*,* $S(t, T_0) = \phi A(T_0 - t)$*,* $\underline{T}(t, T_0) = \phi \underline{B}(T_0 - t)$*, and* $K = 1 + (T_1 - T_f) \frac{100 - K_R}{100}$*. For conventional futures options* $\phi = 1$*, while for pure futures options* $\phi = 0$*. For puts* $\theta = 1$*, and for calls* $\theta = -1$*.*

Proof. Comparing equations (61), (62), (63) and (64) with the general Gaussian option price (46), proposition 19 follows. ■

6.4.2 Example

Table 11 prices 6-month pure puts on 3-month Eurodollar futures (with $T_f = T_0$), for different strikes, and using the $A_1(3)_{DS}$ model of Dai and Singleton (1998, Table II). Exact stochastic volatility option prices are obtained through standard Monte Carlo simulation (with terminal option payoff computed from proposition 11), Gaussian prices are given by equation (64), and the first order approximation results from proposition 19. As before, the proposed approximation is fast and accurate.

7 Conclusions

The main purpose and contribution of this paper consisted in providing (approximate) pricing formulas, under the most general multifactor, mean-reverting, time-homogeneous, and affine term structure model, that only involve one integral with respect to the maturity of the contingent claim under valuation, and are therefore extremely easy to implement in practice.

Starting by fitting a Gaussian-type of model as a "special" (nested) case of the more general Duffie and Kan (1996) model specification, the functional form for Arrow-Debreu prices under such Gaussian version was derived. Then, the exact Gaussian valuation formulas were converted into approximate stochastic volatility ones that involved integrals with respect not only to the maturity of the contingent claim under valuation but also to each one of the model' factors. Finally, and taking advantage of the analytical tractability provided by the "special" model specification adopted, all stochastic volatility pricing formulas were simplified into first order approximate ones that do not involve any integration with respect to the model' state variables.

Such factor-integral independent stochastic volatility valuation formulas were derived for a wide range of interest rate contingent claims: bonds, FRAs, IRSs, interest rate futures, European options on pure discount bonds, caps and floors, yield options, and European futures options on zero-coupon bonds and on short-term interest rates. The empirical results presented in this paper, for different parameter' configurations, have shown that the proposed approximations are extremely fast to implement as well as accurate. In fact, because there is no need to integrate numerically with respect to each state variable, the numerical efficiency of these pricing formulas is still good for high dimensional model specifications. An additional advantage of the first order explicit approximate stochastic volatility pricing formulae proposed in this paper is that they can be easily differentiated with respect to each state variable, and thus enable the implementation of dynamic hedging strategies. As an accessory result, exact pricing solutions were obtained for long-term and short-term interest rate futures, under the "general" specification of the Duffie and Kan (1996) model.

In terms of practical applicability, the proposed explicit approximate stochastic volatility pricing formulae constitute efficient tools to estimate (using, for instance, a non-linear Kalman filter approach) exponential-affine term structure models, based on market information about LIBOR rates, FRAs, short-term interest rate futures and futures options, swaps, caps, floors, and even European swaptions.

A Appendix: Proof of Proposition 3

The Arrow-Debreu security $G\left[\underline{X}(T), T; \underline{X}(t), t\right]$, as any other contingent claim, is, under the deterministic volatility specification of the Duffie and Kan (1996) model, the solution of the following Kolmogorov's backward equation

$$\begin{aligned} 0 &= \mathcal{D}_G G\left[\underline{X}(T), T; \underline{X}(t), t\right] + \frac{\partial G\left[\underline{X}(T), T; \underline{X}(t), t\right]}{\partial t} \\ &\quad - r(t)\, G\left[\underline{X}(T), T; \underline{X}(t), t\right], \end{aligned} \tag{65}$$

$\underline{X}(t) \in \Re^n$, subject to a specific boundary condition

$$G\left[\underline{X}(T), T; \underline{X}(t), T\right] = \delta\left[\underline{X}(t) - \underline{X}(T)\right], \underline{X}(T) \in \Re^n, \tag{66}$$

where $\mathcal{D}_G$ is the infinitesimal generator of $\underline{X}$ under the nested deterministic volatility specification of the Duffie and Kan (1996) model, i.e.

$$\begin{aligned} \mathcal{D}_G G\left[\underline{X}(T), T; \underline{X}(t), t\right] &= \frac{\partial G\left[\underline{X}(T), T; \underline{X}(t), t\right]}{\partial \underline{X}'(t)} \cdot \left[a \cdot \underline{X}(t) + \underline{b}\right] \\ &\quad + \frac{1}{2} tr\left\{ \frac{\partial^2 G\left[\underline{X}(T), T; \underline{X}(t), t\right]}{\partial \underline{X}(t)\, \partial \underline{X}'(t)} \cdot \Theta \right\}, \end{aligned} \tag{67}$$

$tr(A)$ represents the trace of A, and $\delta[\cdot]$ is the Dirac delta function. Similarly, the Fourier transform of $G\left[\underline{X}(T), T; \underline{X}(t), t\right]$,

$$\tilde{G}\left[\underline{\phi}, T; \underline{X}(t), t\right] = \int_{\underline{X}(T) \in \Re^n} d\underline{X}(T) \frac{\exp\left[i\underline{\phi}' \cdot \underline{X}(T)\right]}{\sqrt{(2\pi)^n}} G\left[\underline{X}(T), T; \underline{X}(t), t\right], \tag{68}$$

with $\underline{\phi} \in \Re^n$ and $i^2 = -1$, is the solution of the following PDE

$$\mathcal{D}_G \tilde{G}\left[\underline{\phi}, T; \underline{X}(t), t\right] + \frac{\partial \tilde{G}\left[\underline{\phi}, T; \underline{X}(t), t\right]}{\partial t} - r(t)\, \tilde{G}\left[\underline{\phi}, T; \underline{X}(t), t\right] = 0, \tag{69}$$

$\underline{X}(t) \in \Re^n$, subject to the boundary condition

$$\tilde{G}\left[\underline{\phi}, T; \underline{X}(t), T\right] = \frac{1}{\sqrt{(2\pi)^n}} \exp\left[i\underline{\phi}' \cdot \underline{X}(t)\right]. \tag{70}$$

Substituting the trial solution

$$\tilde{G}\left[\underline{\phi}, T; \underline{X}(t), t\right] = \frac{1}{\sqrt{(2\pi)^n}} \exp\left[\tilde{G}1\left(\tau; \underline{\phi}\right) + \underline{\tilde{G}2}'\left(\tau; \underline{\phi}\right) \cdot \underline{X}(t)\right], \tag{71}$$

with $\tilde{G}1\left(0; \underline{\phi}\right) = 0$ and $\underline{\tilde{G}2}\left(0; \underline{\phi}\right) = i\underline{\phi}$, into equations (69) and (70), the last PDE can be split into one n-dimensional ODE for $\underline{\tilde{G}2}\left(\tau; \underline{\phi}\right) \in \Re^n$,

$$\frac{\partial}{\partial \tau} \underline{\tilde{G}2}'\left(\tau; \underline{\phi}\right) = -\underline{G}' + \underline{\tilde{G}2}'\left(\tau; \underline{\phi}\right) \cdot a,$$

and into another one-dimensional ODE for $\tilde{G}1\left(\tau; \underline{\phi}\right) \in \Re$,

$$\frac{\partial}{\partial \tau} \tilde{G}1\left(\tau; \underline{\phi}\right) = -f + \underline{\tilde{G}2}'\left(\tau; \underline{\phi}\right) \cdot \underline{b} + \frac{1}{2} \underline{\tilde{G}2}'\left(\tau; \underline{\phi}\right) \cdot \Theta \cdot \underline{\tilde{G}2}\left(\tau; \underline{\phi}\right).$$

The first n-dimensional ODE, subject to the terminal condition $\underline{\tilde{G}2}\left(0; \underline{\phi}\right) = i\underline{\phi}$, has the solution

$$\underline{\tilde{G}2}'\left(\tau; \underline{\phi}\right) = i\underline{\phi}' \cdot e^{a\tau} + \underline{B}'(\tau), \tag{72}$$

where the Gaussian duration vector $\underline{B}'(\tau)$ is given by proposition 1. To solve the last one-dimensional ODE, subject to $\tilde{G}1(0;\underline{\phi}) = 0$, result (72) can be used, yielding, after simplifications:

$$\begin{aligned} \tilde{G}1(\tau;\underline{\phi}) &= A(\tau) + i\underline{\phi}' \cdot a^{-1} \cdot (e^{a\tau} - I_n) \cdot \underline{b} - \frac{1}{2}\underline{\phi}' \cdot \Delta(\tau) \cdot \underline{\phi} \\ &\quad + i\underline{\phi}' \cdot \left[a^{-1} \cdot (e^{a\tau} - I_n) \cdot \Theta - \Delta(\tau)\right] \cdot \left(a^{-1}\right)' \cdot \underline{G}, \end{aligned} \tag{73}$$

where $A(\tau)$ is computed under proposition 1, and $\Delta(\tau)$ is given by equations (10) or (11).

Substituting solutions (72) and (73) into equation (71), and inverting equation (68), yields

$$\begin{aligned} G[\underline{X}(T), T; \underline{X}(t), t] &= P_G(t,T) \frac{1}{(2\pi)^n} \int_{\underline{\phi}} d\underline{\phi} \exp\left[-i\underline{\phi}' \cdot \underline{X}(T)\right] \\ &\quad \exp\left[i\underline{\phi}' \cdot \underline{M}(\tau) - \frac{1}{2}\underline{\phi}' \cdot \Delta(\tau) \cdot \underline{\phi}\right]. \end{aligned}$$

Since the second exponential inside the integral is just the characteristic function of a normal n-dimensional random variable with mean $\underline{M}(\tau)$ and variance $\Delta(\tau)$, equation (6) of Shephard (1991) implies the closed form solution (12). ■

B Appendix: Proof of Theorem 1

Under the general specification of the Duffie and Kan (1996) model, the time-t value $V_S[\underline{X}(t), t]$ of any contingent claim with terminal payoff $H[\underline{X}(T)]$ and continuous "dividend yield" $i[\underline{X}(t), t]$ is the solution of the following initial value problem:

$$-i[\underline{X}(t), t] = \mathcal{D}_S V_S[\underline{X}(t), t] + \frac{\partial V_S[\underline{X}(t), t]}{\partial t} - r(t) V_S[\underline{X}(t), t], \tag{74}$$

$\underline{X}(t) \in \mathbf{D}$, subject to

$$V_S[\underline{X}(T), T] = H[\underline{X}(T)], \underline{X}(T) \in \mathbf{D}, \tag{75}$$

where $\mathcal{D}_S$ is the infinitesimal generator of $\underline{X}$ under the "stochastic volatility" specification, i.e.

$$\begin{aligned} \mathcal{D}_S V_S[\underline{X}(t), t] &= \frac{\partial V_S[\underline{X}(t), t]}{\partial \underline{X}'(t)} \cdot [a \cdot \underline{X}(t) + \underline{b}] \\ &\quad + \frac{1}{2} tr \left\{ \frac{\partial^2 V_S[\underline{X}(t), t]}{\partial \underline{X}(t) \partial \underline{X}'(t)} \cdot \Sigma \cdot V^D(t) \cdot \Sigma' \right\}. \end{aligned} \tag{76}$$

Moreover, because

$$\Sigma \cdot V^D(t) \cdot \Sigma' = \Theta + \Sigma \cdot W^D(t) \cdot \Sigma', \tag{77}$$

it is possible to rewrite equation (74) as:

$$\begin{aligned} &-i[\underline{X}(t), t] - \frac{1}{2} tr \left\{ \frac{\partial^2 V_S[\underline{X}(t), t]}{\partial \underline{X}(t) \partial \underline{X}'(t)} \cdot \Sigma \cdot W^D(t) \cdot \Sigma' \right\} \\ &= \mathcal{D}_G V_S[\underline{X}(t), t] + \frac{\partial V_S[\underline{X}(t), t]}{\partial t} - r(t) V_S[\underline{X}(t), t], \underline{X}(t) \in \mathbf{D}. \end{aligned} \tag{78}$$

On the other hand, since the Gaussian *Arrow-Debreu state price*, $G[\underline{X}(T), T; \ \underline{X}(t), t]$, solves the initial value problem (65)-(66), it follows that the *exact* solution of the initial value problem (78)-(75) can be written as:

$$\begin{aligned} V_S[\underline{X}(t), t] &= \int_{\underline{X}(T) \in \mathbf{D}} d\underline{X}(T)\, G[\underline{X}(T), T; \underline{X}(t), t]\, H[\underline{X}(T)] \\ &\quad + \int_t^T dl \int_{\underline{X}(l) \in \mathbf{D}} d\underline{X}(l)\, G[\underline{X}(l), l; \underline{X}(t), t] \\ &\quad \left\{ i[\underline{X}(l), l] + \frac{1}{2} tr \left[\frac{\partial^2 V_S[\underline{X}(l), l]}{\partial \underline{X}(l) \partial \underline{X}'(l)} \cdot \Sigma \cdot W^D(l) \cdot \Sigma' \right] \right\}. \end{aligned} \tag{79}$$

In fact, substituting solution (79) into the right-hand side of equation (78) and using standard differential calculus, yields

$$0=\int_{\underline{X}(T)\in\mathbf{D}} d\underline{X}(T)\,H[\underline{X}(T)]\{\mathcal{D}_G G[\underline{X}(T),T;\underline{X}(t),t]$$
$$+\frac{\partial G[\underline{X}(T),T;\underline{X}(t),t]}{\partial t}-r(t)\,G[\underline{X}(T),T;\underline{X}(t),t]\Big\}+\int_t^T dl\int_{\underline{X}(l)\in\mathbf{D}}$$
$$d\underline{X}(l)\left\{i[\underline{X}(l),l]+\frac{1}{2}tr\left[\frac{\partial^2 V_S[\underline{X}(l),l]}{\partial\underline{X}(l)\,\partial\underline{X}'(l)}\cdot\Sigma\cdot W^D(l)\cdot\Sigma'\right]\right\}$$
$$\left\{\mathcal{D}_G G[\underline{X}(l),l;\underline{X}(t),t]+\frac{\partial G[\underline{X}(l),l;\underline{X}(t),t]}{\partial t}-r(t)\,G[\underline{X}(l),l;\underline{X}(t),t]\right\}$$

which is a true proposition, as implied by equation (65). Concerning the boundary condition, the evaluation of solution (79) at $t=T$,

$$V_S[\underline{X}(T),T]=\int_{\underline{X}(T)\in\mathbf{D}} d\underline{X}(T)\,G[\underline{X}(T),T;\underline{X}(T),T]\,H[\underline{X}(T)],$$

combined with definition (66), generates exactly the terminal payoff function (75).

Assuming that, when the same contingent claim is valued under the nested Gaussian specification of the Duffie and Kan (1996) model, the terminal payoff and the continuous dividend processes are still equal to zero, for $\underline{X}\notin\mathbf{D}$, and given by $H[\underline{X}(T)]$ and $i[\underline{X}(t),t]$, respectively, for $\underline{X}\in\mathbf{D}$,[25] then the corresponding time-t Gaussian price $V_G[\underline{X}(t),t]$ of the contingent claim can be obtained as the solution of

$$-i[\underline{X}(t),t]=\mathcal{D}_G V_G[\underline{X}(t),t]+\frac{\partial V_G[\underline{X}(t),t]}{\partial t}-r(t)\,V_G[\underline{X}(t),t], \tag{80}$$

$\underline{X}(t)\in\mathbf{D}$, subject to

$$V_G[\underline{X}(T),T]=H[\underline{X}(T)],\underline{X}(T)\in\mathbf{D}. \tag{81}$$

And, using again results (65)-(66), such solution can be represented by an integral equation (see, for instance, Jamshidian (1991, equation 37)):

$$\begin{aligned}V_G[\underline{X}(t),t] &= \int_{\underline{X}(T)\in\mathbf{D}} d\underline{X}(T)\,G[\underline{X}(T),T;\underline{X}(t),t]\,H[\underline{X}(T)] \\ &\quad+\int_t^T dl\int_{\underline{X}(l)\in\mathbf{D}} d\underline{X}(l)\,G[\underline{X}(l),l;\underline{X}(t),t]\,i[\underline{X}(l),l].\end{aligned} \tag{82}$$

Combining equations (79) and (82),[26]

$$\begin{aligned}V_S[\underline{X}(t),t] &= V_G[\underline{X}(t),t]+\frac{1}{2}\int_t^T dl\int_{\underline{X}(l)\in\mathbf{D}} d\underline{X}(l) \\ &\quad G[\underline{X}(l),l;\underline{X}(t),t]\,tr\left\{\frac{\partial^2 V_S[\underline{X}(l),l]}{\partial\underline{X}(l)\,\partial\underline{X}'(l)}\cdot\Sigma\cdot W^D(l)\cdot\Sigma'\right\}.\end{aligned} \tag{83}$$

Finally, replacing repeatedly $V_S[\underline{X}(l),l]$ by the right-hand side of (83) evaluated at $t=l$ yields the series expansion (13)-(14). ■

[25] Because this paper only deals with European-style interest rate contingent claims -that is $i[\underline{X}(t),t]=0,\ \forall t$- the only relevant assumption is the one concerning the terminal payoff function.

[26] As pointed out by Qiang Dai, the integral equation (83) can also be stated as

$$\begin{aligned}V_S[\underline{X}(t),t] &= V_G[\underline{X}(t),t] \\ &\quad+\int_t^T dl\int_{\underline{X}(l)\in\mathbf{D}} d\underline{X}(l)\,G[\underline{X}(l),l;\underline{X}(t),t]\,[\mathcal{D}_S-\mathcal{D}_G]\,V_S[\underline{X}(l),l],\end{aligned}$$

where $[\mathcal{D}_S-\mathcal{D}_G]\,V_S[\underline{X}(l),l]$ can be understood as a *perturbation term*.

C Appendix: Proof of Proposition 5

To prove proposition 5 is equivalent to verify that

$$V_S\left[\underline{X}(t),t\right]-V_G\left[\underline{X}(t),t\right]-\frac{1}{2}\tilde{V}_1\left[\underline{X}(t),t\right]=o(\lambda).$$

In order to highlight the dependencies on the perturbation parameter λ, the previous error term can be further rewritten as:

$$\begin{aligned} & V_S\left[\underline{X}(t),t\right]-V_G\left[\underline{X}(t),t\right]-\frac{1}{2}\tilde{V}_1\left[\underline{X}(t),t\right] \\ = & \frac{1}{2}\left\{V_1\left[\underline{X}(t),t\right]-\tilde{V}_1\left[\underline{X}(t),t\right]\right\}+\sum_{p\geq 2}\frac{1}{2^p}V_p\left[\underline{X}(t),t\right] \\ = & \frac{\lambda}{2}\left\{U_1\left[\underline{X}(t),t\right]-\tilde{U}_1\left[\underline{X}(t),t\right]\right\}+\sum_{p\geq 2}\left(\frac{\lambda}{2}\right)^p U_p\left[\underline{X}(t),t\right], \end{aligned}$$

where $U_0\left[\underline{X}(t),t\right]=V_G\left[\underline{X}(t),t\right]$ and, for $p\geq 1$,

$$\begin{aligned} U_p\left[\underline{X}(t),t\right] = & \int_t^T dl\int_{\underline{X}(l)\in\mathbf{D}} d\underline{X}(l)\,G\left[\underline{X}(l),l;\underline{X}(t),t\right] \\ & tr\left\{\frac{\partial^2 U_{p-1}\left[\underline{X}(l),l\right]}{\partial\underline{X}(l)\,\partial\underline{X}'(l)}\cdot\Sigma\cdot\bar{W}^D(l)\cdot\Sigma'\right\}. \end{aligned}$$

Because the $A_m(n)$ canonical specification allows definition (4) to be restated as

$$\mathbf{D}=\left\{\underline{X}\in\Re^n:\alpha_i+\lambda X_i>0,i=1,\ldots,m\right\},$$

it follows that:

$$\begin{aligned} & \lim_{\lambda\to 0}\left|\frac{V_S\left[\underline{X}(t),t\right]-V_G\left[\underline{X}(t),t\right]-\frac{1}{2}\tilde{V}_1\left[\underline{X}(t),t\right]}{\lambda}\right| \qquad (84)\\ = & \lim_{\lambda\to 0}\left|\int_t^T dl\int_{\underline{X}(l)\in\Re^n} d\underline{X}(l)\left(\prod_{j=1}^m 1_{\{\lambda X_j(l)>-\alpha_j\}}-1\right)G\left[\underline{X}(l),l;\underline{X}(t),t\right]\right. \\ & \left.\frac{1}{2}tr\left\{\frac{\partial^2 V_G\left[\underline{X}(l),l\right]}{\partial\underline{X}(l)\,\partial\underline{X}'(l)}\cdot\Sigma\cdot\bar{W}^D(l)\cdot\Sigma'\right\}+\sum_{p\geq 2}\frac{1}{2^p}\lambda^{p-1}U_p\left[\underline{X}(t),t\right]\right|. \end{aligned}$$

Since $\lim_{\lambda\to 0}\prod_{j=1}^m 1_{\{\lambda X_j(l)>-\alpha_j\}}=1$ and $\lim_{\lambda\to 0}U_p\left[\underline{X}(t),t\right]=\tilde{U}_p\left[\underline{X}(t),t\right]$ as long as $\alpha_j>0$, for $j=1,\ldots,m$,[27] then the limit (84) is zero if $\left|\tilde{U}_p\left[\underline{X}(t),t\right]\right|<\infty$ for $p\geq 2$. ■

D Appendix: Conditional Mean and Covariance of $\underline{\tilde{X}}(T)$

For $T\geq t$, and assuming that a^{-1} exists, equation (20) can be rewritten under the following integral form:

$$\begin{aligned} \underline{\tilde{X}}(T) = & e^{a(T-t)}\cdot\underline{\tilde{X}}(t)+\left[e^{a(T-t)}-I_n\right]\cdot a^{-1}\cdot\underline{\tilde{b}} \qquad (85)\\ & +\int_t^T e^{a(T-s)}\cdot\Sigma\cdot\sqrt{\tilde{V}^D(s)}\cdot d\underline{W}^Q(s). \end{aligned}$$

[27] Although Dai and Singleton (1998, definition III.1) normalize α_j to zero for the first m factors, an invariant transformation, along the lines of Dai and Singleton (1998, definition A.1), can always yield the desired condition.

Clearly, because Itô's integrals are martingales, the conditional mean of the state vector is the same for the Gaussian and for the stochastic volatility specifications of the Duffie and Kan (1996) model:

$$\begin{aligned} E_Q^G\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right] &= E_Q^S\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right] \\ &= e^{a(T-t)}\cdot\tilde{\underline{X}}(t)+\left[e^{a(T-t)}-I_n\right]\cdot a^{-1}\cdot\tilde{\underline{b}}, \end{aligned} \tag{86}$$

where $E_Q^G\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right]$ and $E_Q^S\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right]$ denote the expectation of $\tilde{\underline{X}}(T)$, conditional on $\tilde{\underline{X}}(t)$ and computed under the martingale measure Q, for the Gaussian or stochastic volatility versions of the Duffie and Kan (1996) model, respectively.

Using again equation (85), the second conditional moment of the state vector under the nested deterministic volatility specification is

$$COV^G\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right]=\int_t^T e^{a(T-s)}\cdot\Sigma\cdot\tilde{U}^D\cdot\Sigma'\cdot e^{a'(T-s)}ds, \tag{87}$$

where $\tilde{U}^D=diag\left\{\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n\right\}$. For the general stochastic volatility formulation, the conditional covariance matrix corresponds to:

$$COV^S\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right]=\int_t^T e^{a(T-s)}\cdot\Sigma\cdot E_Q^S\left[\tilde{V}^D(s)\middle|\tilde{\underline{X}}(t)\right]\cdot\Sigma'\cdot e^{a'(T-s)}ds, \tag{88}$$

with

$$\begin{aligned} &E_Q^S\left[\tilde{V}^D(s)\middle|\tilde{\underline{X}}(t)\right] \\ =\ &diag\left\{\tilde{\alpha}_1+\underline{\beta_1}'\cdot E_Q^S\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right],\ldots,\tilde{\alpha}_n+\underline{\beta_n}'\cdot E_Q^S\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right]\right\}. \end{aligned}$$

If the following rude approximation is made,

$$\tilde{V}^D(s)\cong\tilde{V}^D(t),\forall s\in[t,T], \tag{89}$$

then

$$\underline{u}=\underline{X}(t)\Rightarrow COV^G\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right]=COV^S\left[\tilde{\underline{X}}(T)\middle|\tilde{\underline{X}}(t)\right].$$

E Appendix: Proof of Corollary 1 (items 2 and 3)

Concerning the last two items of Corollary 1, combining equation (34) with the Gaussian *Arrow-Debreu state price* (12), and with the definition (4) of the state variables' domain under the general Duffie and Kan (1996) model, yields:

$$\begin{aligned} V_1\left[\underline{X}(t),t\right] &= \int_t^T dlP_G(t,l)\exp\left[\varphi(l,T)+\frac{1}{2}\underline{\psi}'(l,T)\cdot\Delta(l-t)\right. \\ &\left.\cdot\underline{\psi}(l,T)+\underline{\psi}'(l,T)\cdot\underline{M}(l-t)\right]\left[\sum_{k=1}^{n}\left(\underline{\psi}'(l,T)\cdot\underline{\varepsilon_k}\right)^2\right]U, \end{aligned} \tag{90}$$

with

$$\begin{aligned} U &= \int_{\underline{X}(l)\in\Re^n} d\underline{X}(l)\frac{\left[\underline{\beta_k}'\cdot\underline{X}(l)\right]\prod_{j=1}^n 1_{\left\{\underline{\beta_j}'\cdot\underline{X}(l)\geq-\alpha_j\right\}}}{\sqrt{(2\pi)^n}\left|\Delta(l-t)\right|}\exp\left\{-\frac{1}{2}\right. \\ &\left.\left[\underline{X}(l)-\Delta(l-t)\cdot\underline{\mu}(l)\right]'\cdot\Delta^{-1}(l-t)\cdot\left[\underline{X}(l)-\Delta(l-t)\cdot\underline{\mu}(l)\right]\right\}. \end{aligned}$$

If $n = 1$, then

$$U = \int_{-\alpha_k}^{\infty} dy \frac{y}{\sqrt{2\pi \underline{\beta_k}' \cdot \Delta(l-t) \cdot \underline{\beta_k}}} \exp\left[-\frac{1}{2} \frac{\left(y - \underline{\beta_k}' \cdot \Delta(l-t) \cdot \underline{\mu}(l)\right)^2}{\underline{\beta_k}' \cdot \Delta(l-t) \cdot \underline{\beta_k}} \right],$$

and the exact solution (30) follows.

In order to set an upper bound for $V_1[\underline{X}(t), t]$, Schwarz inequality can be applied:

$$\begin{aligned} U^2 \leq\ & \underline{\beta_k}' \cdot E\left[\underline{X}(l) \cdot \underline{X}'(l) \middle| \mathcal{F}_t\right] \cdot \underline{\beta_k} \int_{\underline{X}(l) \in \Re^n} d\underline{X}(l) \\ & \frac{\prod_{j=1}^{n} 1^2_{\left\{\underline{\beta_j}' \cdot \underline{X}(l) \geq -\alpha_j\right\}}}{\sqrt{(2\pi)^n |\Delta(l-t)|}} \exp\left\{ -\frac{1}{2} \left[\underline{X}(l) - \Delta(l-t) \cdot \underline{\mu}(l)\right]' \right. \\ & \left. \cdot \Delta^{-1}(l-t) \cdot \left[\underline{X}(l) - \Delta(l-t) \cdot \underline{\mu}(l)\right] \right\}. \end{aligned} \tag{91}$$

Because the factor-integral on the right-hand side of (91) is surely positive, another application of Schwarz inequality can be made and square roots can be taken:

$$\begin{aligned} U^2 \leq\ & \underline{\beta_k}' \cdot E\left[\underline{X}(l) \cdot \underline{X}'(l) \middle| \mathcal{F}_t\right] \cdot \underline{\beta_k} \sqrt{\Pr\left[\underline{\beta_1}' \cdot \underline{X}(l) \geq -\alpha_1\right]} \\ & \left(\int_{\underline{X}(l) \in \Re^n} d\underline{X}(l) \frac{\prod_{j=2}^{n} 1^4_{\left\{\underline{\beta_j}' \cdot \underline{X}(l) \geq -\alpha_j\right\}}}{\sqrt{(2\pi)^n |\Delta(l-t)|}} \exp\left\{ -\frac{1}{2} \right. \right. \\ & \left. \left. \left[\underline{X}(l) - \Delta(l-t) \cdot \underline{\mu}(l)\right]' \cdot \Delta^{-1}(l-t) \cdot \left[\underline{X}(l) - \Delta(l-t) \cdot \underline{\mu}(l)\right] \right\} \right)^{\frac{1}{2}}. \end{aligned}$$

Repeating successively the same reasoning, it can be shown that

$$U^2 \leq \underline{\beta_k}' \cdot E\left[\underline{X}(l) \cdot \underline{X}'(l) \middle| \mathcal{F}_t\right] \cdot \underline{\beta_k} \prod_{j=1}^{n} \left\{ \Pr\left[\underline{\beta_j}' \cdot \underline{X}(l) \geq -\alpha_j\right] \right\}^{\frac{1}{2^j}}. \tag{92}$$

Finally, imposing a zero lower bound to U, computing explicitly the previous expectation and all the probabilities, and taking square roots from both sides of inequality (92), inequality (32) arises.

The lower bound (33) follows from (90), imposing

$$\left[\underline{\beta_k}' \cdot \underline{X}(l)\right] \prod_{j=1}^{n} 1_{\left\{\underline{\beta_j}' \cdot \underline{X}(l) \geq -\alpha_j\right\}} = -\alpha_k \prod_{j=1}^{n} 1_{\left\{\underline{\beta_j}' \cdot \underline{X}(l) \geq -\alpha_j\right\}}, \forall \underline{X}(l),$$

and assuming the independence amongst the events $\left\{\underline{\beta_j}' \cdot \underline{X}(l) \geq -\alpha_j\right\}$, for all j. ■

F Appendix: Proof of Proposition 8

Using equations (74) and (75), considering the zero-endowment nature of futures contracts, and the well known convergence of the terminal futures price to its underlying spot price, it follows that the futures price $FP_S(t, T_f, T_1)$ is the solution of the following initial value problem:

$$0 = \mathcal{D}_S FP_S(t, T_f, T_1) + \frac{\partial FP_S(t, T_f, T_1)}{\partial t}, \tag{93}$$

subject to

$$FP_S(T_f, T_f, T_1) = P_S(T_f, T_1). \tag{94}$$

Clearly, solution (37) satisfies the boundary condition (94). Moreover, substituting (37) into the PDE (93), rearranging terms as well as adding and subtracting the time-t instantaneous interest rate, $r(t)$,

$$\begin{aligned}
0 \;=\; & \left\{\underline{B}'(T_1-t)\cdot[a\cdot\underline{X}(t)+\underline{b}]+\left[\frac{\partial A(T_1-t)}{\partial t}+\frac{\partial\underline{B}'(T_1-t)}{\partial t}\cdot\underline{X}(t)\right]\right. \\
& \left.+\frac{1}{2}tr\left[\underline{B}(T_1-t)\cdot\underline{B}'(T_1-t)\cdot\Sigma\cdot V^D(t)\cdot\Sigma'\right]-r(t)\right\} \\
& -\left\{\underline{B}'(T_f-t)\cdot[a\cdot\underline{X}(t)+\underline{b}]+\left[\frac{\partial A(T_f-t)}{\partial t}+\frac{\partial\underline{B}'(T_f-t)}{\partial t}\cdot\underline{X}(t)\right]\right. \\
& \left.+\frac{1}{2}tr\left[\underline{B}(T_f-t)\cdot\underline{B}'(T_f-t)\cdot\Sigma\cdot V^D(t)\cdot\Sigma'\right]-r(t)\right\} \\
& +\left\{\underline{D}'(t,T_f,T_1)\cdot[a\cdot\underline{X}(t)+\underline{b}]+\left[\frac{\partial C(t,T_f,T_1)}{\partial t}+\frac{\partial\underline{D}'(t,T_f,T_1)}{\partial t}\right.\right. \\
& \left.\cdot\underline{X}(t)\right]-\frac{1}{2}\underline{B}'(T_1-t)\cdot\Sigma\cdot V^D(t)\cdot\Sigma'\cdot\underline{B}(T_1-t)+\frac{1}{2}\underline{B}'(T_f-t) \\
& \cdot\Sigma\cdot V^D(t)\cdot\Sigma'\cdot\underline{B}(T_f-t)+\frac{1}{2}\left[\underline{B}(T_1-t)-\underline{B}(T_f-t)+\underline{D}(t,T_f,T_1)\right]' \\
& \left.\cdot\Sigma\cdot V^D(t)\cdot\Sigma'\cdot\left[\underline{B}(T_1-t)-\underline{B}(T_f-t)+\underline{D}(t,T_f,T_1)\right]\right\}.
\end{aligned}$$

The first two terms on the right-hand-side of the previous equation are equal to zero, since they are just the PDEs satisfied by the pure discount bond prices $P_S(t,T_1)$ and $P_S(t,T_f)$, respectively. Therefore, simplifying some terms, and since $\Sigma\cdot V^D(t)\cdot\Sigma'=\sum_{k=1}^{n}\underline{\varepsilon_k}\cdot\underline{\varepsilon_k}'\left[\alpha_k+\underline{\beta_k}'\cdot\underline{X}(t)\right]$, then the right-hand-side of the last equation can be rewritten as an affine function of $\underline{X}(t)$:

$$\begin{aligned}
0 \;=\; & \left\{\underline{D}'(t,T_f,T_1)\cdot a+\frac{\partial\underline{D}'(t,T_f,T_1)}{\partial t}+\sum_{k=1}^{n}\underline{B}'(T_f-t)\cdot\underline{\varepsilon_k}\cdot\underline{\varepsilon_k}'\right. \\
& \cdot\left[\underline{B}(T_f-t)-\underline{B}(T_1-t)\right]\underline{\beta_k}'+\frac{1}{2}\sum_{k=1}^{n}\underline{D}'(t,T_f,T_1)\cdot\underline{\varepsilon_k}\cdot\underline{\varepsilon_k}' \\
& \left.\cdot\left[2\underline{B}(T_1-t)-2\underline{B}(T_f-t)+\underline{D}(t,T_f,T_1)\right]\underline{\beta_k}'\right\}\cdot\underline{X}(t) \\
& +\left\{\underline{D}'(t,T_f,T_1)\cdot\underline{b}+\frac{\partial C(t,T_f,T_1)}{\partial t}+\sum_{k=1}^{n}\underline{B}'(T_f-t)\cdot\underline{\varepsilon_k}\cdot\underline{\varepsilon_k}'\right. \\
& \cdot\left[\underline{B}(T_f-t)-\underline{B}(T_1-t)\right]\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\underline{D}'(t,T_f,T_1)\cdot\underline{\varepsilon_k}\cdot\underline{\varepsilon_k}' \\
& \left.\cdot\left[2\underline{B}(T_1-t)-2\underline{B}(T_f-t)+\underline{D}(t,T_f,T_1)\right]\alpha_k\right\}.
\end{aligned}$$

The previous PDE can now be split into the n-dimensional Riccati differential equation (38) and into the first order ODE (39). ■

G Appendix: Proof of Proposition 10

By convention, the futures price is quoted on an annualized basis, and therefore the terminal futures price corresponds to

$$FR_G(T_f,T_f,T_1)=100\left[1-R_G(T_f,T_1)\right], \tag{95}$$

where $R_G(T_f,T_1)=\frac{1}{T_1-T_f}\left[\frac{1}{P_G(T_f,T_1)}-1\right]$ is the Gaussian time-T_f nominal spot interest rate for the period (T_1-T_f).

Because -see for instance Cox, Ingersoll and Ross (1981, equation 46)- a futures price is just the expectation of the spot price on the delivery date, under the martingale measure Q, and using the exponential-affine formula (1),

$$FR_G(t,T_f,T_1) = 100 + \frac{100}{T_1 - T_f} - \frac{100}{T_1 - T_f} \exp\left[-A(T_1 - T_f)\right] \tag{96}$$
$$E_Q\left\{\exp\left[-\underline{B}'(T_1 - T_f)\cdot \underline{X}(T_f)\right] \middle| \mathcal{F}_t\right\},$$

where $E_Q(Y|\mathcal{F}_t)$ denotes the time-t expected value of the random variable Y, computed under the probability measure Q. Moreover, the expectation appearing in the right-hand-side of the last equation is the moment generating function of the random variable $\left[\underline{B}'(T_1 - T_f)\cdot \underline{X}(T_f)\right]$, with a coefficient of -1.

On the other hand, since matrix a is time-homogeneous and assuming that matrix a is also nonsingular, Arnold (1992, corollary (8.2.4)) provides the following *strong* solution for equation (7), with $t \geq t_0$:

$$\underline{X}(t) = e^{a(t-t_0)}\cdot \underline{X}(t_0) + \left[e^{a(t-t_0)} - I_n\right]\cdot a^{-1}\cdot \underline{b} + \int_{t_0}^{t} e^{a(t-v)}\cdot S\cdot d\underline{W}^Q(v).$$

Consequently, $\underline{X}(T_f) \cap N^n\left(\underline{u}(T_f - t), \Delta(T_f - t)\right)$, where

$$\underline{u}(t - t_0) = e^{a(t-t_0)}\cdot \underline{X}(t_0) + \left[e^{a(t-t_0)} - I_n\right]\cdot a^{-1}\cdot \underline{b}, \tag{97}$$

and therefore

$$E_Q\left\{\exp\left[-\underline{B}'(T_1 - T_f)\cdot \underline{X}(T_f)\right] \middle| \mathcal{F}_t\right\} = \exp\left[-\underline{B}'(T_1 - T_f)\right. \tag{98}$$
$$\left.\cdot \underline{u}(T_f - t) + \frac{1}{2}\underline{B}'(T_1 - T_f)\cdot \Delta(T_f - t)\cdot \underline{B}(T_1 - T_f)\right].$$

Finally, combining equations (8), (9), (96), (97), and (98), it is trivial to obtain the exact Gaussian solution (41). ■

H Appendix: Proof of Corollary 2

Using equations (46) to (49), the functional form of the "gamma matrix" $\frac{\partial^2 V_p[\underline{X}(l),l]}{\partial \underline{X}(l)\partial \underline{X}'(l)}$ can be computed, and it can be shown that equations (13) and (14) yield the following first order approximation:

$$V_S[\underline{X}(t),t] \cong V_G[\underline{X}(t),t] + \frac{q}{2}\int_t^{T_0} dl\left[V_{11}(l) + V_{12}(l) + V_{13}(l)\right], \tag{99}$$

where[28]

$$V_{11}(l) = \theta \exp\left[U(l,\cdot)\right]\int_{\underline{X}(l)} d\underline{X}(l)\, G\left[\underline{X}(l), l; \underline{X}(t), t\right] \tag{100}$$
$$\exp\left[\underline{Q}'(l,\cdot)\cdot \underline{X}(l)\right]\Phi\left[\theta d_1(l)\right]\left[\sum_{k=1}^{n}\left(\underline{Q}'(l,\cdot)\cdot \underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\right]\cdot \underline{X}(l),$$

$$V_{12}(l) = -\theta K \exp\left[S(l,T_0)\right]\int_{\underline{X}(l)} d\underline{X}(l)\, G\left[\underline{X}(l), l; \underline{X}(t), t\right] \tag{101}$$
$$\exp\left[\underline{T}'(l,T_0)\cdot \underline{X}(l)\right]\Phi\left[\theta d_0(l)\right]\left[\sum_{k=1}^{n}\left(\underline{T}'(l,T_0)\cdot \underline{\varepsilon_k}\right)^2 \underline{\beta_k}'\right]\cdot \underline{X}(l),$$

and

$$V_{13}(l) = \frac{K e^{S(l,T_0)}}{\sigma(l)\sqrt{2\pi}}\int_{\underline{X}(l)} d\underline{X}(l)\, G\left[\underline{X}(l), l; \underline{X}(t), t\right] e^{\underline{T}'(l,T_0)\cdot \underline{X}(l)} \tag{102}$$
$$\exp\left\{-\frac{1}{2}\left[d_0(l)\right]^2\right\}\left\{\sum_{k=1}^{n}\left[\left(\underline{Q}'(l,\cdot) - \underline{T}'(l,T_0)\right)\cdot \underline{\varepsilon_k}\right]^2 \underline{\beta_k}'\right\}\cdot \underline{X}(l).$$

[28] In this appendix, all factor-integrals refer to integration over $\Re^n$.

The next step consists in eliminating all factor-integrals from the above equations. Beginning with $V_{13}(l)$, using the definition (12) of Gaussian Arrow-Debreu prices, and because $d_0(l)$ can be written as an explicit function of $\underline{X}(l)$,

$$d_0(l) = \frac{1}{\sigma(l)}\left\{\left[\underline{Q}'(l,\cdot) - \underline{T}'(l,T_0)\right]\cdot \underline{X}(l) + d_0^*(l)\right\}, \tag{103}$$

then

$$\begin{aligned} V_{13}(l) &= \exp\left\{S(l,T_0) - \frac{[d_0^*(l)]^2}{2\sigma^2(l)} - \frac{1}{2}\underline{M}'(l-t)\cdot\Delta^{-1}(l-t)\cdot\underline{M}(l-t)\right. \\ &\left. + \frac{1}{2}\underline{m}'(l)\cdot\varphi^{-1}(l)\cdot\underline{m}(l)\right\}\frac{KP_G(t,l)}{\sigma(l)\sqrt{2\pi}}\sqrt{\frac{|\varphi^{-1}(l)|}{|\Delta(l-t)|}} \\ &\int_{\underline{X}(l)} d\underline{X}(l)\left\{\sum_{k=1}^{n}\left[\left(\underline{Q}'(l,\cdot)-\underline{T}'(l,T_0)\right)\cdot\underline{\varepsilon_k}\right]^2\underline{\beta_k}'\right\}\cdot\underline{X}(l) \\ &\frac{\exp\left\{-\frac{1}{2}\left[\underline{X}(l)-\varphi^{-1}(l)\cdot\underline{m}(l)\right]'\cdot\varphi(l)\cdot\left[\underline{X}(l)-\varphi^{-1}(l)\cdot\underline{m}(l)\right]\right\}}{\sqrt{(2\pi)^n|\varphi^{-1}(l)|}}. \end{aligned}$$

But, the last integral is just the expectation of the random variable $\left\{\sum_{k=1}^{n}\left[\left(\underline{Q}'(l,\cdot) \quad -\underline{T}'(l,T_0)\right)\cdot\underline{\varepsilon_k}\right]^2\underline{\beta_k}'\right\}\cdot \underline{X}(l)$, with $\underline{X}(l)\cap N^n\left(\varphi^{-1}(l)\cdot\underline{m}(l),\varphi^{-1}(l)\right)$. Computing such expected value explicitly, the factor-integral independent analytical formula (52) is finally obtained.

In order to simplify $V_{11}(l)$, it is convenient to express $\Phi[\theta d_1(l)]$ as a n-dimensional integral with respect to $\underline{X}(T_0)$. Evaluating (46) at $t=T_0$,

$$V_G[\underline{X}(T_0),T_0] = q\left\{\theta\exp\left[U(T_0,\cdot)+\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\right]-\theta K\right\}^+,$$

and using result (82),

$$\begin{aligned} V_G[\underline{X}(t),t] &= \theta q\int_{\underline{X}(T_0)} d\underline{X}(T_0)\,G[\underline{X}(T_0),T_0;\underline{X}(t),t]\,1\varepsilon \\ &\quad\left\{\exp\left[U(T_0,\cdot)+\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\right]-K\right\}, \end{aligned}$$

where $\varepsilon = \left\{\underline{X}(T_0) : \theta\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\geq\theta K^*\right\}$. Solving the above integral equation, and comparing each term with (46), it can be shown that:

$$\begin{aligned} \Phi[\theta d_1(l)] &= \int_{\underline{X}(T_0)} d\underline{X}(T_0)\frac{1}{\sqrt{(2\pi)^n|\Delta(T_0-l)|}}1\varepsilon \\ &\exp\left\{-\frac{1}{2}\left[\underline{X}(T_0)-\underline{M}(T_0-l)-\Delta(T_0-l)\cdot\underline{Q}(T_0,\cdot)\right]'\right. \\ &\left.\cdot\Delta^{-1}(T_0-l)\cdot\left[\underline{X}(T_0)-\underline{M}(T_0-l)-\Delta(T_0-l)\cdot\underline{Q}(T_0,\cdot)\right]\right\}. \end{aligned} \tag{104}$$

Combining this last result with (100),

$$\begin{aligned} V_{11}(l) &= \theta P_G(t,l)\sqrt{\frac{|\Omega^{-1}(l)|}{|\Delta(l-t)|}}\exp\left[U(l,\cdot)-\frac{1}{2}\underline{M}'(l-t)\cdot\Delta^{-1}(l-t)\right. \\ &\left.\cdot\underline{M}(l-t)+\frac{1}{2}\underline{\mu_1}'(l)\cdot\Omega^{-1}(l)\cdot\underline{\mu_1}(l)\right]\int_{\underline{X}(T_0)} d\underline{X}(T_0)\frac{1}{\sqrt{(2\pi)^n|\Delta(T_0-l)|}}1\varepsilon \\ &\exp\left\{-\frac{1}{2}\left[\underline{X}(T_0)-\underline{MC_1}(T_0-l)\right]'\cdot\Delta^{-1}(T_0-l)\cdot\left[\underline{X}(T_0)\right.\right. \\ &\left.\left.-\underline{MC_1}(T_0-l)\right]\right\}\int_{\underline{X}(l)} d\underline{X}(l)\frac{\left[\sum_{k=1}^{n}\left(\underline{Q}'(l,\cdot)\cdot\underline{\varepsilon_k}\right)^2\underline{\beta_k}'\right]\cdot\underline{X}(l)}{\sqrt{(2\pi)^n|\Omega^{-1}(l)|}} \\ &\exp\left\{-\frac{1}{2}\left[\underline{X}(l)-\Omega^{-1}(l)\cdot\underline{\mu_1}(l)\right]'\cdot\Omega(l)\cdot\left[\underline{X}(l)-\Omega^{-1}(l)\cdot\underline{\mu_1}(l)\right]\right\}, \end{aligned}$$

where

$$\underline{\mu_1}(l) = \Delta^{-1}(l-t)\cdot\underline{M}(l-t)+\underline{Q}(l,\cdot) \\ +e^{a'(T_0-l)}\cdot\Delta^{-1}(T_0-l)\cdot\left[\underline{X}(T_0)-\underline{MC_1}(T_0-l)\right].$$

Because the integral with respect to $\underline{X}(l)$ is just the expectation of $\left[\sum_{k=1}^{n}\left(\underline{Q}'(l,\cdot)\ \cdot\underline{\varepsilon_k}\right)^2\underline{\beta_k}'\right]\cdot\underline{X}(l)$, with $\underline{X}(l)\cap N^n\left(\Omega^{-1}(l)\cdot\underline{\mu_1}(l),\Omega^{-1}(l)\right)$, then, and after some linear algebra manipulations,

$$V_{11}(l)=\theta P_G(t,l)\sqrt{\frac{|\Omega^{-1}(l)\cdot\Psi^{-1}(l)|}{|\Delta(l-t)\cdot\Delta(T_0-l)|}}\exp\left[U(l,\cdot)-\frac{1}{2}\underline{M}'(l-t)\right. \\ \cdot\Delta^{-1}(l-t)\cdot\underline{M}(l-t)+\frac{1}{2}\underline{\mu_{C_1}}'(l)\cdot\Omega^{-1}(l)\cdot\underline{\mu_{C_1}}(l)-\frac{1}{2}\underline{MC_1}'(T_0-l) \\ \left.\cdot\Delta^{-1}(T_0-l)\cdot\underline{MC_1}(T_0-l)+\frac{1}{2}\underline{N_1}'(l)\cdot\Psi^{-1}(l)\cdot\underline{N_1}(l)\right] \\ \int_{\underline{X}(T_0)}d\underline{X}(T_0)\frac{1}{\sqrt{(2\pi)^n|\Psi^{-1}(l)|}}1\varepsilon\left[\sum_{k=1}^{n}\left(\underline{Q}'(l,\cdot)\cdot\underline{\varepsilon_k}\right)^2\underline{\beta_k}'\right] \\ \cdot\Omega^{-1}(l)\cdot\left[\underline{\mu_{C_1}}(l)+e^{a'(T_0-l)}\cdot\Delta^{-1}(T_0-l)\cdot\underline{X}(T_0)\right] \\ \exp\left\{-\frac{1}{2}\left[\underline{X}(T_0)-\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]'\cdot\Psi(l)\cdot\left[\underline{X}(T_0)-\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]\right\}.$$

Noticing that the expectation of an indicator function results in a probability,

$$V_{11}(l) = \theta P_G(t,l)\exp\left[U(l,\cdot)-\frac{1}{2}\underline{M}'(l-t)\cdot\Delta^{-1}(l-t)\cdot\underline{M}(l-t)\right. \\ +\frac{1}{2}\underline{\mu_{C_1}}'(l)\cdot\Omega^{-1}(l)\cdot\underline{\mu_{C_1}}(l)-\frac{1}{2}\underline{MC_1}'(T_0-l)\cdot\Delta^{-1}(T_0-l) \\ \left.\cdot\underline{MC_1}(T_0-l)+\frac{1}{2}\underline{N_1}'(l)\Psi^{-1}(l)\underline{N_1}(l)\right]\sqrt{\frac{|\Omega^{-1}(l)\cdot\Psi^{-1}(l)|}{|\Delta(l-t)\cdot\Delta(T_0-l)|}} \\ \left\{\eta+\left[\sum_{k=1}^{n}\left(\underline{Q}'(l,\cdot)\cdot\underline{\varepsilon_k}\right)^2\underline{\beta_k}'\right]\cdot\Omega^{-1}(l)\cdot\underline{\mu_{C_1}}(l)\right. \\ \left.\Pr\left[\theta\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\geq\theta K^*\right]\right\}, \tag{105}$$

where

$$\eta = \int_{\underline{X}(T_0)}d\underline{X}(T_0)\frac{\left[\underline{C_1}'(l)\cdot\underline{X}(T_0)\right]1_{\left\{\theta\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\geq\theta K^*\right\}}}{\sqrt{(2\pi)^n|\Psi^{-1}(l)|}} \\ \exp\left\{\frac{1}{2}\left[\underline{X}(T_0)-\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]'\cdot\Psi(l)\cdot\left[\underline{X}(T_0)-\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]\right\},$$

and $\Pr(A)$ denotes the probability of occurrence of the event A.

Because $\underline{X}(T_0)\cap N^n\left(\Psi^{-1}(l)\cdot\underline{N_1}(l),\Psi^{-1}(l)\right)$ implies that the random variable $\left[\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\right]$ possesses a univariate normal distribution with mean $\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{N_1}(l)$ and variance $\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{Q}(T_0,\cdot)$, the probability contained in equation (105) corresponds to

$$\Pr\left[\theta\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\geq\theta K^*\right]=\Phi\left[\theta\frac{\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{N_1}(l)-K^*}{\sqrt{\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{Q}(T_0,\cdot)}}\right]. \tag{106}$$

Concerning the term η, and for reasons of analytical tractability, $\underline{C_1}(l)$ is going to be approximated by the vector $\left[\lambda_1(l)\underline{Q}(T_0,\cdot)\right]$, where $\lambda_1(l)$ is chosen as to minimize the Euclidean distance between the two vectors:

$$\eta \cong \int_{\underline{X}(T_0)} d\underline{X}(T_0) \frac{\lambda_1(l)\left[\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\right] 1_{\{\theta\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\geq\theta K^*\}}}{\sqrt{(2\pi)^n\left|\Psi^{-1}(l)\right|}}$$
$$\exp\left\{\frac{1}{2}\left[\underline{X}(T_0)-\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]'\cdot\Psi(l)\cdot\left[\underline{X}(T_0)-\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]\right\},$$

with[29]

$$\lambda_1(l) : Min_{\lambda_1(l)}\left\|\underline{C_1}(l)-\lambda_1(l)\,\underline{Q}(T_0,\cdot)\right\|.$$

The last integral is equal to the expectation of the random variable $\lambda_1(l)\left[\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\right] 1_{\{\theta\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\geq\theta K^*\}}$, subject to $\underline{X}(T_0)\cap N^n\left(\Psi^{-1}(l)\cdot\underline{N_1}(l),\Psi^{-1}(l)\right)$. To evaluate such expectation, it is simpler to use the density of the random variable $\theta\left[\underline{Q}'(T_0,\cdot)\cdot\underline{X}(T_0)\right]\equiv y$, because the integral under consideration becomes one-dimensional:

$$\begin{aligned}\eta &\cong \int_{\theta K^*}^{\infty} dy\frac{\lambda_1(l)}{\theta}y\frac{1}{\sqrt{2\pi\theta^2\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{Q}(T_0,\cdot)}}\\
&\quad\exp\left\{-\frac{1}{2}\frac{\left[y-\theta\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]^2}{\theta^2\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{Q}(T_0,\cdot)}\right\}\\
&\cong \frac{\lambda_1(l)}{\theta}\sqrt{\frac{\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{Q}(T_0,\cdot)}{2\pi}}\\
&\quad\exp\left\{-\frac{1}{2}\frac{\left[K^*-\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]^2}{\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{Q}(T_0,\cdot)}\right\}+\lambda_1(l)\\
&\quad\left[\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{N_1}(l)\right]\Phi\left[\theta\frac{\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{N_1}(l)-K^*}{\sqrt{\underline{Q}'(T_0,\cdot)\cdot\Psi^{-1}(l)\cdot\underline{Q}(T_0,\cdot)}}\right].\end{aligned}$$

Combining this last result with equations (105) and (106) yields the "explicit" solution (51) for $i=1$.

Following exactly the same steps as for $V_{11}(l)$, equation (51) can also be derived for $i=2$. Alternatively, such "explicit" formula for $V_{12}(l)$ also arises by comparing the analytical forms of $V_{11}(l)$ and $V_{12}(l)$ under equations (100) and (101), as well as the definitions of $d_1(l)$ and $d_0(l)$. In fact, $V_{12}(l)$ can be obtained from $-KV_{11}(l)$ when $U(l,\cdot)$, $\underline{Q}(l,\cdot)$, and $\underline{M}(T_0-l)$ are replaced by $S(l,T_0)$, $\underline{T}(l,T_0)$, and $\left[\underline{M}(T_0-l)-\Delta(T_0-l)\cdot\underline{Q}(T_0,\cdot)\right]$, respectively. Performing these substitutions in equation (51) with $i=1$ yields equation (51) for $i=2$. ■

I Appendix: Proof of Proposition 17

Using equation (41), the (intrinsic) terminal value of the call option is

$$\begin{aligned}&c^G_{T_0}\left[FR_G(T_0,T_f,T_1);K_R;T_0\right] \\ &= \frac{100}{T_1-T_f}\left\{\left[1+(T_1-T_f)\frac{100-K_R}{100}\right]-\frac{\exp\left[L(T_0)\right]}{P_G(T_0,T_f,T_1)}\right\}^+,\end{aligned} \tag{107}$$

where $P_G(T_0,T_f,T_1)=\frac{P_G(T_0,T_1)}{P_G(T_0,T_f)}$ is the Gaussian time-T_0 forward price, for delivery at date T_f, of a zero-coupon bond expiring at time T_1.

Denoting by Q_0 the probability measure (equivalent to Q) obtained when the "reference bond" $P_G(t,T_0)$ is taken as numeraire, it is well known that the discounted call value is a Q_0-martingale:

$$\frac{c^G_t\left[FR_G(t,T_f,T_1);K_R;T_0\right]}{P_G(t,T_0)}=E_{Q_0}\left\{c^G_{T_0}\left[FR_G(T_0,T_f,T_1);K_R;T_0\right]\middle|\mathcal{F}_t\right\}. \tag{108}$$

[29] Note that in the univariate case ($n=1$), this is not an approximation but an exact result. However, the focus of this paper is on the multivariate case.

In order to compute the last expectation, it is necessary to find the stochastic process followed by the forward price $P_G(T_0, T_f, T_1)$ under the risk-neutral measure Q_0. As shown in Nunes (1998, subsection 2.2),

$$d\underline{W}^{Q_0}(t) = -S' \cdot \underline{B}(\tau_0) \cdot dt + d\underline{W}^{Q}(t) \tag{109}$$

is still a vector of n independent Brownian motion increments (with the same standard filtration as $d\underline{W}^{Q}(t)$), but under measure Q_0. Hence, combining equations (7) and (109), and using Itô's lemma, it can be easily shown that

$$P_G(T_0, T_f, T_1) = \frac{P_G(t, T_1)}{P_G(t, T_f)} \exp[\nu(t) - z], \tag{110}$$

where

$$\begin{aligned} \nu(t) &= \underline{G}' \cdot a^{-1} \cdot \left\{ e^{a(T_1 - T_0)} \cdot \Delta(\tau_0) \cdot \left[I_n - \frac{1}{2} e^{a'(T_1 - T_0)} \right] \right. \\ &\quad \left. + e^{a(T_f - T_0)} \cdot \Delta(\tau_0) \cdot \left[\frac{1}{2} e^{a'(T_f - T_0)} - I_n \right] \right\} \cdot (a^{-1})' \cdot \underline{G}, \end{aligned}$$

and

$$z = \int_t^{T_0} \left[\underline{B}'(T_f - u) - \underline{B}'(T_1 - u) \right] \cdot S \cdot d\underline{W}^{Q_0}(u).$$

Because $z \cap N^1(0, \sigma_R^2(t))$, and combining equations (107) (108) and (110), yields

$$\begin{aligned} & c_t^G [FR_G(t, T_f, T_1); K_R; T_0] \\ = & \frac{100 P_G(t, T_0)}{T_1 - T_f} \int_{-\infty}^{\infty} dz \frac{1}{\sigma_R(t)\sqrt{2\pi}} \exp\left[-\frac{z^2}{2\sigma_R^2(t)} \right] 1_{\{z < z^*\}} \\ & \left\{ \left[1 + (T_1 - T_f) \frac{100 - K_R}{100} \right] - \exp[L(T_0) - \nu(t)] \frac{P_G(t, T_f)}{P_G(t, T_1)} e^z \right\}, \end{aligned}$$

with

$$z^* = \ln \left[\frac{1 + (T_1 - T_f) \frac{100 - K_R}{100}}{\frac{P_G(t, T_f)}{P_G(t, T_1)}} \right] - L(T_0) + \nu(t).$$

Solving the last integral explicitly and defining $\rho(t) = \frac{\sigma_R^2(t)}{2} - \nu(t)$, equation (61) is easily obtained. The put option solution (62) can be also derived along the same lines. ■

References

Arnold, L. (1992). *Stochastic Differential Equations: Theory and Applications.* Florida: Krieger Publishing Company.

Barone, E. and L. Mengoni (1997). "Futures-Style Options on Euro-Deposit Futures: Nihil Sub Sole Novi?," *European Financial Management* 3, 99–126.

Baxter, M. and A. Rennie (1996). *Financial Calculus: An Introduction to Derivative Pricing.* Cambridge: Cambrigde University Press.

Beaglehole, D. and M. Tenney (1991). "General Solutions of some Interest Rate-Contingent Claim Pricing Equations," *Journal of Fixed Income* (September), 69–83.

Brace, A. and M. Musiela (1994). "A Multifactor Gauss Markov Implementation of Heath, Jarrow, and Morton," *Mathematical Finance* 4, 259–283.

Chen, L. (1994). "Stochastic Mean and Stochastic Volatility: A Three-Factor Model of the Term Structure and its Application in Pricing of Interest Rate Derivatives," Working Paper. Federal Reserve Board.

Chen, L. (1996). *Interest Rate Dynamics, Derivatives Pricing and Risk Management.* Lecture Notes in Economics and Mathematical Systems 435. New York: Spinger-Verlag.

Chen, R.-R. and L. Scott (1992). "Pricing Interest Rate Options in a Two-Factor Cox-Ingersoll-Ross Model of the Term Structure," *The Review of Financial Studies* 5, 613–636.

Chen, R.-R. and L. Scott (1993). "Pricing Interest Rate Futures Options with Futures-Style Margining," *Journal of Futures Markets* 13, 15–22.

Chen, R.-R. and L. Scott (1995). "Interest Rate Options in Multifactor Cox-Ingersoll-Ross Models of the Term Structure," *Journal of Derivatives* (Winter), 53–72.

Cox, J., J. Ingersoll and S. Ross (1981). "The Relation Between Forward Prices and Futures Prices," *Journal of Financial Economics* 9, 321–346.

Cox, J., J. Ingersoll and S. Ross (1985). "A Theory of the Term Structure of Interest Rates," *Econometrica* 53, 385–407.

Dai, Q. and K. Singleton (1998). "Specification Analysis of Affine Term Structure Models," Working Paper. New York University and Stanford University.

Duan, J.-C. and J.-G. Simonato (1995). "Estimating and Testing Exponential-Affine Term Structure Models by Kalman Filter," Working Paper. McGill University.

Duffie, D. (1989). *Futures Markets.* Englewood Cliffs, New Jersey: Prentice-Hall.

Duffie, D. and K. Singleton (1997). "An Econometric Model of the Term Structure of Interest-Rate Swap Yields," *Journal of Finance* 52, 1287–1321.

Duffie, D. and R. Kan (1994). "Multi-Factor Term Structure Models," *Phil. Trans. R. Soc. Lond.* 347, 577–586.

Duffie, D. and R. Kan (1996). "A Yield-Factor Model of Interest Rates," *Mathematical Finance* 6, 379–406.

Duffie, D., J. Pan and K. Singleton (1998). "Transform Analysis and Option Pricing for Affine Jump - Diffusions," Working Paper. Graduate School of Business, Stanford University.

El Karoui, N. and J.-C. Rochet (1989). "A Pricing Formula for Options on Coupon Bonds," Working Paper 72. SEEDS.

El Karoui, N., C. Lepage, R. Myneni, N. Roseau and R. Viswanathan (1991). "The Valuation and Hedging of Contingent Claims with Markovian Interest Rates," Working Paper. Université de Paris VI.

Fong, H. and O. Vasicek (1991). "Interest Rate Volatility as a Stochastic Factor," Working Paper. Gifford Fong Associates.

Jamshidian, F. (1991). "Bond and Option Evaluation in the Gaussian Interest Rate Model," *Research in Finance* 9, 131–170.

Jamshidian, F. (1993). "Option and Futures Evaluation with Deterministic Volatilities," *Mathematical Finance* 3, 149–159.

Kloeden, P. and E. Platen (1992). *Numerical Solution of Stochastic Differential Equations.* New York: Springer-Verlag.

Langetieg, T. (1980). "A Multivariate Model of the Term Structure," *Journal of Finance* 35, 71–97.

Leblanc, B. and O. Scaillet (1998). "Path Dependent Options on Yields in the Affine Term Structure Model," *Finance and Stochastics* 2, 349–367.

Longstaff, F. and E. Schwartz (1992). "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model," *Journal of Finance* 47, 1259–1282.

Lund, J. (1994). "Econometric Analysis of Continuous-Time Arbitrage-Free Models of the Term Structure of Interest Rates," Working paper. The Aarhus School of Business.

Munk, C. (1998). "Stochastic Duration and Fast Coupon Bond Option Pricing in Multi-Factor Models," Working Paper. Odense University.

Murdock, J. (1991). *Perturbations: Theory and Methods.* New York: John Wiley & Sons.

Nayfeh, A. (1973). *Perturbation Methods.* New York: John Wiley & Sons.

Nunes, J. (1998). "Interest Rate Options in a Duffie-Kan Model with Deterministic Volatility," *Revista de Mercados e Activos Financeiros* 1, 63–101.

Press, W., B. Flannery, S. Teukolsky and W. Vetterling (1994). *Numerical Recipes in Pascal: The Art of Scientific Computing.* Cambridge: Cambrigde University Press.

Schlogl, E. and D. Sommer (1997). "Factor Models and the Shape of the Term Structure," Discussion Paper B-395. University of Bonn.

Schlogl, E. and D. Sommer (1998). "Factor Models and the Shape of the Term Structure," *Journal of Financial Engineering* 7, 79–88.

Schobel, R. (1990). "Options on Short Term Interest Rate Futures," Working Paper. Universitat Luneburg.

Shephard, N. (1991). "From Characteristic Function to Distribution Function: A Simple Framework for the Theory," *Econometric Theory* 7, 519–529.

Van Loan, C. (1978). "Computing Integrals Involving the Matrix Exponential," *IEEE Transactions on Automatic Control* 23, 395–404.

Wei, J. (1997). "A Simple Approach to Bond Option Pricing," *Journal of Futures Markets* 17, 131–160.

Table 1: Pricing of pure discount bonds and swaps using the same parameter values as in the three-factor CIR model of Schlogl and Sommer (1998, Figure 5), for different affine invariant transformations

PDB's Expiry (years)	Exact price (P_S)	Percentage Pricing Errors				
		$\underline{u} = -a^{-1} \cdot \underline{b}$		$\underline{u} = \underline{X}(t)$		
		$\frac{P_G - P_S}{P_S}$	$\frac{P_G + \frac{V_1}{2} - P_S}{P_S}$	$\frac{P_G - P_S}{P_S}$	$\frac{P_G + \frac{V_1}{2} - P_S}{P_S}$	MAPE[b]
0.5	0.970442	0.0000%	0.0000%	0.0000%	0.0000%	0.0001%
1	0.941755	0.0000%	0.0000%	0.0000%	0.0000%	0.0007%
1.5	0.913916	-0.0001%	0.0000%	0.0000%	0.0000%	0.0023%
2	0.886900	0.0000%	0.0000%	0.0001%	0.0000%	0.0049%
2.5	0.860686	0.0000%	0.0000%	0.0001%	0.0000%	0.0090%
				...		
18	0.338659	0.1220%	-0.0035%	0.0502%	-0.0027%	0.8489%
18.5	0.328568	0.1303%	-0.0037%	0.0530%	-0.0027%	0.8963%
19	0.318776	0.1380%	-0.0044%	0.0552%	-0.0031%	0.9442%
19.5	0.309272	0.1461%	-0.0049%	0.0579%	-0.0032%	0.9929%
20	0.300049	0.1543%	-0.0053%	0.0600%	-0.0033%	1.0417%
IRS[a]	6.1045%	-0.0972%	0.0029%	-0.0398%	0.0020%	
Time	44.71s	1.05s		1.16s		

P_S is the exact stochastic volatility price, computed from equations (5) and (6).
P_G is the exact Gaussian price, computed from proposition 1.
$P_G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 6.
$\underline{X}(t)$ is the current state-vector, a and $\underline{b}$ are model' parameters, and $\underline{u}$ defines the affine transformation under use.
[a] 20-years swap rate with semiannually compounding.
[b] $MAPE = \frac{1}{2}\frac{\max(|Minimum V_1 - Estimated V_1|, |Maximum V_1 - Estimated V_1|)}{P_S}$ is maximum absolute percentage error for the V_1 estimate. Maximum/Minimum V_1 are computed from Corollary 1.

Table 2: Pricing of pure discount bonds and swaps using the parameters corresponding to an $A_2(3)$ model, for different affine invariant transformations

PDB's Expiry (years)	Exact price (P_S)	Percentage Pricing Errors				
		$\underline{u} = 0.5\underline{X}(t)$		$\underline{u} = \underline{X}(t)$		
		$\frac{P_G - P_S}{P_S}$	$\frac{P_G + \frac{V_1}{2} - P_S}{P_S}$	$\frac{P_G - P_S}{P_S}$	$\frac{P_G + \frac{V_1}{2} - P_S}{P_S}$	MAPE[b]
0.5	0.975063	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
1	0.947129	-0.0004%	0.0000%	0.0002%	0.0000%	0.0005%
1.5	0.919966	-0.0018%	0.0000%	0.0005%	0.0000%	0.0019%
2	0.893223	-0.0046%	0.0000%	0.0013%	0.0000%	0.0050%
2.5	0.866454	-0.0092%	0.0000%	0.0022%	0.0000%	0.0097%
			...			
18	0.172002	-10.7382%	0.9599%	-5.4174%	0.4076%	6.3996%
18.5	0.159963	-11.8327%	1.0796%	-6.0239%	0.4639%	7.1025%
19	0.148605	-12.9962%	1.2052%	-6.6727%	0.5235%	7.8528%
19.5	0.137907	-14.2292%	1.3364%	-7.3651%	0.5866%	8.6517%
20	0.127847	-15.5326%	1.4697%	-8.1011%	0.6516%	9.4977%
IRS[a]	8.8908%	3.9567%	-0.3314%	1.9450%	-0.1424%	
Time	474.06s	1.53s		2.92s		

P_S is the exact stochastic volatility price, computed from equations (5) and (6).

P_G is the exact Gaussian price, computed from proposition 1.

$P_G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 6.

$\underline{X}(t)$ is the current state-vector and $\underline{u}$ defines the affine transformation under use.

[a] 20-years swap rate with semiannually compounding.

[b] $MAPE = \frac{1}{2}\frac{\max(|MinimumV_1 - EstimatedV_1|, |MaximumV_1 - EstimatedV_1|)}{P_S}$ is maximum absolute percentage error of the V_1 estimate. Maximum/Minimum V_1 are computed from Corollary 1.

Table 3: Pricing of bond futures with a maturity of 6 months using the same parameter values as in the three-factor CIR model of Schlogl and Sommer (1998, Figure 5), for different affine invariant transformations

PDB's Expiry (years)	Exact price (FP_S)	Percentage Pricing Errors: $\underline{u}$: VAR match $\frac{FP_G - FP_S}{FP_S}$	$\underline{u}$: VAR match $\frac{FP_G + \frac{V_1}{2} - FP_S}{FP_S}$	$\underline{u} = \underline{X}(t)$ $\frac{FP_G - FP_S}{FP_S}$	$\underline{u} = \underline{X}(t)$ $\frac{FP_G + \frac{V_1}{2} - FP_S}{FP_S}$	MAPE[b]
1	0.970434	0.0000%	0.0000%	0.0000%	0.0000%	0.0003%
1.5	0.941742	0.0000%	0.0000%	0.0000%	0.0000%	0.0009%
2	0.913899	0.0001%	0.0001%	0.0001%	0.0001%	0.0019%
2.5	0.886883	0.0001%	0.0001%	0.0001%	0.0001%	0.0030%
3	0.860667	0.0002%	0.0002%	0.0002%	0.0002%	0.0043%
			...			
18.5	0.338553	0.0520%	0.0520%	0.0526%	0.0526%	0.0264%
19	0.328463	0.0545%	0.0542%	0.0551%	0.0548%	0.0262%
19.5	0.318670	0.0571%	0.0568%	0.0577%	0.0574%	0.0264%
20	0.309167	0.0595%	0.0592%	0.0598%	0.0598%	0.0268%
20.5	0.299944	0.0620%	0.0617%	0.0623%	0.0623%	0.0269%
FCBB[a]	121.7132	0.0265%	0.0263%	0.0268%	0.0268%	
Time	1499.02s	1.76s		1.76s		

FP_G is the exact Gaussian price, computed from proposition 7.
FP_S is the exact stochastic volatility price, computed from proposition 8.
$FP_G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 9.
$\underline{X}(t)$ is the current state-vector and $\underline{u}$ defines the affine transformation under use.
[a]6-month future on a benchmark bond with a maturity of 20.5 years, a semi-annual coupon of 8%, and a face value of 100. Delivery options are ignored.
[b]$MAPE = \frac{1}{2}\frac{\max(|MinimumV_1 - EstimatedV_1|, |MaximumV_1 - EstimatedV_1|)}{FP_S}$ is maximum absolute percentage error for the V_1 estimate. Maximum/Minimum V_1 are given by Corollary 1.

Table 4: Pricing of 3-month Eurodollar futures using the parameters corresponding to the $A_1(3)_{DS}$ Dai and Singleton (1998, Table II) model, for different affine invariant transformations

Futures' Maturity (years)	Exact price (FR_S)	Percentage Pricing Errors: $\underline{u} = -a^{-1} \cdot \underline{b}$ $\frac{FR_G}{FR_S} - 1$	$\underline{u} = -a^{-1} \cdot \underline{b}$ $\frac{FR_G - \frac{50V_1}{0.25}}{FR_S} - 1$	$\underline{u} = \underline{X}(t)$ $\frac{FR_G}{FR_S} - 1$	$\underline{u} = \underline{X}(t)$ $\frac{FR_G - \frac{50V_1}{0.25}}{FR_S} - 1$	MAPE[a]
1/12	88.3744	0.0008%	0.0008%	0.0000%	0.0000%	0.0001%
2/12	88.4027	0.0007%	0.0008%	-0.0001%	-0.0001%	0.0001%
3/12	88.4491	0.0007%	0.0008%	-0.0001%	-0.0001%	0.0002%
4/12	88.4981	0.0006%	0.0008%	-0.0001%	-0.0001%	0.0003%
5/12	88.5459	0.0005%	0.0007%	-0.0001%	-0.0001%	0.0004%
			...			
8	89.1890	-0.0022%	-0.0008%	0.0024%	0.0002%	0.0035%
8.25	89.1738	-0.0022%	-0.0008%	0.0026%	0.0003%	0.0035%
8.5	89.1586	-0.0022%	-0.0008%	0.0027%	0.0004%	0.0035%
8.75	89.1436	-0.0021%	-0.0008%	0.0028%	0.0004%	0.0035%
9	89.1288	-0.0021%	-0.0008%	0.0029%	0.0005%	0.0035%
Time	43393s	29s		40.97s		

FR_G is the exact Gaussian price, computed from proposition 10.
FR_S is the exact stochastic volatility price, computed from proposition 11.
$FR_G - \frac{50V_1}{0.25}$ is the first order approximate stochastic volatility price, given by proposition 12.
$\underline{X}(t)$ is the current state-vector, a and $\underline{b}$ are model' parameters, and $\underline{u}$ defines the affine transformation under use.
[a]$MAPE = \frac{50}{0.25}\frac{\max(|MinimumV_1 - EstimatedV_1|, |MaximumV_1 - EstimatedV_1|)}{FR_S}$ is maximum absolute percentage error for V_1 estimate. Maximum/Minimum V_1 are given by Corollary 1.

Table 5: Pricing of a five-year ATM floor with quarterly compounding using the three-factor CIR model of Schlogl and Sommer (1998, Figure 5), for different affine invariant transformations

Call Expiry (years)	Money-ness[a]	Exact MCIR price(c_0^S)	Percentage Pricing Errors: Duffie *et al.* (1998)	$\underline{u}=-a^{-1}\cdot\underline{b}$: $\frac{c_0^G-c_0^S}{c_0^S}$	$\underline{u}=-a^{-1}\cdot\underline{b}$: $\frac{c_0^G+\frac{V_1}{2}-c_0^S}{c_0^S}$	$\underline{u}=\underline{X}(t)$: $\frac{c_0^G-c_0^S}{c_0^S}$	$\underline{u}=\underline{X}(t)$: $\frac{c_0^G+\frac{V_1}{2}-c_0^S}{c_0^S}$
0.25	-0.0001%	0.046329	-0.0005%	-0.677%	0.0408%	0.107%	0.0443%
0.5	-0.0002%	0.063169	0.0001%	-0.535%	0.0768%	0.213%	0.0839%
0.75	-0.0003%	0.074636	0.0003%	-0.391%	0.1115%	0.322%	0.1228%
1	-0.0003%	0.083181	-0.0021%	-0.247%	0.1454%	0.434%	0.1634%
1.25	-0.0002%	0.089795	-0.0060%	-0.101%	0.1784%	0.549%	0.2044%
1.5	-0.0002%	0.095013	-0.0061%	0.045%	0.2103%	0.661%	0.2405%
				...			
3.5	0.0002%	0.111222	0.0831%	1.205%	0.4277%	1.453%	0.3930%
3.75	0.0003%	0.111500	0.0861%	1.347%	0.4499%	1.554%	0.4142%
4	0.0003%	0.111559	0.0866%	1.486%	0.4709%	1.657%	0.4367%
4.25	0.0003%	0.111429	0.0849%	1.624%	0.4908%	1.760%	0.4599%
4.5	0.0002%	0.111134	0.0815%	1.761%	0.5096%	1.862%	0.4836%
4.75	0.0002%	0.110696	0.0767%	1.896%	0.5273%	1.964%	0.5074%
Floor		1.890898	0.0456%	1.130%	0.3363%	1.186%	0.3742%
Time			43625.8s	71.35s		56.9s	

The floor rate is set equal to the 5-year forward swap rate (with quarterly compounding): $k = 6.0456\%$. Floor prices are for \$100 of Notional Value.

[a]Difference between forward price of underlying PDB and strike $(1+0.25k)^{-1}$, over strike.

Exact MCIR prices, c_0^S, are computed from Chen and Scott (1995) formulae.

Duffie, Pan and Singleton (1998) approach implemented by evaluating numerically the characteristic function and inverting each Fourier transform through a 10-point Gaussian quadrature.

c_0^G is the exact Gaussian price, computed from proposition 13.

$c_0^G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 14.

$\underline{X}(t)$ is the current state-vector, a and $\underline{b}$ are model' parameters, and $\underline{u}$ defines the affine transformation under use.

Table 6: Pricing of a five-year OTM floor with quarterly compounding using the three-factor CIR model of Schlogl and Sommer (1998, Figure 5), for different affine invariant transformations

Call		Exact	Percentage Pricing Errors				
Expiry	Money-	MCIR	Duffie *et*	$\underline{u}$: VAR match		$\underline{u} = \underline{X}(t)$	
(years)	ness[a]	price(c_0^S)	*al.* (1998)	$\frac{c_0^G - c_0^S}{c_0^S}$	$\frac{c_0^G + \frac{V_1}{2} - c_0^S}{c_0^S}$	$\frac{c_0^G - c_0^S}{c_0^S}$	$\frac{c_0^G + \frac{V_1}{2} - c_0^S}{c_0^S}$
0.25	-0.258%	0.000436	-0.0899%	49.699%	-3.9281%	50.258%	-4.0793%
0.5	-0.258%	0.003148	0.1813%	30.826%	-0.1865%	31.369%	-0.3581%
0.75	-0.258%	0.006929	-0.1872%	24.868%	0.5117%	25.540%	0.3875%
1	-0.258%	0.010796	-0.4358%	22.015%	0.8221%	22.776%	0.7036%
1.25	-0.258%	0.014413	-0.0053%	20.377%	1.0055%	21.223%	0.8916%
1.5	-0.258%	0.017674	-0.0031%	19.350%	1.1382%	20.276%	1.0261%
				...			
3.5	-0.257%	0.032466	-0.0028%	17.566%	1.7256%	18.936%	1.6001%
3.75	-0.257%	0.033306	-0.0054%	17.616%	1.7737%	19.020%	1.6454%
4	-0.257%	0.033997	-0.0081%	17.688%	1.8181%	19.121%	1.6869%
4.25	-0.257%	0.034556	-0.0107%	17.779%	1.8590%	19.236%	1.7247%
4.5	-0.257%	0.034998	-0.0132%	17.884%	1.8966%	19.361%	1.7591%
4.75	-0.257%	0.035336	-0.0155%	18.002%	1.9311%	19.493%	1.7902%
Floor		0.450374	-0.0158%	18.282%	1.5733%	19.542%	1.4487%
Time			42486.2s	56.25s		53.77s	

The floor rate is set equal to $k = 5\%(< 6.0456\%)$. Floor prices are for \$100 of Notional Value.
[a]Difference between forward price of underlying PDB and strike $(1 + 0.25k)^{-1}$, over strike.
Exact MCIR prices, c_0^S, are computed from Chen and Scott (1995) formulae.
Duffie, Pan and Singleton (1998) approach implemented by evaluating numerically the characteristic function and inverting each Fourier transform through a 10-point Gaussian quadrature.
c_0^G is the exact Gaussian price, computed from proposition 13.
$c_0^G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 14.
$\underline{X}(t)$ is the current state-vector and $\underline{u}$ defines the affine transformation under use.

Table 7: Pricing of a five-year ITM floor with quarterly compounding using the three-factor CIR model of Schlogl and Sommer (1998, Figure 5), for different affine invariant transformations

Call Expiry (years)	Money-ness[a]	Exact MCIR price(c_0^S)	Percentage Pricing Errors: Duffie et al. (1998)	$\underline{u}=-a^{-1}\cdot\underline{b}$: $\frac{c_0^G-c_0^S}{c_0^S}$	$\underline{u}=-a^{-1}\cdot\underline{b}$: $\frac{c_0^G+\frac{V_1}{2}-c_0^S}{c_0^S}$	$\underline{u}=\underline{X}(t)$: $\frac{c_0^G-c_0^S}{c_0^S}$	$\underline{u}=\underline{X}(t)$: $\frac{c_0^G+\frac{V_1}{2}-c_0^S}{c_0^S}$
0.25	0.235%	0.228866	-0.0001%	-0.696%	-0.6960%	-0.142%	-0.0053%
0.5	0.235%	0.230697	-0.0005%	-3.096%	-3.0955%	-0.454%	0.0052%
0.75	0.235%	0.233313	0.0027%	-5.776%	-5.7762%	-0.700%	0.0243%
1	0.235%	0.235610	0.0094%	-8.271%	-8.2710%	-0.877%	0.0454%
1.25	0.235%	0.237351	0.0126%	-10.504%	-10.5022%	-1.007%	0.0670%
1.5	0.235%	0.238519	0.0001%	-12.489%	-12.4785%	-1.101%	0.0884%
				...			
3.5	0.235%	0.233367	0.0001%	-22.749%	-21.8815%	-1.280%	0.2410%
3.75	0.235%	0.231575	0.0001%	-23.611%	-22.5237%	-1.268%	0.2571%
4	0.235%	0.229631	-0.0066%	-24.417%	-23.0907%	-1.253%	0.2725%
4.25	0.235%	0.227557	0.0000%	-25.173%	-23.5915%	-1.236%	0.2871%
4.5	0.235%	0.225372	0.0000%	-25.885%	-24.0341%	-1.216%	0.3010%
4.75	0.235%	0.223092	0.0000%	-26.558%	-24.4251%	-1.194%	0.3142%
Floor		4.517682	0.0010%	-16.669%	-16.1096%	-1.080%	0.1610%
Time			41782s	56.63s		56.79s	

The floor rate is set equal to $k = 7\%(> 6.0456\%)$. Floor prices are for \$100 of Notional Value.

[a]Difference between forward price of underlying PDB and strike $(1+0.25k)^{-1}$, over strike.

Exact MCIR prices, c_0^S, are computed from Chen and Scott (1995) formulae.

Duffie, Pan and Singleton (1998) approach implemented by evaluating numerically the characteristic function and inverting each Fourier transform through a 10-point Gaussian quadrature.

c_0^G is the exact Gaussian price, computed from proposition 13.

$c_0^G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 14.

$\underline{X}(t)$ is the current state-vector, a and $\underline{b}$ are model' parameters, and $\underline{u}$ defines the affine transformation under use.

Table 8: Pricing of a five-year ATM floor with quarterly compounding using an $A_2(3)$ model

Call Expiry (years)	Money-ness[a]	Standard Monte Carlo price (c_0^S)	Standard Monte Carlo % std. error	Percentage Pricing Errors: Duffie *et al.* (1998)	$\underline{u} = \underline{X}(t)$: $\frac{c_0^G - c_0^S}{c_0^S}$	$\underline{u} = \underline{X}(t)$: $\frac{c_0^G + \frac{V_1}{2} - c_0^S}{c_0^S}$
0.25	0.2016%	0.197400	0.1104%	0.2258%	0.0246%	0.1986%
0.5	0.1363%	0.157100	0.1925%	-0.1155%	0.4464%	0.0483%
0.75	0.1286%	0.166500	0.2131%	-0.2253%	2.2049%	0.1358%
1	0.1317%	0.177400	0.2201%	-0.2563%	3.5054%	0.2368%
1.25	0.1299%	0.182500	0.2271%	-0.5878%	4.0310%	-0.0240%
1.5	0.1196%	0.180800	0.2363%	-0.5354%	4.5817%	0.0256%
			...			
3.5	-0.1128%	0.112500	0.3523%	-0.0485%	1.8958%	0.4934%
3.75	-0.1473%	0.104700	0.3690%	0.4500%	1.6021%	1.1187%
4	-0.1821%	0.097700	0.3850%	0.7523%	1.0691%	1.5610%
4.25	-0.2169%	0.091600	0.3998%	0.6557%	0.1064%	1.6086%
4.5	-0.2519%	0.086000	0.4150%	0.4969%	-0.9362%	1.5973%
4.75	-0.2869%	0.080700	0.4306%	0.4609%	-1.8656%	1.7165%
Floor		2.710338		-0.1013%	2.7013%	0.3778%
Time		60 hours		45543.21s	67.29s	

The floor rate is set equal to the 5-year forward swap rate (with quarterly compounding): $k = 6.3933\%$. Floor prices are for \$100 of Notional Value.

[a]Difference between forward price of underlying PDB and strike, divided by strike price.

Monte Carlo: 200,000 simulations with 1,000 time steps per year; % std. error is standard error divided by option price estimate.

Duffie, Pan and Singleton (1998) approach implemented by evaluating numerically the characteristic function and inverting each Fourier transform through a 10-point Gaussian quadrature.

c_0^G is the exact Gaussian price, computed from proposition 13.

$c_0^G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 14.

$\underline{X}(t)$ is the current state-vector and $\underline{u}$ defines the affine transformation under use.

Table 9: Pricing of a 6-month European call on a 5-year coupon-bearing bond (CBB), using an $A_2(3)$ model

call on CBB (6% annual coupon)				1st-order approximation (with $\underline{u} = \underline{X}(t)$)			
Strike (X)	Money-	Standard Monte Carlo		call on PDB		call on CBB	
	ness[a]	price	%std. error	$\frac{X}{\zeta}$	$c_0^G + \frac{V_1}{2}$	$\zeta\left(c_0^G + \frac{V_1}{2}\right)$	% error
99	1.668%	2.064355	0.2162%	0.76617	0.016007	2.068404	0.1962%
99.5	1.157%	1.720938	0.2410%	0.77004	0.013353	1.725422	0.2605%
100	0.652%	1.409296	0.2698%	0.77391	0.010941	1.413725	0.3143%
100.6515	0.000%	1.054243	0.3151%	0.77895	0.008192	1.058508	0.4045%
101	-0.345%	0.888957	0.3439%	0.78164	0.006912	0.893177	0.4747%
101.5	-0.836%	0.682297	0.3920%	0.78551	0.005311	0.686210	0.5734%
102	-1.322%	0.510303	0.4505%	0.78938	0.003978	0.514039	0.7320%
102.5	-1.803%	0.371407	0.5220%	0.79325	0.002899	0.374629	0.8676%
Time		63148s			14.83s		

[a]Difference between 6-month forward price of CBB (100.6515) and strike, divided by strike.
Monte Carlo: 200,000 simulations with 1,000 time steps per year; % std. error is standard error divided by option price estimate.
ξ is the forward price of the CBB for its stochastic duration: 129.2149.
The stochastic duration of the CBB is the maturity of a PDB with the same instantaneous variance of relative price changes: 4.460377 years.
c_0^G and V_1 are computed from propositions 13 and 14.
The European call on the CBB, with strike X, is approximated by ξ times an European call, with strike $\frac{X}{\xi}$, on a PDB with maturity equal to the stochastic duration of the CBB.

Table 10: Pricing of 3-month European calls on 6-month pure discount bond futures with a maturity of 2.5 years for the underlying bond, using the three-factor CIR model of Schlogl and Sommer (1998, Figure 5)

				$\underline{u} = \underline{X}(t)$			
Strike	Money-	Standard Monte Carlo		Gaussian model		SV model	
	ness[a]	price (c_0^S)	% std. error	c_0^G	$\frac{c_0^G - c_0^S}{c_0^S}$	$c_0^G + \frac{V_1}{2}$	$\frac{c_0^G + \frac{V_1}{2} - c_0^S}{c_0^S}$
0.88	-0.776%	0.007449	0.1788%	0.007398	-0.6789%	0.007436	-0.1798%
0.8825	-0.494%	0.005510	0.2177%	0.005467	-0.7860%	0.005497	-0.2270%
0.885	-0.212%	0.003846	0.2691%	0.003820	-0.6663%	0.003835	-0.2916%
0.88688	0.000%	0.002806	0.3194%	0.002798	-0.2828%	0.002796	-0.3511%
0.8875	0.070%	0.002507	0.3387%	0.002505	-0.0742%	0.002498	-0.3696%
0.89	0.352%	0.001510	0.4358%	0.001530	1.3483%	0.001503	-0.4456%
0.8925	0.633%	0.000830	0.5778%	0.000865	4.2088%	0.000827	-0.4024%
0.895	0.915%	0.000412	0.7962%	0.000450	9.2744%	0.000411	-0.2209%
Time		64236s			17.63s		

[a]Difference between underlying futures price and strike price, divided by strike price.
Monte Carlo: 200,000 simulations with 1,000 time steps per year; % std. error is standard error divided by option price estimate.
c_0^G is the exact Gaussian price, computed from proposition 15.
$c_0^G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 16.
$\underline{X}(t)$ is the current state-vector and $\underline{u}$ defines the affine transformation under use.

Table 11: Pricing of 6-month pure put options on 3-month Eurodollar futures (also with a maturity of 6 months). using the $A_1(3)_{DS}$ Dai and Singleton (1998, Table II) model

Strike	Money-ness[a]	Standard Monte Carlo price (Fp_0^S)	% std. error	Percentage Pricing Errors (for the affine transformation $\underline{u} = \underline{X}(t)$) $\frac{Fp_0^G - Fp_0^S}{Fp_0^S}$	$\frac{Fp_0^G + \frac{V_1}{2} - Fp_0^S}{Fp_0^S}$
88.00	-0.668%	0.186360	0.4908%	-1.3784%	0.2939%
88.25	-0.386%	0.268631	0.4101%	-1.2087%	-0.1089%
88.50	-0.104%	0.373441	0.3465%	-1.0245%	-0.3221%
88.592	0.000%	0.417997	0.3265%	-0.9686%	-0.3780%
88.75	0.178%	0.502075	0.2955%	-0.8788%	-0.4452%
89.00	0.461%	0.654135	0.2541%	-0.7286%	-0.4710%
89.25	0.743%	0.827910	0.2202%	-0.5620%	-0.4158%
89.50	1.025%	1.021050	0.1920%	-0.4181%	-0.3395%
Time		34306s		23.34s	

[a]Difference between underlying futures price and strike price, divided by strike price.
Monte Carlo: 200,000 simulations with 1,000 time steps per year; % std. error is standard error divided by option price estimate.
Fp_0^G is the exact Gaussian price, computed from proposition 18
$Fp_0^G + 0.5V_1$ is the first order approximate stochastic volatility price, given by proposition 19.
$\underline{X}(t)$ is the current state-vector and $\underline{u}$ defines the affine transformation under use.