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# The Sampling Properties of Volatility Cones


## ABSTRACT

In this research, we extend the original work on volatility cones by Burghardt and Lane (1990) to consider of the sampling properties of the variance of variance (and the standard deviation of volatility) under a rich class of models that includes stochastic volatility and conditionally fat-tailed distributions.

Because the volatility cone examines volatility at quite long horizons, the estimation requires the use of overlapping data. This theory confirms the casual observation that the estimation of the variance of variance is downward biased when estimation is done on an overlapping basis. Our principal contribution is to identify what this bias is and derive an adjustment factor that approximates an unbiased estimate of the true variance of variance when overlapping data is used. Another contribution is the derivation of a formula that describes the variance of the quadratic variation over different time horizons.

Using the theory presented, we tested the bias adjustments to the standard deviation of volatility using simulations. Two cases were examined: a GBM i.i.d. process and a non-i.i.d. process associated with the stochastic volatility model suggested by Heston (1993). In both cases, the bias introduced by estimation of volatility with overlapping data becomes insignificant after making the theoretical adjustments..

These results are relevant to those who must sell options and must understand the nature of quadratic variation in asset prices. This will provide clearer insights into the nature of hedging errors when dynamically hedging options.

This research also suggests a new method for the estimation of stochastic volatility models, where estimation over a long horizon is likely to provide robustness not associated with current methods.



JEL classifications: C13, G13

Keywords: Overlapping Data Observations, Volatility Estimation, Quadratic variation, Volatility Cones, Stochastic Volatility

## 1. INTRODUCTION

The seminal work on option pricing by Black and Scholes (1973) launched the field of contingent claims analysis. Ever since, the sampling properties of realized volatility over different time horizons has been of some concern to academics and practitioners alike. However, the first published empirical examination of the average levels of volatilities at different time horizons seems to lie in the 1990 work of Burghardt and Lane. Since then the One critical assumption of this paper was that the volatility of the underlying asset was known and constant. Volatility is defined as the square root of the prospective (annualised) quadratic variation. Quadratic variation is the integral [or approximated as the sum] of squared returns over some time period. Despite the success of the Black Scholes approach, it is generally agreed that volatility is neither known nor constant and that forecasting it is paramount in successful derivatives trading.

Merton (1973) relaxed the assumption of constant volatility. He showed that the Black Scholes formula still yields a unique option price and the appropriate riskless hedge by incorporating the (time varying) quadratic variation of the underlying price process integrated over the life of the option. In subsequent research, Hull & White (1987) showed that for a certain class of stochastic volatility situations, option values can be obtained as integrals over the probability distribution of future (realised) volatility. These papers establish the importance of the prospective quadratic variation in both deterministic and stochastic volatility settings.

Neuberger (1992) demonstrated the strength of the link between realised quadratic variation and the profit or loss on a hedged options position. He shows that by dynamic hedging against a static options position, it is possible to engineer the future profit [or loss] as an exact linear function of the realised quadratic variation. Therefore, option traders must understand the sampling properties of the volatility for the time horizon of the option, not only to determine the expected average level but also to estimate the possible gains or losses.

Burghardt and Lane (1990) were the first to examine empirically the average levels of volatilities at different time horizons and consider the variability of the realisations. Using sample these samples of realised volatilities (measured on an overlapping basis), they determined the average levels and the maximum/minimum ranges. Neither Burghardt and Lane (1990) nor subsequent researchers examined the sample properties of these estimates. It is apparent that to apply this approach for forecasting, we must be sensitive to the biases introduced by the use of overlapping data. The purpose of this paper is to examine these biases. We will restrict our analysis initially to the variance of the quadratic variation and use these results to explore the effects of overlapping biases on the standard deviation of the volatility.

We develop a theory of the sampling properties of volatility estimated on an overlapping basis. The resulting estimate of the bias can be used to determine an adjustment to the standard deviation of the volatility estimated using overlapping data. This yields a truer picture of the future variability of asset prices. The major contribution of this research is to develop a theory of the variance (or standard deviations) of quadratic variation (volatility) in asset price processes over different time horizons.

The sampling properties of cones will be analyzed under a variety of different models. These models will include jump processes and stochastic volatility. We argue that cones are relevant as they reflect the sampling properties of volatility at a range of time horizons. In contrast, most other methods used in understanding volatility dynamics are estimated directly from daily returns (e.g. ARCH, GARCH models estimated using maximum likelihood methods) which makes them less robust to specification error. Our analysis is a prerequisite for the development of estimation procedures that fit simultaneously across different time horizons [see Tompkins (2000)].

The paper is organised as follows. The second section provides a brief literature review and develops a theory of sampling properties of variance estimation using overlapping data. Initially, we will focus on the variance of quadratic variation in asset price processes over different time horizons. This will entail a detailed description of the implicit weighting scheme for observations when data is overlapping. This is developed for a general model related to the Heston (1993) volatility process [which is similar to the Cox, Ingersoll and Ross (1985) paper on interest rates]. Nested in our model are all i.i.d. processes, which are considered ahead of the stochastic volatility case. In this section, we precisely quantify the bias in estimating the variance of quadratic variation and approximately quantify the bias for the standard deviation of volatility. The third section tests this model by simulating the sampling properties of the standard deviation of volatility for an i.i.d. model and a stochastic volatility model. Using our sampling theory derived in the second section, we assess how effectively we can form unbiased estimates of the standard deviation of volatility at different time horizons. A summary and implications of these results follow.

## 2. A THEORY OF THE SAMPLING PROPERTIES OF VOLATILITY CONES

One crucial assumption of Black and Scholes (1973) analysis of option pricing is that the volatility of the underlying asset is known and constant. Volatility is defined as the square root of the prospective (annualised) quadratic variation. Quadratic variation is the integral [or approximated as the sum] of squared returns over some time period. Despite the success of the Black Scholes approach, it is generally agreed that volatility is neither known nor constant and that forecasting it is paramount in successful derivatives trading.

Merton (1973) relaxed the assumption of constant volatility. He showed that the Black Scholes formula still yields a unique option price and the appropriate riskless hedge by incorporating the (time varying) quadratic variation of the underlying price process integrated over the life of the option. In subsequent research, Hull & White (1987) showed that for a certain class of stochastic volatility situations, option values can be obtained as integrals over the probability distribution of future (realised) volatility. These papers establish the importance of the prospective quadratic variation in both deterministic and stochastic volatility settings.

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Burghardt and Lane (1990) were the first to examine empirically the average levels of volatilities at different time horizons and consider the variability of the realisations. They achieved this by use of the volatility cone technique. This method measures the unconditional volatility for different time horizons using a given set of underlying returns. With these samples of realised volatilities (measured on an overlapping basis), they determined the average levels and the maximum/minimum ranges. The reason why their technique is referred to as a volatility cone is that the difference between the maximum and minimum levels narrows as the time horizon of volatility estimation is extended.

The choice of the range as a measure of variability is somewhat arbitrary (although it suited the purpose of their research). Neither Burghardt and Lane (1990) nor subsequent researchers examined the sample properties of volatility cones, yet it is readily apparent that the use of overlapping data must introduce biases into these estimates.

Consider a typical procedure in which, following from Burghardt and Lane (1990), volatility cones are estimated using daily price relatives (natural logarithm of closing prices) for the entire period of analysis. Assuming the time period of ten years of trading days (250 trading days per year), the number of daily price relatives will be approximately $T$=2500. These price relatives will be grouped into periods of analysis from a minimum of one day ($h$=1) to a maximum period of two years ($h$=500) in one day increments. With these selected horizon periods, standard deviations (or the volatility) can be estimated for the 500 horizon periods.

For the estimates of the standard deviation to be unbiased, observations must be independent. The usual manner to achieve this is to restrict the analysis solely to non-overlapping data. When this is done, the sampling properties of such an approach are well known. An equally well known problem is that as the time horizon of estimation is extended, the number of non-overlapping samples is reduced ($n = T/h$), sampling properties are unable to be determined as the degrees of freedom are also reduced. Said simply, the estimation method runs out of samples. The penultimate problem exists for the time horizon $h = T$, where only one estimate is possible. An alternative is to estimate the standard deviation on an overlapping basis. Now, the number of observed samples, $n$, is increased ($n = T-h+1$) but they are no longer independent. Unfortunately, the sampling properties of the standard deviation will be biased due to interdependence in the samples. This is what is examined in this paper.

In this research, we focus initially on the variance of the variance. The high degree of correlation between such overlapping samples will dampen the true variability of the variance and will provide little insight into its true nature. A number of authors have considered this problem. One method of addressing the problem of overlapping data in variance estimation is the use of panel regression techniques. Hansen and Hodrick (1980) first suggested this approach. Dunis and Keller (1995) modified this for the examination of currency option volatilities. Other proposed solutions to the overlapping data problem include the bootstrap method proposed by Efron (1983) and the jack-knife method suggested by Yang and Robinson (1986).

The approach adopted here is to develop a theory of the biases of the estimated variances for overlapping and non-overlapping periods. We examine these variances for a range of time horizons under a rich case of processes, which nest stochastic volatility and all i.i.d. processes (including potentially non-normal ones).

## 2.1 Assumptions

We will consider the sampling properties of volatility cone analysis conducted on a time series containing $t = 1,.., T$ observations of single period security returns $r_t$. The analysis focuses on the variance properties of these returns. With daily (or even weekly) returns data, variance estimates are unaffected by the level of the mean return. Without any real loss of generality we can therefore make the convenient assumption that each return $r_t$ is drawn from a distribution with zero mean. We assume the distribution of $r_t$, conditioned on the information at the beginning of the return period, is constant in shape, but that its variance $v_t$ changes through time. In particular we assume that its kurtosis exists as a finite constant $K$. The fact that $v_t$ may have a complex autocorrelation structure precludes the possibility of devising any simple bootstrapping method.

Formally, we will assume that $v_t$ is a stationary stochastic process with long run mean of $\bar{v}$. We can write the squared returns as:

$$s_t = r_t^2 = v_t \left(1 + \sqrt{K-1}\,\varepsilon_t\right), \text{ where } \varepsilon_t \text{ has zero mean, unit variance, is i.i.d. and}$$

$$v_t = \bar{v} + u_t, \text{ where } u_t \text{ is independent of } \varepsilon_t. \tag{1}$$

Note that if our returns are measured over time intervals of length $\Delta t$ then

$$\bar{v} = \sigma^2\, \delta t, \text{ where } \sigma^2 \text{ is the average variance rate.}$$

## 2.2 Weighting Of Observations

We will consider the sampling properties of variance estimates based on overlapping (contiguous) sub-series of data of length $h$. For a given horizon of length $h$, the number of distinct sub-series available is $n = T - h + 1$. Our analysis is restricted to cases where $h < T/2$, which is also a sensible practical restriction. If we restricted ourselves to non-overlapping sub-series we would have at most $T/h$ observations available on the quadratic variation (measured as the average variance rate) to horizon $h$.

Each variance estimate for the cone is obtained as a weighted average of the single period squared returns, $s_t = r_t^2$. The variance estimate, $\phi_i$, from the $i^{\text{th}}$ sub-series is simply

$$\phi_i = \frac{1}{h}\sum_{t=i}^{i+h-1} s_t. \qquad \text{where } i = 1, \ldots, n. \tag{2.1}$$

We can more conveniently regard it as the vector product

$$\phi_i = \mathbf{w}_i' \mathbf{s} \tag{2.2}$$

where $\mathbf{w}_i$ and $\mathbf{s}$ are $T\times 1$ vectors, and $\mathbf{w}_i$ has values of $1/h$ in elements $i$ through $i+h-1$ and zeros elsewhere. The mean of these variance estimates, $\bar{\phi}$, weights the squared returns with a mean weighting vector, $\mathbf{m}$, as

$$\bar{\phi} = \frac{1}{n}\sum_{i=1}^{n}\phi_i = \left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i\right]' \mathbf{s} = \mathbf{m}'\mathbf{s}. \tag{3}$$

The deviations of these estimates from their mean, $z_i$, are given by

$$z_i = \phi_i - \bar{\phi} = (\mathbf{w}_i - \mathbf{m})'\mathbf{s} = \mathbf{x}_i'\mathbf{s}.$$

(4.1)

Finally, by stacking the $n$ $(1 \times T)$ row-vectors $\mathbf{x}_i' = (\mathbf{w}_i - \mathbf{m})'$ vertically on top of each other to form an $n \times T$ matrix $\mathbf{X}'$, we can represent the $n \times 1$ vector $\mathbf{z}$ of deviations, $z_i$, from the mean as

$$\mathbf{z} = \mathbf{X}'\mathbf{s}. \tag{4.2}$$

### 2.3 The Cross-Sample Variance of Estimated Variances

By construction, the $z_i$ sum to zero. The cross-sample variance of $h$-period variances obtained from the overlapping volatility cone technique is therefore given by

$$\hat{\theta} = \frac{1}{n}\mathbf{z}'\mathbf{z}. \tag{5.1}$$

We are interested in the behaviour of this, and in particular in its bias.

$$\hat{\theta} = \frac{1}{n}\mathbf{z}'\mathbf{z} = \frac{1}{n}\mathbf{s}'\mathbf{X}\mathbf{X}'\mathbf{s} = \frac{1}{n}\mathbf{1}'(\mathbf{s}\mathbf{s}') \odot (\mathbf{X}\mathbf{X}')\mathbf{1}, \tag{5.2}$$

where $\odot$ denotes the Hadamard product $c_{ij} = a_{ij} \times b_{ij}$ of corresponding elements of a matrix and $\mathbf{1}$ is a $T \times 1$ vector of ones.

The expected variance of variance is obtained as:

$$E[\hat{\theta}] = \frac{1}{n}\mathbf{1}'\mathbf{V} \odot \mathbf{W}\mathbf{1}, \tag{6}$$

where $\mathbf{V} = E[\mathbf{s}\mathbf{s}']$ and $\mathbf{W} = \mathbf{X}\mathbf{X}'$. In other words, we form the two $T \times T$ matrices $\mathbf{V}$ and $\mathbf{W}$, sum the products of their corresponding elements and divide by $n$.

We will consider the **V** and **W** matrices separately. We will first describe the structure of **X′** and **W** in a little more detail. We will examine **V** and the $E[\hat{\theta}]$ which results from the product with **W**. We do this first for the constant volatility (i.i.d.) case as this is the most tractable, and then provide an extension to the stochastic volatility one.

| **Table 1** **Summary of Principal Notation and Assumptions** |
|---|
| Single period returns $r_t$ (where $t$ = 1,..., $T$) are drawn from a distribution with mean zero, variance $v_t$, and kurtosis $K$. These are all conditioned on the information at $t$–1. $v_t$ is a stationary stochastic process with long run mean of $\bar{v}$. |
| $s_t = r_t^2 = v_t \left(1+\sqrt{K-1}\,\varepsilon_t\right)$ where $\varepsilon_t$ has zero mean, unit variance and is i.i.d. |
| $v_t = \bar{v} + u_t$, where $u_t$ is independent of $\varepsilon_t$. |
| $\mathbf{z} = \mathbf{X}'\mathbf{s}$ is the $n \times 1$ vector of deviations of the variances from the sample mean where $\mathbf{X}'$ is a $n \times T$ matrix of fixed weights which are applied to the $T \times 1$ vector $\mathbf{s}$ of squared returns. The cross-sample variance of the variance is: |
| $\hat{\theta} = \frac{1}{n}\mathbf{z}'\mathbf{z} = \frac{1}{n}\mathbf{s}'\mathbf{X}\mathbf{X}'\mathbf{s} = \frac{1}{n}\mathbf{1}'(\mathbf{s}\mathbf{s}') \odot (\mathbf{X}\mathbf{X}')\mathbf{1},$ |
| where $\odot$ denotes the Hadamard product $c_{ij} = a_{ij} \times b_{ij}$ of corresponding elements of a matrix. The expected variance of variance is obtained as: |
| $E[\hat{\theta}] = \frac{1}{n}\mathbf{1}'\mathbf{V} \odot \mathbf{W}\mathbf{1},$ |
| where $\mathbf{V} = E[\mathbf{s}\mathbf{s}']$ and $\mathbf{W} = \mathbf{X}\mathbf{X}'$. |

## 2.4 The Structure of The X′ Matrix

The basic structure of the **X′** matrix has already been outlined above, but because the form of **XX′** is central to our analysis, it is worth providing rather more detail. Figure 1 shows the nature of the weights involved: it shows the construction of the 4$^{\text{th}}$ row of the matrix **X′** for the case $T$ = 20, $h$ = 5.

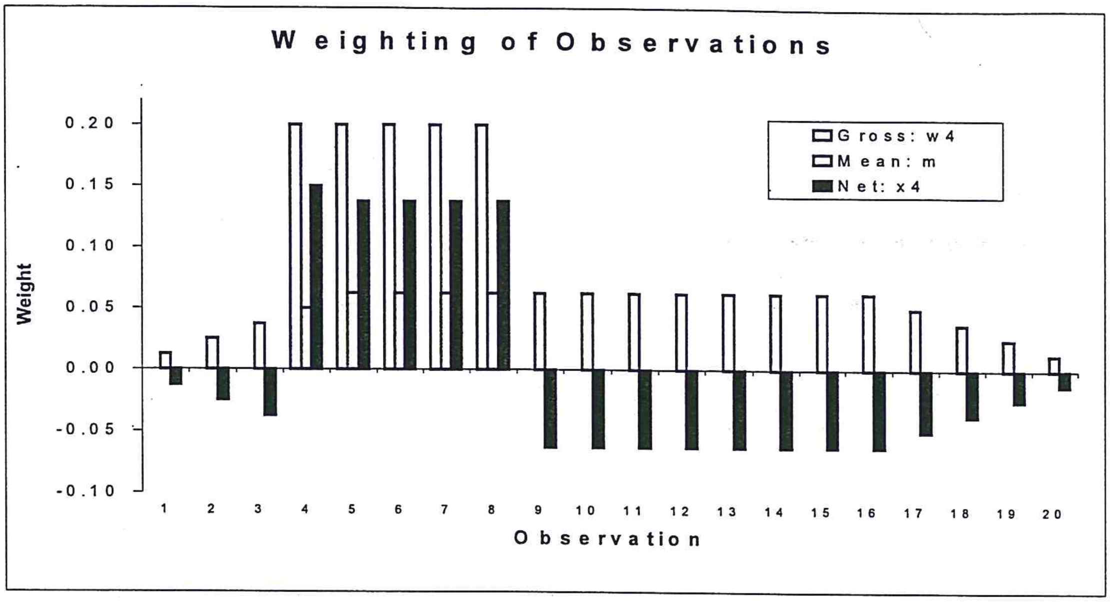


*Figure 1, Weighting of Observations*

For any row $i$ of the matrix, the "raw" weights, $\mathbf{w}'_i$, contains a group of $h$ contiguous $1/h$'s, and the remaining $T$–$h$ entries are zeros. The average of these, $\mathbf{m}'$, is subtracted from each $\mathbf{w}'_i$ to obtain the corresponding $\mathbf{x}'_i$ row of $\mathbf{X}'$. Table 2 shows the form of the coefficients in this matrix. A property we shall later make use of is that both the column sums and the row sums of this matrix are zero.

**Table 2 Form of the X′ Matrix (multiplied by *nh*)**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n-1$ | $n-2$ | $n-h$ | $-h$ | $-h$ | $-h$ | $-h$ | $-2$ | $-1$ |
| $-1$ | $n-2$ | $n-h$ | $n-h$ | $-h$ | $-h$ | $-h$ | $-2$ | $-1$ |
| $-1$ | $-2$ | $n-h$ | $n-h$ | $n-h$ | $-h$ | $-h$ | $-2$ | $-1$ |
| $-1$ | $-2$ | $-h$ | $n-h$ | $n-h$ | $n-h$ | $-h$ | $-2$ | $-1$ |
| $-1$ | $-2$ | $-h$ | $-h$ | $n-h$ | $n-h$ | $n-h$ | $-2$ | $-1$ |
| $-1$ | $-2$ | $-h$ | $-h$ | $-h$ | $n-h$ | $n-h$ | $n-2$ | $-1$ |
| $-1$ | $-2$ | $-h$ | $-h$ | $-h$ | $-h$ | $n-h$ | $n-2$ | $n-1$ |

($h$ –1) columns ($n$–$h$+1) columns ($h$ – 1) columns

[Dimensions correspond to the case $T = 9$, $h = 3$, $n = 7$, with $n$ rows and $T = n + h - 1$ columns.]

## 2.5 The Structure Of The W Matrix

We next consider the form of the **W** matrix ( = **XX′**). Table 3 continues the previous example to show the general form of the **W** matrix ( = **XX′**) multiplied by $nh^2$. The matrix is symmetric, and the upper triangle provides a guide to the general formulae which are given below the table.

**Table 3** **Form of the W (=XX′) Matrix (multiplied by $nh^2$)**

[This table also with $T = 9,\ h = 3, n = 7$, and **W** is $T \times T$.]

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6 | $e_{11}$ | $e_{12}$ | $b_1$ | $b_1$ | $b_1$ | $b_1$ | $a_{12}$ | $a_{11}$ |
| 5 | 10 | $e_{21}$ | $f_{22}$ | $b_2$ | $b_2$ | $b_2$ | $a_{22}$ | $a_{21}$ |
| 4 | 8 | 12 | $g_1$ | $g_2$ | $c$ | $c$ | $b_2$ | $b_1$ |
| -3 | 1 | 5 | 12 | $g_1$ | $g_2$ | $c$ | $b_2$ | $b_1$ |
| -3 | -6 | -2 | 5 | 12 | $g_1$ | $g_2$ | $b_2$ | $b_1$ |
| -3 | -6 | -9 | -2 | 5 | 12 | $g_1$ | $f_{22}$ | $b_1$ |
| -3 | -6 | -9 | -9 | -2 | 5 | 12 | $e_{21}$ | $e_{12}$ |
| -2 | -4 | -6 | -6 | -6 | 1 | 8 | 10 | $e_{11}$ |
| -1 | -2 | -3 | -3 | -3 | -3 | 4 | 5 | 6 |

The formulae for these cell values are:

$$a_{ij} = -ij,\ b_i = -ih,\ c = -h^2,\ e_{id} = i\,(n - i\ - d),\ f_{id} = h\,(n - i\,) - d\,n^2, \text{ and}$$

$$g_d = h\,(n - h) - d\,n^2,$$

where $i, j < h$ denote a row or column distance from the edge of the matrix, and $d < h$ denotes a distance from the diagonal of the matrix.

The values of the diagonal itself are $i\,(n - i)$ for $i \leq h$, and $h\,(n - h)$ elsewhere.

As before, this example is for $T = 9,\ h = 3, n = 7$.

### 2.6 The I.I.D. Case

In the case where the underlying volatility is constant, we have $v_t = \bar{v}$ for all $t$, and $s_t$ simplifies to:

$$s_t = r_t^2 = \bar{v}\left(1 + \sqrt{K-1}\,\varepsilon_t\right) \tag{7}$$

where $\varepsilon_t$ has zero mean, unit variance and is i.i.d. This makes V a diagonal matrix with entries on the diagonal equal to $(K-1)\bar{v}^2$. (8)

We will consider first the true variance of the rate of quadratic variation in samples of length $h$, and then the variance obtained from overlapping estimates.

The annualized variance rate $\sigma^2$ is $\bar{v}$ divided by the length of the time interval $\Delta t$. For a sequence of $h$ returns, the realized (annual) rate of quadratic variation is

$$\frac{1}{h}\sum_{t=1}^{h} s_t / \Delta t. \tag{9}$$

Its mean is $\sigma^2$ and its variance is $(K-1)\,\sigma^4 / h$.

We next derive the comparable expression for estimates derived from volatility cones employing $n$ overlapping sub-series each containing $h$ returns.

In this i.i.d. case, since **V** is diagonal with constant elements, we require the sum of the diagonal elements of **W** (or its trace).

From the expressions derived previously, it is clear that this sum amounts to:

$$trace(\mathbf{W}) = \frac{2\sum_{i=1}^{h-1} i(n-i) + h(n-h)(n-h+1)}{nh^2} \tag{10.1}$$

which simplifies to

$$trace(\mathbf{W}) = \frac{3n^2 - 3nh + h^2 - 1}{3nh}. \tag{10.2}$$

The variance of the estimated annualized quadratic variation is this multiplied by $(K-1)\,\sigma^4 / n$, which gives

$$E[\hat{\theta}] = \frac{(K-1)\sigma^4}{h}\left(1 - \frac{h}{n} + \frac{h^2-1}{3n^2}\right), \tag{11}$$

so the variance obtained from the cone is too small by a factor of the expression on the right hand side, which is approximately $(1 - h/n)$, or roughly $\left(1 - \frac{h}{T-h}\right)$ in terms of the total number of observations, $T$. From this we can derive the following rule of thumb: to maintain a (proportional) bias (before adjustment) of less than $1/k$, we should restrict $k$ not to exceed $\frac{1}{k+1}$.

Under the hypothesis of an i.i.d. process, we can adjust for this bias by multiplying quadratic variation estimated from overlapping data by the adjustment factor:

$$\frac{1}{1 - \frac{h}{n} + \frac{h^2-1}{3n^2}}. \tag{12}$$

### 2.7 The Stochastic Volatility Case

We have in mind the class of stochastic volatility models which provide the a negative exponential covariance structure for the covariance of. the asset's returns. A particularly suitable example was first suggested by Cox, Ingersoll and Ross (1976) for interest rates and subsequently applied to variances by Heston (1993). The continuous-time variance process can be expressed as:

$$dv(t) = \alpha[\sigma^2 - v(t)]dt + \xi\sqrt{v(t)}\, dz_2(t) \tag{13}$$

and the spot asset at time $t$ follows the diffusion

$$dS(t) = \mu S dt + \sqrt{v(t)}\, S\, dz_1(t) \tag{14}$$

where $\alpha$, $\xi$ and $\mu$ are constants and $dz_1(t), dz_2(t)$ are increments of Brownian Motions with correlation $\rho$. Although for these diffusion processes, the disturbance term for the spot asset price is Gaussian, under our discrete time formulation we permit a more general distribution with kurtosis $K$. Under the process we have just described, the unconditional covariance between $v_t$ at any two of our (discretely labelled) dates $s$ and $t$ is given by:

$$Cov[v_s, v_t] = E[u_s\, u_t] = \frac{\sigma^2\xi^2}{2\alpha} e^{-\alpha|t-s|\delta t}. \tag{15}$$

In this stochastic volatility case, we now have to work with the full structure of both the **V** and **W** matrices. The above equation provides the form of the off-diagonal entries of **V**. The term $\frac{\sigma^2\xi^2}{2\alpha}$ is the steady state variance of the variance and $\alpha$ is the rate of mean reversion.

It should be noted that this is ignoring transient effects from having a single starting point. For known $v_0$ we would instead have[1]:

$$E[u_s\, u_t] = \frac{\sigma^2\xi^2}{2\alpha} \cdot e^{-\alpha|t-s|\Delta t} \times \left(1 - e^{-2\alpha s \Delta t}\right) \text{ for } s < t. \tag{16}$$

However, this effect dies out rapidly from the mean reversion of the model and thus, we concentrate on the analysis of equation (15) from this point forward.

The form of the **V** matrix is therefore:

[1] For O-U processes only: C-I-R seems more complicated

$$v_{ij} = (K\frac{\sigma^2\xi^2}{2\alpha}e^{-\alpha|j-i|\delta t} + (K-1)\sigma^4)\Delta t^2 \text{, for } i = j \text{, and}$$
$$= \frac{\sigma^2\xi^2}{2\alpha}e^{-\alpha|j-i|\delta t}\Delta t^2 \text{, for } i \neq j. \tag{17}$$

This has a banded diagonal structure.

As before, the variance of the annualized quadratic variation comes from $\mathbf{1'V\ 1}/h^2$, where for this purpose $\mathbf{V}$ is taken to be $h \times h$.

This gives:

$$E[\hat{\theta}] = [(K-1)\sigma^4 + Kc + \frac{2c}{k}\sum_{i=1}^{h-1}(h-i)a^i]/h$$
$$= [(K-1)\sigma^4 + Kc + \frac{2ac}{h}\frac{a^h + h(1-a) - 1}{(1-a)^2}]/h\text{, where} \tag{18}$$
$$c = \frac{\sigma^2\xi^2}{2\alpha}\text{, and } a = e^{-\alpha\Delta t}.$$

This equation if of importance in its own right, and provides a temporal pattern for the time profile of variance quite different from the i.i.d case.

In the case of overlapping observations we have to combine the $\mathbf{V}$ and $\mathbf{W}$ matrices. The particular negative exponential form of $\mathbf{V}$ means that in order to obtain the variance of the quadratic variation, besides the terms from the diagonal, we need to evaluate sums of the form:

$$\sum_d a^d \sum_i w_{i,i+d}\,. \tag{19}$$

The sums we require can all be expressed analytically. The expressions they give rise to are rather complicated, so we have given them in Appendix B rather than in the main body of the paper. They provide us with analytic expressions for the variance of the annualized quadratic variation from overlapping samples in the presence of both stochastic volatility and conditional excess kurtosis.

### 2.8 The Standard Deviation of Volatility

So far we have provided formulae for the variance (and hence the standard deviation) of the quadratic variation, expressed as an annual rate. In this section we show how the preceding results may be (approximately) re-expressed in terms of the standard deviation of volatility by using a Taylor series expansion.

We have obtained estimates of the mean, M, and standard deviation S of a quadratic variation measure, $q$. We now need the mean and standard deviation of the volatility $\sigma = \sqrt{q}$. Write:

$$q = M\left(1 + \frac{S}{M}\varepsilon\right) \tag{20}$$

where $\varepsilon$ has zero mean and unit standard deviation. Expanding $\sigma$ as a Taylor series we obtain:

$$\sigma = \sqrt{M}\sqrt{1 + \frac{S}{M}\varepsilon} = \sqrt{M}\left(1 + \frac{S}{2M}\varepsilon - \frac{S^2}{8M^2}\varepsilon^2 + \frac{S^3}{16M^3}\varepsilon^3 + ..\right) \tag{21}$$

We therefore have the approximate result that:

$$\mathrm{SD}(\sigma) \approx \frac{S}{2\sqrt{M}}. \tag{22}$$

The closeness of this approximation depends both on the ratio of $S/M$ and on the higher moments of $\varepsilon$. We can examine it directly for particular cases. The approximation is often rather poor. However, because of the way we apply it to two variances to obtain the ratio of a true to a biased standard deviation, most of the error cancels out and it gives rise to only a second order error in our estimates.

## 3. TESTING THE MODELS

To assess the effectiveness of the adjustment to the standard deviation of volatility in correcting the biases introduced by overlapping estimation, models were tested by simulation.

In this simulation, two cases were considered: an i.i.d. (GBM) case[2] and a stochastic volatility process consistent with equations (13) and (14). Upon fixing model parameters, a 100 year long time series of daily prices was simulated (with 252

[2] We also considered alternative i.i.d. processes, which included a Student-t process. Results were similar to those found for the i.i.d. GBM process.

trading days in each year).[3] For both simulations, the variance$(\sigma^2)$ used for the generation of prices was set equal to 0.04 (or 20% volatility). For the simulations, prices were generated using the standard Euler approach (for discrete increments in time of one day, i.e. 1/252) consistent with equation (14):

$$\Delta S = S\mu\Delta t + S\sigma\Delta z_1 \tag{23}$$

In this equation, we assumed the interest rate was zero, so the only adjustment to the drift came from the risk neutral adjustment associated with a GBM process.

For the stochastic volatility case, a time series of daily variances for the same period of 100 years was estimated. Again, this assumes discrete daily increments and also relied upon an Euler approach. The discrete time version of equation (13) can be expressed as:

$$\Delta v_t = \alpha\left[\sigma^2 - v_{t-1}\right]\Delta t + \xi\sqrt{v_{t-1}}\Delta z_2 \tag{24}$$

The two disturbance terms $(\Delta z_1, \Delta z_2)$ are independent draws from a Gauss-Wiener process. These were determined using a standard Box-Müller method and used the antithetic approach suggested by Boyle (1977). For the stochastic volatility model, the rate of mean reversion $\alpha$ was set to 4.00, the volatility of the variance $\xi$ was set to 0.6 and the long-term variance $\sigma^2$ was set equal to 0.04. As indicated previously, the effects of the stochastic volatility will be to increase the kurtosis of the unconditional returns. Within our sample, the unconditional kurtosis of daily returns was 6.374.

Using this data set of 100 years, the true sampling properties of the volatility cone were estimated with good precision[4]. The returns of the daily time series were estimated using differences in the natural logarithm of prices. This result was then annualised. This can be expressed as:

$$\ln (S_t/S_{t-1}) \times \sqrt{252} \tag{25}$$

[3] Actually, 102 years of data were determined. However, The first two years of simulated prices were eliminated so that the starting values were randomised with the correct distribution.

[4] For the IID case, the theoretical standard deviation of volatility is approximately $\sigma/\sqrt{(2h)}$. In Table 1A in Appendix A, these have been presented along with the standard deviation of volatility from the simulated 100 years. The errors in the simulation are within 3% of the theoretical values.

With this series of daily returns, the standard deviation was determined at time horizons from 20 days to 500 days in 20-day increments. Thereafter, we determined the standard deviation of the standard deviation (volatility) at each time horizon.

Then the 100 years of data was split into 20 samples of five years each. In each of these sub-periods, the volatility cones were re-estimated using overlapping data. From these 20 sub-periods, the standard deviations of the volatility were determined for the same time horizons. The average of the standard deviations of volatility (across the 20 sample periods) was compared to the true standard deviation of volatility determined using the entire 100-year sample. Finally, the square root of the adjustment factors derived previously for the variance of variance were multiplied by the average standard deviations of volatility to assess if the bias had been corrected. The results of these simulations appear in Figure 2 for the IID case and in Figure 3 for the stochastic volatility case. In Appendix A, Tables A1 and A2 appear and provide the numerical summaries of these cases, including the adjustment factor and significance tests of the biases prior and post correction.

For the 100-year period, the standard deviation of volatility was estimated for all time horizons (using overlapping data). For each of the 20 sub-periods, the standard deviations of volatility were also estimated using overlapping data. These were averaged and this was compared to the estimate of the standard deviation for the 100-year period. Both absolute and percentage differences were computed. In Figure 2, results are presented for the IID case. In this Figure, the left two panels consider the unadjusted case. The top panel plots the relationship between the true standard deviation of volatility (as a dashed line) and the unadjusted average of the standard deviations of volatility (as a solid line) relative to the time horizon of estimation. It is clear that the unadjusted average is systematically biased downwards and is directly related to the time horizon of estimation. The bottom panel considers the absolute difference in these two standard deviations and further indicates the relationship between the degree of bias and the time horizon.

The two panels on the right-hand side show the adjusted average standard deviation of volatility. This is determined by multiplying the averaged standard deviation by the adjustment factor formula [equation (12)]. In the upper panel, the two series are plotted as absolute levels of volatility, and they coincide almost exactly. The lower panel plots the percentage differences between the two series. The maximum deviation is +3.42%. In Table A1, t-tests indicate that a statistically significant bias exists when overlapping the data ($|t| > 1$ for $h \geq 60$, and $|t| > 2$ for $h \geq 260$). After correction, there are no cases where the bias is significant.

Figure 3 provides the same information for the stochastic volatility case. We can see clearly that the presence of stochastic volatility has two effects. First, the standard deviation of the volatility now decreases at a slower rate as the time horizon is increased: the $1/\sqrt{n}$ rule no longer applies. Second, the bias from overlapping observations has increased. At the 500$^{th}$ time horizon, the bias is –34.83%, whereas for the i.i.d. case it was –28.91%. Retaining the same scales, the right-hand side panels of Figure 3, display the adjusted average standard deviations of volatility. This difference varies around zero and is no longer monotonically decreasing in the time horizon. The most extreme percentage difference is now only –2.69%. In Table A2, t-tests again indicate significant biases in the unadjusted estimates ($|t| > 1$ for all $h$, and $|t| > 2$ for $h \geq 200$), but not in the adjusted estimates.

## 4. SUMMARY AND CONCLUSIONS

The sampling properties of realized volatility over different time horizons concern both academics and practitioners. For academic research, this will provide information about volatility dynamics beyond what is currently modelled using daily data. For traders of option contracts, a better understanding of the sampling properties of quadratic variation will provide a better forecast of possible gains or losses when dynamically hedging these products.

In this research, we have described how volatility cones can provide information about the sampling properties of volatility measured at different horizon dates. We extend the original work on volatility cones by Burghardt and Lane (1990) and present the sampling properties of the variance of variance (and the standard deviation of volatility). The original approach gave biased information about the variability of volatility because of its use of overlapping data. By developing a theory of the sampling properties of volatility cone estimates, we are able to address and correct for this bias.

We derive expressions for the variance of the variance for a general model, which nests stochastic volatility models and alternative price processes to Gauss-Wiener diffusions. This theory confirms the casual observation that the estimation of the variance of variance is downward biased when estimation is done on an overlapping basis. Our main contribution is to identify what this bias is and derive an adjustment factor that approximates an unbiased estimate of the true variance of variance when overlapping data is used. Equation (18) is also potentially important: it describes the variance of the quadratic variation over different time horizons and under a rich class of models that includes stochastic volatility and conditionally fat-tailed distributions.

To put our work into the same metric as Burghardt and Lane (and most derivatives practitioners), we have also extended the theoretical analysis to the standard deviation of volatility estimated at the various horizons. This part of our work relies on an approximation. The bias adjustments to the standard deviation of volatility were testing using simulations. Two cases were examined: a GBM i.i.d. process and a non-i.i.d. process associated with the stochastic volatility model suggested by Heston (1993). For both cases, the bias in the estimates was significant before the adjustment was made and insignificant afterwards.

This research has a number of implications. Clearly our results are relevant to those who must sell options and must understand the nature of quadratic variation in asset prices. This should lead to clearer insights into the nature of hedging errors when dynamically hedging options.

Another application is the use of unbiased sampling properties of long-term volatilities to estimate stochastic volatility models. This research allows an unbiased estimate of the volatility of volatility, which is so critical to these models. Most other methods for estimation rely directly on daily returns, which makes them less robust to specification error. Our analysis is a prerequisite for the development of estimation procedures that fit simultaneously across longer time horizons.

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**Appendix B**

The $T \times T$ matrix **W** is summed along each of its $d = j - i = 1, \ldots, T-1$ diagonal bands, and these sums are then summed as geometric progressions in increasing powers of $a$.

The summations were checked with the aid of a computer algebra program, and then output as code for numerical work. The key results are summarized here.

The sums below have all been divided by a factor of n.

$\boldsymbol{d = 0}$

This is the sum of the diagonal ($i = j$) entries:

$$hn^2\left[1 - \frac{h}{n} + \frac{h^2 - 1}{3n^2}\right].$$

$\boldsymbol{d = 1}$

This and the following ones occur above and below the diagonal and so must be multiplied by two. We require $a$ times the value of the following sum:

$$n^2(h-1) - h^2 n + \frac{h}{3}(h^2 + 2).$$

$\boldsymbol{d = 2,.., h-1}$, gives $a^2$ times the following sums over $i = 0,.., I = h-3$:

$$-\frac{1}{3}\sum_{i=0}^{I} i^3 a^i + (h-2)\sum_{i=0}^{I} i^2 a^i + (4h - n^2 - 11/3)\sum_{i=0}^{I} i\, a^i$$

$$+ (n^2(h-2) - h^2 n + (h^3 + 11h - 6)/3)\sum_{i=0}^{I} a^i .$$

$\boldsymbol{d = h,.., n-h+1}$, gives $a^h$ times the following sums over $i = 0,.., I = n-2h+1$:

$$h^2 \sum_{i=0}^{I} i\, a^i - h^2 (n-h)\sum_{i=0}^{I} a^i .$$

$\boldsymbol{d = n-h+2,.., n}$, gives $a^{n-h+2}$ times the following sums over $i = 0,.., I = h-$
2:

$$-\frac{1}{6}\sum_{i=0}^{I} i^3 a^i - \sum_{i=0}^{I} i^2 a^i + (h^2 - 11/6)\sum_{i=0}^{I} i\, a^i + (-h^3 + 2h^2 - 1)\sum_{i=0}^{I} a^i .$$

$\boldsymbol{d = n+1,.., n+h-2}$, gives $a^{n+1}$ times the following sums over $i = 0,.., I = h-3$:

$$\frac{1}{6}\sum_{i=0}^{I} i^3 a^i - \frac{h-1}{2}\sum_{i=0}^{I} i^2 a^i + \frac{3h(h-2)+2}{6}\sum_{i=0}^{I} i\, a^i - \frac{h(h-1)(h-2)}{6}\sum_{i=0}^{I} a^i .$$

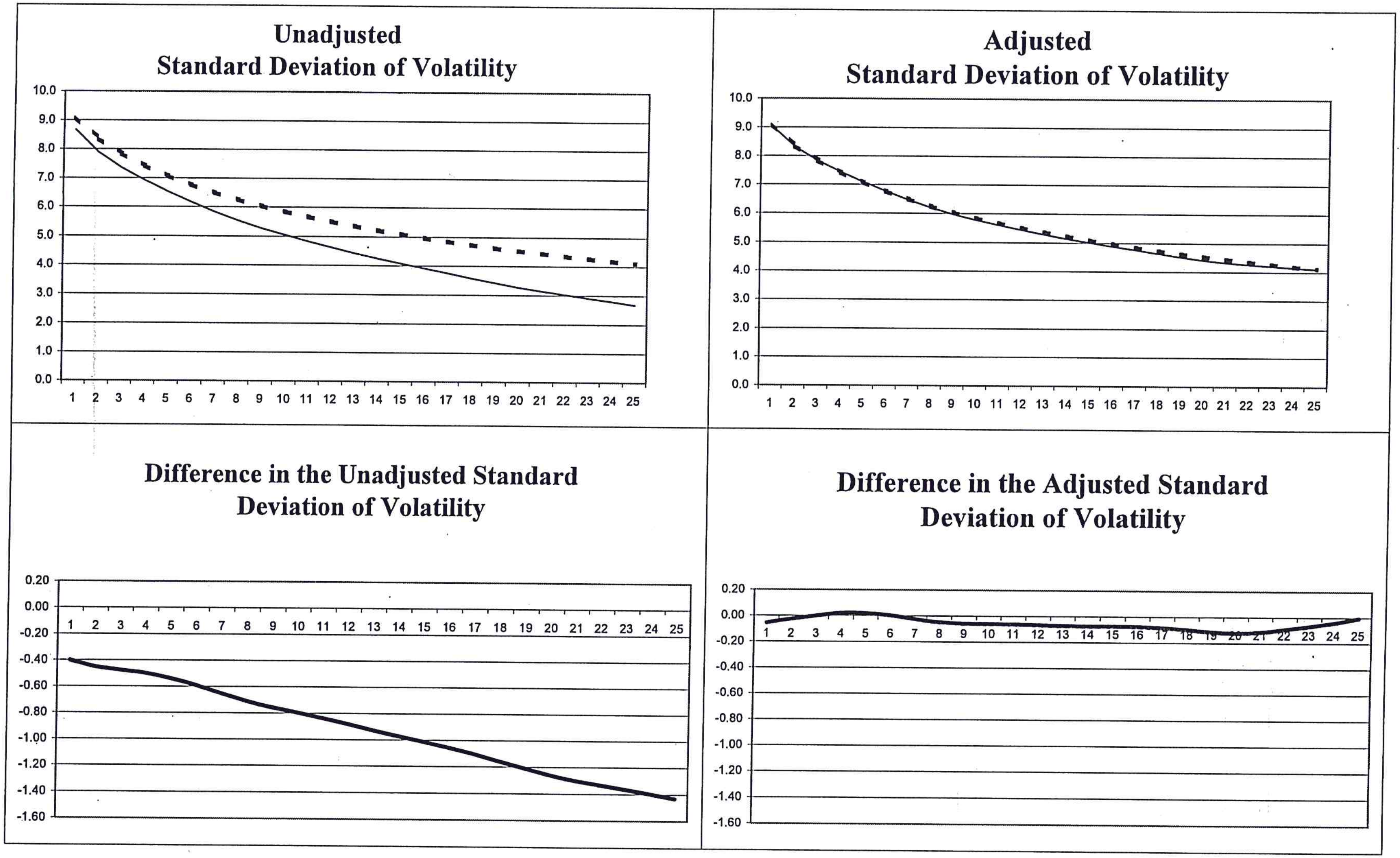


Figure 3, Differences in Unadjusted & Adjusted Standard Deviation of Volatility (Stochastic Volatility Case)

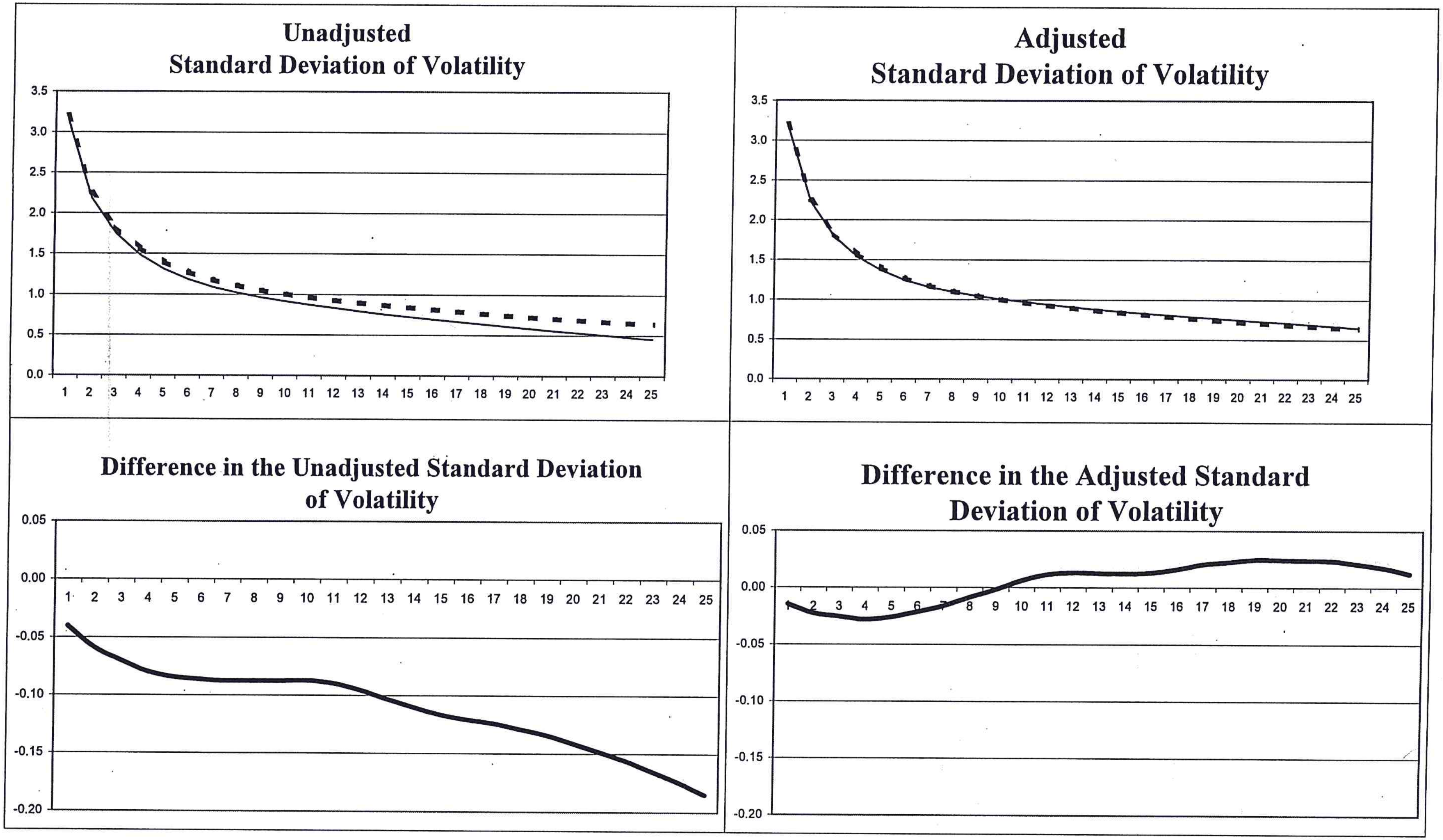


Figure 2, Differences in Unadjusted & Adjusted Standard Deviation of Volatility (I.I.D. Case)

Stochastic Volatility case
True Standard Deviation of Volatility

| | testall | 20 | 40 | 60 | 80 | 100 | 120 | 140 | 160 | 180 | 200 | 220 | 240 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| svall | Stdev | 9.0617 | 8.3317 | 7.8305 | 7.4218 | 7.0656 | 6.7546 | 6.4829 | 6.2433 | 6.0294 | 5.8372 | 5.6625 | 5.5036 |
| sv1 | Stdev | 8.9932 | 8.4017 | 7.8649 | 7.2808 | 6.6696 | 6.0961 | 5.4913 | 4.9042 | 4.3561 | 3.8605 | 3.4424 | 3.0856 |
| sv2 | Stdev | 7.3091 | 6.4180 | 5.6864 | 5.1415 | 4.7134 | 4.3626 | 4.0287 | 3.7395 | 3.5359 | 3.4019 | 3.3067 | 3.2424 |
| sv3 | Stdev | 11.0655 | 10.5739 | 10.2676 | 9.9939 | 9.7139 | 9.4325 | 9.1759 | 8.9566 | 8.7582 | 8.5836 | 8.4365 | 8.2882 |
| sv4 | Stdev | 11.1675 | 10.3929 | 9.7743 | 9.2126 | 8.6980 | 8.2382 | 7.8764 | 7.5718 | 7.3103 | 7.0654 | 6.8013 | 6.5040 |
| sv5 | Stdev | 9.4406 | 8.4825 | 7.8934 | 7.4369 | 7.0655 | 6.7002 | 6.2958 | 5.8960 | 5.5362 | 5.1843 | 4.8192 | 4.4869 |
| sv6 | Stdev | 10.2413 | 9.3608 | 8.7712 | 8.2336 | 7.6896 | 7.1823 | 6.7657 | 6.4321 | 6.1205 | 5.8043 | 5.4353 | 5.0503 |
| sv7 | Stdev | 5.2758 | 4.5186 | 4.0570 | 3.7607 | 3.4882 | 3.2422 | 2.9954 | 2.7195 | 2.4572 | 2.2383 | 2.0336 | 1.8674 |
| sv8 | Stdev | 8.3046 | 7.5677 | 7.0241 | 6.5633 | 6.0291 | 5.4827 | 4.9544 | 4.4995 | 4.1427 | 3.8839 | 3.6960 | 3.5802 |
| sv9 | Stdev | 6.7098 | 5.7433 | 5.1518 | 4.6119 | 4.0772 | 3.5985 | 3.1567 | 2.8009 | 2.5719 | 2.4555 | 2.3962 | 2.3791 |
| sv10 | Stdev | 7.9687 | 7.2147 | 6.7624 | 6.4224 | 6.1274 | 5.8617 | 5.6346 | 5.4156 | 5.1953 | 4.9720 | 4.7319 | 4.4720 |
| sv11 | Stdev | 10.9757 | 10.2919 | 9.7342 | 9.3067 | 8.9759 | 8.6434 | 8.2802 | 7.9409 | 7.6658 | 7.4089 | 7.1360 | 6.8638 |
| sv12 | Stdev | 9.3753 | 8.0893 | 7.2029 | 6.5474 | 5.9585 | 5.4238 | 4.9848 | 4.6678 | 4.4356 | 4.2691 | 4.1539 | 4.0495 |
| sv13 | Stdev | 9.0310 | 8.3803 | 7.9987 | 7.6950 | 7.3381 | 6.9328 | 6.4739 | 6.0249 | 5.5807 | 5.1718 | 4.8171 | 4.4743 |
| sv14 | Stdev | 6.7707 | 5.9187 | 5.3172 | 4.8861 | 4.5456 | 4.2822 | 4.0887 | 3.9585 | 3.8504 | 3.7320 | 3.5990 | 3.4561 |
| sv15 | Stdev | 9.2160 | 8.3723 | 7.8253 | 7.3939 | 6.9804 | 6.6446 | 6.3700 | 6.1249 | 5.9090 | 5.7033 | 5.5080 | 5.3071 |
| sv16 | Stdev | 8.8273 | 7.8777 | 7.4118 | 6.9747 | 6.5403 | 6.1134 | 5.7345 | 5.3752 | 5.0157 | 4.6845 | 4.3575 | 4.0588 |
| sv17 | Stdev | 8.8093 | 8.1550 | 7.6133 | 7.1546 | 6.7969 | 6.4585 | 6.1234 | 5.8113 | 5.5189 | 5.2598 | 5.0326 | 4.8207 |
| sv18 | Stdev | 8.3111 | 7.8561 | 7.5647 | 7.3243 | 7.1269 | 6.9430 | 6.7869 | 6.6513 | 6.5339 | 6.4288 | 6.3096 | 6.1890 |
| sv19 | Stdev | 6.2774 | 5.3623 | 4.8138 | 4.4090 | 4.0908 | 3.8742 | 3.7290 | 3.6257 | 3.5120 | 3.3833 | 3.2569 | 3.1355 |
| sv20 | Stdev | 9.1272 | 8.6483 | 8.3365 | 8.1115 | 7.9173 | 7.7782 | 7.6593 | 7.5410 | 7.4208 | 7.3110 | 7.2174 | 7.1342 |
| Unadjusted | Average | 8.6599 | 7.8813 | 7.3536 | 6.9230 | 6.5271 | 6.1646 | 5.8303 | 5.5329 | 5.2714 | 5.0401 | 4.8244 | 4.6223 |
| | Difference | -0.4018 | -0.4504 | -0.4770 | -0.4988 | -0.5385 | -0.5900 | -0.6526 | -0.7104 | -0.7581 | -0.7971 | -0.8382 | -0.8813 |
| | % Difference | -4.43% | -5.41% | -6.09% | -6.72% | -7.62% | -8.74% | -10.07% | -11.38% | -12.57% | -13.66% | -14.80% | -16.01% |
| | T-Test | -1.09141 | -1.18508 | -1.23469 | -1.28473 | -1.38121 | -1.51388 | -1.67118 | -1.81124 | -1.92768 | -2.03133 | -2.1489 | -2.28483 |
| Adjusted | Average | 9.0050 | 8.3050 | 7.8284 | 7.4412 | 7.0837 | 6.7572 | 6.4571 | 6.1946 | 5.9694 | 5.7764 | 5.5992 | 5.4364 |
| | Difference | -0.0567 | -0.0268 | -0.0022 | 0.0194 | 0.0180 | 0.0026 | -0.0257 | -0.0487 | -0.0600 | -0.0609 | -0.0634 | -0.0672 |
| | % Difference | -0.63% | -0.32% | -0.03% | 0.26% | 0.26% | 0.04% | -0.40% | -0.78% | -1.00% | -1.04% | -1.12% | -1.22% |
| | T-Test | -0.154 | -0.07039 | -0.00566 | 0.049923 | 0.046247 | 0.006584 | -0.06587 | -0.12409 | -0.15257 | -0.15508 | -0.16245 | -0.17412 |
| | adjustment | 1.039856 | 1.053755 | 1.064566 | 1.074849 | 1.085265 | 1.09613 | 1.10752 | 1.119598 | 1.132431 | 1.14608 | 1.160603 | 1.176138 |

| | testall | 260 | 280 | 300 | 320 | 340 | 360 | 380 | 400 | 420 | 440 | 460 | 480 | 500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| svall | Stdev | 5.3559 | 5.2168 | 5.0863 | 4.9625 | 4.8445 | 4.7325 | 4.6276 | 4.5303 | 4.4397 | 4.3542 | 4.2716 | 4.1910 | 4.1115 |
| sv1 | Stdev | 2.7686 | 2.4872 | 2.2383 | 2.0186 | 1.8405 | 1.7082 | 1.6543 | 1.6738 | 1.7393 | 1.8079 | 1.8233 | 1.7916 | 1.7179 |
| sv2 | Stdev | 3.1878 | 3.1167 | 3.0359 | 2.9489 | 2.8462 | 2.7152 | 2.5477 | 2.3625 | 2.1981 | 2.0813 | 1.9863 | 1.9125 | 1.8448 |
| sv3 | Stdev | 8.1273 | 7.9562 | 7.7695 | 7.5733 | 7.3744 | 7.1720 | 6.9703 | 6.7657 | 6.5602 | 6.3281 | 6.0505 | 5.7273 | 5.3648 |
| sv4 | Stdev | 6.1735 | 5.8401 | 5.5165 | 5.1695 | 4.7958 | 4.4154 | 4.0728 | 3.7954 | 3.5713 | 3.4249 | 3.3243 | 3.2657 | 3.2250 |
| sv5 | Stdev | 4.1980 | 3.9499 | 3.7467 | 3.5724 | 3.4021 | 3.2187 | 3.0290 | 2.8265 | 2.6182 | 2.4131 | 2.2559 | 2.1382 | 2.0306 |
| sv6 | Stdev | 4.6898 | 4.3719 | 4.1023 | 3.8661 | 3.6404 | 3.4224 | 3.1972 | 2.9965 | 2.8327 | 2.7085 | 2.6027 | 2.5163 | 2.4549 |
| sv7 | Stdev | 1.7407 | 1.6285 | 1.5299 | 1.4343 | 1.3347 | 1.2351 | 1.1543 | 1.0837 | 1.0081 | 0.9357 | 0.8696 | 0.8236 | 0.8098 |
| sv8 | Stdev | 3.5157 | 3.4609 | 3.4019 | 3.3049 | 3.1889 | 3.0590 | 2.9220 | 2.7826 | 2.6509 | 2.5304 | 2.4175 | 2.3130 | 2.2063 |
| sv9 | Stdev | 2.3299 | 2.2708 | 2.1898 | 2.1103 | 2.0428 | 1.9765 | 1.9043 | 1.8409 | 1.7925 | 1.7489 | 1.6958 | 1.6177 | 1.5431 |
| sv10 | Stdev | 4.2065 | 3.9422 | 3.6779 | 3.4196 | 3.1682 | 2.9489 | 2.7581 | 2.5875 | 2.4404 | 2.3192 | 2.2113 | 2.1226 | 2.0471 |
| sv11 | Stdev | 6.6048 | 6.3476 | 6.0845 | 5.8273 | 5.5831 | 5.3482 | 5.1228 | 4.9082 | 4.7053 | 4.5097 | 4.3152 | 4.1123 | 3.9068 |
| sv12 | Stdev | 3.9377 | 3.8194 | 3.7047 | 3.5976 | 3.4936 | 3.3884 | 3.2844 | 3.1891 | 3.1073 | 3.0232 | 2.9240 | 2.8117 | 2.6852 |
| sv13 | Stdev | 4.1220 | 3.7812 | 3.4719 | 3.1958 | 2.9481 | 2.7330 | 2.5580 | 2.4295 | 2.3456 | 2.2976 | 2.2550 | 2.1914 | 2.1094 |
| sv14 | Stdev | 3.3243 | 3.2226 | 3.1283 | 3.0296 | 2.9100 | 2.7664 | 2.6077 | 2.4503 | 2.3179 | 2.2292 | 2.1534 | 2.0887 | 2.0303 |
| sv15 | Stdev | 5.1146 | 4.9579 | 4.8230 | 4.7044 | 4.5929 | 4.4738 | 4.3435 | 4.2104 | 4.0783 | 3.9470 | 3.8125 | 3.6713 | 3.5136 |
| sv16 | Stdev | 3.7734 | 3.5083 | 3.2671 | 3.0377 | 2.7915 | 2.5349 | 2.3131 | 2.1341 | 2.0289 | 1.9367 | 1.8591 | 1.7938 | 1.7364 |
| sv17 | Stdev | 4.6065 | 4.3952 | 4.2072 | 4.0512 | 3.9253 | 3.8042 | 3.6802 | 3.5475 | 3.3959 | 3.2271 | 3.0484 | 2.8653 | 2.6778 |
| sv18 | Stdev | 6.0725 | 5.9608 | 5.8371 | 5.7243 | 5.6241 | 5.5127 | 5.3993 | 5.2810 | 5.1573 | 5.0400 | 4.9011 | 4.7527 | 4.5969 |
| sv19 | Stdev | 3.0305 | 2.9565 | 2.9053 | 2.8478 | 2.7764 | 2.6948 | 2.6072 | 2.5127 | 2.4249 | 2.3454 | 2.2729 | 2.2032 | 2.1389 |
| sv20 | Stdev | 7.0540 | 6.9680 | 6.8639 | 6.7309 | 6.5734 | 6.4064 | 6.2297 | 6.0328 | 5.8172 | 5.5929 | 5.3708 | 5.1560 | 4.9515 |
| Unadjusted | Average | 4.4289 | 4.2471 | 4.0751 | 3.9082 | 3.7426 | 3.5767 | 3.4178 | 3.2705 | 3.1395 | 3.0223 | 2.9075 | 2.7937 | 2.6796 |
| | Difference | -0.9270 | -0.9697 | -1.0112 | -1.0543 | -1.1018 | -1.1558 | -1.2099 | -1.2598 | -1.3002 | -1.3319 | -1.3641 | -1.3973 | -1.4319 |
| | % Difference | -17.31% | -18.59% | -19.88% | -21.24% | -22.74% | -24.42% | -26.14% | -27.81% | -29.29% | -30.59% | -31.93% | -33.34% | -34.83% |
| | T-Test | -2.43266 | -2.57523 | -2.72121 | -2.87793 | -3.05071 | -3.24711 | -3.45886 | -3.68308 | -3.9121 | -4.1484 | -4.42558 | -4.75263 | -5.14337 |
| Adjusted | Average | 5.2827 | 5.1411 | 5.0104 | 4.8848 | 4.7596 | 4.6328 | 4.5134 | 4.4083 | 4.3242 | 4.2608 | 4.2026 | 4.1501 | 4.1058 |
| | Difference | -0.0733 | -0.0757 | -0.0759 | -0.0777 | -0.0849 | -0.0997 | -0.1142 | -0.1220 | -0.1155 | -0.0935 | -0.0690 | -0.0409 | -0.0057 |
| | % Difference | -1.37% | -1.45% | -1.49% | -1.57% | -1.75% | -2.11% | -2.47% | -2.69% | -2.60% | -2.15% | -1.62% | -0.98% | -0.14% |
| | T-Test | -0.19227 | -0.20111 | -0.20428 | -0.21208 | -0.23496 | -0.2802 | -0.32654 | -0.35671 | -0.34752 | -0.29113 | -0.2239 | -0.13912 | -0.02064 |
| | adjustment | 1.19277 | 1.210496 | 1.229512 | 1.24988 | 1.271731 | 1.295261 | 1.320568 | 1.347887 | 1.377353 | 1.409752 | 1.445441 | 1.485497 | 1.532253 |

Table A2, Comparions of True Standard Deviation of Volatility to Sample Standard Deviations of Volatility Using Overlapping Data (For Stochastic Volatility Case)