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Mark C W Wong
and
Stewart D Hodges

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Financial Options Research Centre
Warwick Business School
University of Warwick
Coventry
CV4 7AL
Phone: (0)24 76 524118

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Mark C. W. Wong
Warwick Business School, University of Warwick,
Coventry CV4 7AL, UK.
Mark.Wong@warwick.ac.uk

Stewart D. Hodges
Financial Options Research Centre
Warwick Business School, University of Warwick,
Coventry CV4 7AL, UK.
forcsh@razor.wbs.warwick.ac.uk

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1 Introduction

The valuation of corporate debt is central to theoretical and empirical work in corporate finance. The literature on pricing of risky debt has evolved into two main directions: the structural approach and the reduced-form approach. As in Merton [1974], the structural approach has taken the dynamics of the assets of the issuing firm as given, and priced corporate bonds as contingent claims on the assets. Merton [1974] adopts the Black and Scholes [1973] option pricing model to the pricing of risky discount bonds. Assuming a constant interest rate economy, Merton's model provides an important insight into the determinants of the risk structure, and shows how the default risk premium is affected by changes in the firm's business risk, debt maturity and the prevailing interest rate.

Black and Cox [1976] and Geske [1977] provide generalizations that take into account the effects of coupon and bond indenture provisions. Geske [1977] applies the technique for valuing compound options to the problem of risky coupon bonds. He derives an analytical formula which consists of multivariate normal integrals with dimensions up to the total number of contractual payments. It is shown that with a special auto-correlation structure, an application of an integral reduction may simplify the numerical computations.

The application of stochastic interest rate in the valuation of corporate debt using Merton's framework [1974] is discussed in Shimko, Tjima, and Van Deventer [1993]. The model of Longstaff and Schwartz [1995] is an adaptation of the Black and Cox model to a more realistic setting. Firstly,

[1995], Jarrow, Lando, and Turnbull [1997], Lando [1995], Madan and Unal [2000], Duffee [1999], and Duffie and Singleton [1999]. This approach provides us with a model that is close to the data, and it is always possible to fit some version of the model. However, the fitted model may not perform well on "out of sample" analysis.

A middle-way approach has been suggested in the literature. Cathcart and El-Jahel [1999] propose a framework situated between structural and reduced-form approaches, within which a default event occurs in an expected or unexpected manner when the value of a signalling process reaches a certain lower barrier or at the first jump time of a hazard-rate process. Although the model can generate strictly positive credit spreads for small maturities, the simple assumption that the firm goes bankrupt immediately when a jump in the asset value occurs for the first time needs empirical justification. Zhou [1997] proposes a numerical model for pricing discount bond in much the same spirit as in Longstaff and Schwartz [1995], when the underlying asset value follows a jump diffusion process which is similar to the stock price process in Merton [1976]. Coupon bonds are not considered in the paper. Instead of employing a jump process as a determinant of default mechanism, Duffie and Lando [1999] study the implications of imperfect accounting information for modelling corporate bonds. They suppose that bond investors cannot directly observe the issuer's assets, and receive only periodic and imperfect accounting reports. As a consequence of the uncertainty in asset values, bounding short spreads away from zero can be obtained in their model.

There is a basic incompatibility in default mechanisms between the Duffie simplicity, we assume that a default event[3] can only happen on payment dates. The firm goes into bankruptcy *expectedly* when the asset level hits a certain lower barrier through a continuous diffusion crossing, or *unexpectedly* when its value drops precipitously below the barrier.[4] Consistent with Geske [1977], Leland and Toft [1996], and Leland [1994], [1998], the default boundary is determined endogenously by requiring the value of equity to be at least the amount of the coupon just paid, in order to avoid bankruptcy. As investors in corporate bonds are subject to state and local taxes, we also consider the effects of tax premium in an economy where jump risk is correlated to a market portfolio.

The objectives of this paper are as follows. We show several significant implications of the jump process for the level and the term structure of credit spreads. For example, it is interesting to note that while the presence of jumps in asset values eliminates the undesirable qualitative feature of credit spreads decreasing to zero at the short end, negative jumps can have significant and persistent effects on spread levels. The jump effects on spread levels are conspicuous for short maturities. For long maturities, credit spreads are indistinguishable from those generated by a pure diffusion model. However, when downward jumps are of higher volatility while the total variance of the firm's asset value remains the same, the effects on credit spreads become more persistent. Other important factors include taxes and dividends. As suggested by Elton, Gruber, Agrawal, and Mann [2000], we

[3]The method is flexible enough to be modified and allow bankruptcy events to happen in any between-payment dates.

[4]Hence one would expect to see a marked increase in volatility of bond returns and a sudden drop in equity prices.

2 Theoretical Models

2.1 A Continuous Time Model with Non-Systematic Jump Risk

Zhou's model [1997] is in much the same spirit as that of Longstaff and Schwartz [1995]. The underlying process of the firm's asset value is modelled as a jump-diffusion process where the jump risk is assumed to be diversifiable. Such an approach is analogous to the modelling of the stock price process in Merton [1976].

Let V_t be the total market value of the assets of the firm at time t. Under an equivalent martingale measure, Zhou [1997] assumes that the dynamics of the firm's asset value process V_t follows the following jump-diffusion process:

$$dV_t/V_t = (r - \lambda_J \overline{m})\ dt + \sigma\ dZ_t + m\ dJ, \tag{1}$$

where

> r is the contant spot rate of interest based on continuous componding;
>
> σ is the instantanous volatility conditional on no jumps;
>
> Z_t is a standard Brownian motion under the risk-neutral measure;
>
> m is the random percentage change in firm value if a Poisson jump occurs: $1+m$ is log-normally distributed, $\log(1+m) \sim N[\gamma_m - \frac{1}{2}\sigma_m^2, \sigma_m^2]$, $E(m) = \overline{m} = e^{\gamma_m} - 1$; λ_J is the intensity of the Poisson jump process $J : P[dJ = 1] = \lambda_J\ dt$.

The process most often resembles geometric Brownian motion, but on average λ_J times per year the price jumps discretely by a random amount.

firm's asset value process are discussed in the paper.

2.2 A Continuous Time Model with General Jump Risk

The jump diffusion processes as described by Merton [1976] and Zhou [1997] are perhaps the simplest type of models to include jumps in asset prices. The crucial assumption is that the jump risk is diversifiable and non-systematic. This assumption is questionable, as asset prices appear to be correlated with market movements.

In an empirical study of an economy where stock prices are assumed to follow a jump diffusion process, Jarrow and Rosenfeld [1984] investigate the satisfaction of assuming jumps to be diversifiable. Evidence has been found to show that the jump component of stock's returns has a strong correlation with the market portfolio, that is, market portfolio appears to contain a jump component. A similar conclusion is drawn by Kim, Oh and Brooks [1994], who study 20 component stocks of the Major Market Index. They find that Poisson-type jumps observed from both the index and its component stocks constitute non-diversifiable risk. This implies that the standard assumption in option pricing as in Merton [1976] that those jumps are not priced may be invalid.

Relationships of common risk factors between the returns on stocks and bonds have been investigated in Fama and French [1993], and Elton, Gruber, Agrawal, and Mann [2000]. Based on the Fama-French three-factor model [1993], Elton, Gruber, Agrawal, and Mann [2000] find that expected default accounts for a surprisingly small fraction of the premium in credit spreads of corporate bonds. They conclude that while state taxes explain a

$A1$: Frictionless Markets.

$A2$: Optimally invested wealth W_t follows a jump diffusion,

$$dW_t/W_t = (\mu_w - \lambda \overline{k}_w - C/W)\ dt + \sigma_w\ dB_t' + k_w\ dq,$$

where μ_w is constant and k_w is the percentage change in wealth when the Poisson jump happens. $1 + k_w$ is log-normally distributed, $\log(1 + k_w) \sim N[\gamma_w - \frac{1}{2}\sigma^2_{k_w}, \sigma^2_{k_w}], E(k_w) = \overline{k}_w = e^{\gamma_w} - 1$, and $Cov[\log(1 + k), \log(1 + k_w)] = \sigma_{vw}$.

$A3$: The representative consumer has time-separable power utility.

$$E_\tau \int_\tau^\infty e^{-\rho t} U(C_\tau)\ dt,\ U(C) = (C^{1-R} - 1)/(1 - R).$$

Assuming that jump risk is systematic, all asset prices and wealth jump simultaneously, possibly by different amounts. Analogous to Bates' model [1991] of systematic jump risk, we assume that under a risk-neutral meausre the jump diffusion model (2) takes the following form

$$dV_t/V_t = (r - \lambda^* \overline{k}^* - \delta)\ dt + \sigma\ dB_t + k^*\ dq^*, \tag{3}$$

where

σ and δ are as before,

$\lambda^* = \lambda E[1 + \triangle J_w/J_w] = \lambda \exp[-R\gamma_w + \frac{1}{2}R(1 + R)\sigma^2_{k_w}]$,

q^* is a Poisson counter with intensity λ^*,

k^* is the random percentage change in firm value if a Poisson jump occurs: $1+k^*$ is log-normally distributed, $\log(1+k^*) \sim N[\gamma^* - \frac{1}{2}\sigma^2_k, \sigma^2_k], E(k^*) = \overline{k}^* = e^{\gamma^*} - 1$, and $\gamma^* = \gamma - R\sigma_{vw}$.

of the firm under the risk neutral measure is of the form:

$$V_t = V_0 \exp\Big([r - \frac{1}{2}\sigma^2 - \lambda^* \overline{k}^* - \delta]t + \sigma B_t + \sum_{j=0}^{NJ} \log(1 + k_j^*)\Big), \quad (4)$$

where NJ is the total number of Poisson jumps over time t, and the Poisson jump sizes $(1+k_j)$'s are independent and identically log-normally distributed random variables with parameters $N[\gamma^* - \frac{1}{2}\sigma_k^2, \sigma_k^2]$. For computational convenience, we turn V_t into a logarithmic scale and define the drift of the logarithm of the asset value in equation (4) by

$$\alpha = r - \delta - \frac{1}{2}\sigma^2 - \lambda^* \overline{k}^*.$$

The process (3) can be rewritten in the following form.

$$X_t = \log(V_t/V_0) = \alpha t + \sigma B_t + \sum_{j=0}^{NJ} \log(1 + k_j^*).$$

To approximate X_t, we adopt a method used by Amin [1993] to discretize the process. The discrete time formulation is based on Cox, Ross and Rubinstien [1976] as the starting point. Multivariate jumps are superimposed on the model to obtain the model with a limiting jump diffusion process.

Let T be the maturity of a coupon bond. For a fixed positive integer n, we divide the interval $[0, T]$ into n subintervals of width $h_n = T/n$. Let $i = 1, \ldots, n$. At each date ih_n, the value of the approximate process X_i is shifted upward by αh_n relative to the grid at time $(i-1)h_n$. Therefore, the asset value at time ih_n and in state j relative to date 0 is given by $V_i = V_0 \exp(\alpha i h_n + j\sigma\sqrt{h_n})$. Any point at time $(i-1)h_n$ can move to any other points at time ih_n. As discussed in Amin [1993], there are two types of movements as to the dynamics of changes in the asset values over time framework.[7]

Let $B_s(i)$ be the value of the risky bond at time ih_n and in state s. Then for any state k, the bond price between two consecutive payment dates is given by the iterative formula

$$B_k(i) = e^{-rh_n}\Big(\lambda^* h_n E_Y\Big[B_{k+y}(i{+}1)\Big] + \frac{1}{2}(1{-}\lambda^* h_n)\Big(B_{k-1}(i{+}1){+}B_{k+1}(i{+}1)\Big)\Big).$$

The first term on the right hand side represents a fraction of the bond price as a consequence of non-local changes in the asset prices, whereas the second term is the expected value of two bond prices resulted from local movements of the asset value. Here we assume that the probability of a jump in the discrete model at any time is equal to $\lambda^* h_n$. We also assume that h_n is so small that multiple jumps cannot occur within the same period. At each coupon date $t = ih_n$, the bond prices immediately before coupon payment is given by:

$$B_k(i^-) = Min\Big(V_t, coupon + B_k(i^+)\Big).$$

We assume Geske's condition [1977] that coupon payments are financed by issues of new equity. The firm goes bankrupt only when its stock value immediately after a coupon date is less than the total coupon payment. Black and Cox [1976] argue that this situation will happen whenever the value of the equity, after payment is made, is less than the value of the payment. The argument is intuitive, in that the firm will find no takers

[7] Amin [1993] shows that the discrete time process converges weakly to the continuous time process. This guarantees that the prices of European options computed from the discrete time model will converge to their corresponding continuous time values under fairly mild regularity conditions. For example the option payoff must be uniformly integrable in h_n.

[Please Insert Figure 1 Here.]

[Please Insert Figure 2 Here.]

Figure 1 shows the corresponding term structures of credit spreads for three jump diffusion cases and a mis-specified pure diffusion case. In the jump diffusion cases, the total variances under the risk-neutral measure are kept the same, and the contributions of jump components to total variance are (i) 55% and (ii) 9% respectively.[9] We assume that the mis-specified pure diffusion process has a constant variance equal to the total variance of the jump diffusion case. We observe that while the term structures are similar in shape for both jump-diffusion and pure diffusion cases, the gap between credit spreads in the two cases narrows with maturities. In the presence of jumps, the possibility of sudden default only raises the levels of credit spread for short maturities.[10] This effect becomes more evident for bonds with shorter maturities when jumps occur more frequently and have higher variability. As maturity increases, the differences in credit spreads dwindle, implying that the effect of a jump becomes less prominent for bonds with long maturities. Similar results are shown in Figure 2, where the term structure of credit spreads for a different face value $K = 70$ is plotted. The rationale behind these results is given as follows.

Recall that the total variance per unit time remains constant in all cases. When the maturity is small, the diffusion volatility in the pure diffusion

[9]The total variance is $\sigma^2 + \lambda^* \left(\sigma_k^2 + (\gamma^* - \frac{1}{2}\sigma_k^2)^2 \right)$, where the first and second terms are due to the diffusion and jump components respectively.

[10]Analytically, we can prove that there is a positive instantaneous default hazard at time 0, which is equal to λ^* times probability of default in case of a jump.

resemblance in shape to the corresponding risk-neutral ones shows that the empirical property of fat-tailness in asset return can also be observed in the real world.

Note that our comparison analysis is based on the assumption of constant total variance, measured in the risk-neutral world. There is a point of paramount importance in our credit-spread analysis. In case of non-systematic jumps, the total variances of the jump-diffusion process in the real and risk-neutral world are the same. This implies that the theoretical levels of credit spread due to default risk can be estimated by the observable parameters in the real world. However, if the jump risk is systematic, estimation of spread levels becomes subtle. This is particularly the case if the jumps in the asset value process and market portfolio are negative, that is γ, $\gamma_w < 0$. Under this assumption, it is trivial to see that the total variance measured in the risk-neutral world is higher than the one observed in the real world, because of $\gamma^* < \gamma < 0$ and $\lambda^* > \lambda$. Jumps in the risk-neutral world tend to be more influential as they become more negative. This is also true in times of economic recession where investors become more averse to risk. The implication is that in presence of systematic jumps the theoretical levels of spread cannot be computed by using the observable parameters without making specific assumptions about risk aversion and market portfolio. In fact, without knowledge of the risk aversion and market portfolio the jumps would not be priced correctly, and so there is a tendency to underestimate the spread levels.

By comparing with empirical properties of credit spread, it is evident that the spread levels generated in Figures 1 and 2 do not quite resemble

Since state tax is deductible from income for the purpose of federal tax, state tax is reduced by the federal tax rate. Hence, the effective tax rate is of the form:

$$\tau = \tau_s(1 - \tau_g),$$

where τ_s is a state tax, and τ_g is the federal tax rate. We assume $\tau = 4.875\%$ by following the arguments in Elton, Gruber, Agrawal, and Mann [2000] that we choose $\tau_s = 7.5\%$ as the mid-point of maximum marginal state taxes, and $\tau_g = 35\%$ as the maximum federal tax rate. It is easy to modify the model in the last section to fit with this tax factor. As in the last section, we assume that default can only happen on coupon payment dates. There are two cases where bond price will be affected. On each coupon payment date, if default does not happen, then the actual bond value is the original bond value less the total amount of tax on the interest payment. If the bond defaults, then the bond price becomes the residual asset value plus the tax refund due to a capital loss. We take the after-tax coupon rate to be 5%.

[Please Insert Figure 4 Here.]

[Please Insert Figure 5 Here.]

The tax effects on spread levels are shown in Figure 4 and 5. Figure 5 shows that the term structures of credit spread of two risky bonds[12] with different face values and their spread component generated by pure tax effects. Note that the pure tax level is the same in both cases.[13] As expected, tax effects contribute to considerable portion of total spread levels. The

[12]We assume that they are issued by two identical firms.

[13]Spread levels are computed when the firm's asset value is large relative to the amount of debt.

stock price immediately after a coupon date is at least worth the coupon payment. Bankruptcy boundary is determined at the asset level where stock price is equal to the coupon payment. In case such a default is imminent, total debt value increases slightly because of the refund of capital loss in the event of bankruptcy. As a consequence, the rise in effective tax rate raises the bankruptcy barrier. Furthermore, as the firm's value drops to a low level, tax shelter for coupon payments will not be fully realized.[15] The result suggests that a decrease in the federal tax rate may precipitate earlier default of low-grade bonds.

Finally, the effect of dividends is shown in Figure 7, where we plot the term structures of credit spread under a jump diffusion process for different values of dividend payout rate $\delta = 7\%$ and 5%. It is evident that a small change in dividend rates can have significant and persistent effects on spread levels.

[Please Insert Figure 6 Here.]

[Please Insert Figure 7 Here.]

6 Conclusion

This paper has compared the structural framework of bond pricing models under a jump diffusion process with those under a pure diffusion process. We have developed a tractable, discrete time model for the valuation of

[15]Under U.S. tax codes, to benefit fully with tax shelter the firm must have earnings before interest and taxes that is not less than total coupon payments. When default is imminent, it is quite possible that profits will be less than the coupon payout and tax savings will not be fully realized. See Leland [1994].

spreads become more persistent.

Other important factors include taxes and dividends. State taxes have been ignored in almost all modelling of defaultable bonds. As a further contribution, this paper has introduced the important factor of tax into the model. As motivated by Elton, Gruber, Agrawal, and Mann [2000], we have shown that taxes do have significant and persistent effects for bonds with long maturities. In fact, credit spread increases with effective tax rate on coupon payments. Tax effects appear to be the second most important factor for spread levels, as documented in Elton, Gruber, Agrawal, and Mann [2000]. We have also found that while downward jumps in firm value increase the probability of default, the bankruptcy boundary does not seem to be affected.

We have also investigated the effects of state and federal taxes on default mechanism. Assuming the distribution of state taxes remains unchanged, we have shown that a decrease in the federal tax rate may precipitate earlier default of low-grade bonds. Finally, we have found that dividend payout rates can have significant and persistent effects on spread levels. With deployment of the additional factors of taxes and dividends, the jump-diffusion model has been shown to be more flexible than the pure diffusion ones in fitting empirical spreads. It remains to be seen whether it is sufficiently flexible and sufficiently easy to fit for it to be useful in empirical work.

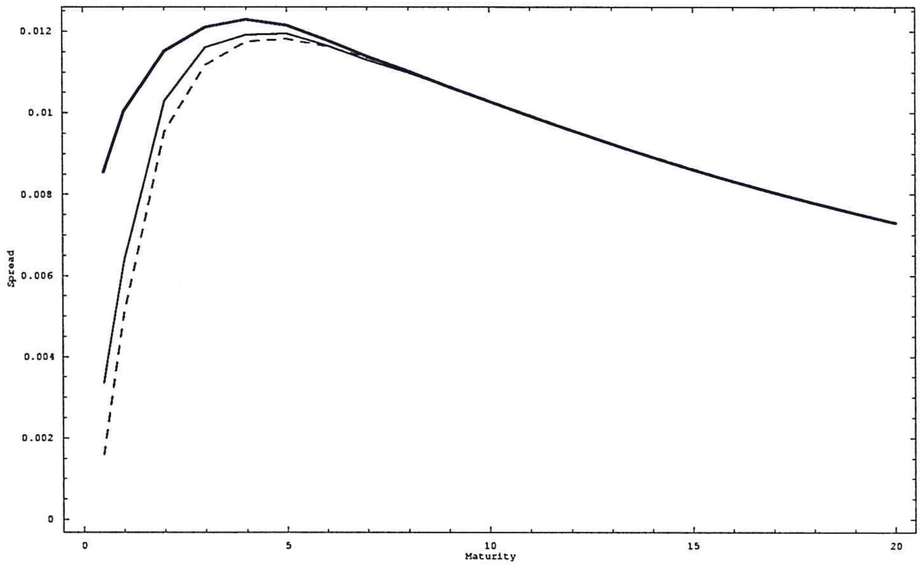

Figure 2: **Term structures of credit spreads under a jump diffusion process.** This figure shows the term structures of credit spreads of a 5% coupon bond when the underlying process follows a jump diffusion process with the same total variance but different jump components. (i) $\lambda^* = 0.3$: (thick solid line); (ii) $\lambda^* = 0.05$: (thin solid line). The corresponding term structure (dashed line) under a pure diffusion process with a constant variance equal to the total variance of the jump diffusion case is shown for comparison. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $K = 70$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$, except stated otherwise. Total variance per unit time remains constant in all cases.

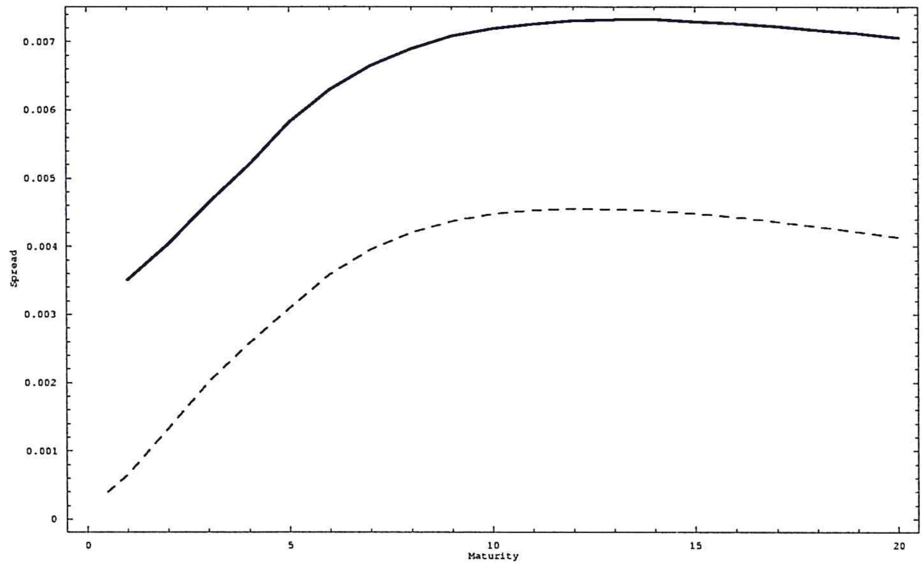

Figure 4: **Term structure of credit spread under a jump diffusion process for face values K =50 with tax effects.** This figure shows the term structure of credit spreads (solid line) of a 5% (*after-tax rate*) coupon bond when the underlying process follows a jump diffusion process. The level of credit spread without taking tax effects into account is shown in dashed line. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax $c = 0.05$, $V_0 = 100$, $\lambda^* = 0.05$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$. Total variance per unit time remains constant in all cases.

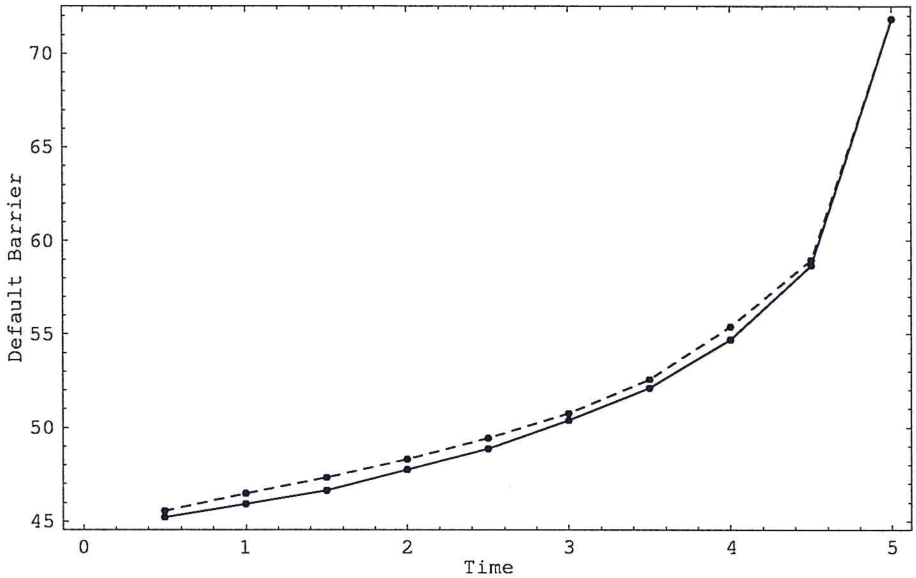

Figure 6: **Default boundaries under a jump diffusion process for different values of τ = 4.875% and 5.25%.** This figure shows the default boundaries of a 5-year 5% (*after-tax rate*) coupon bonds with different values of τ:(i) τ = 4.875% (solid line), (ii) τ = 5.25% (dashed line), when the underlying process follows a jump diffusion process. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax rate $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, $\sigma_k = 0.25$.

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