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Abstract

Motivated by the empirical observation that there exists some degree of predictability in asset returns, this paper investigates the theoretical constraints on the time variation in the risk premia of the market portfolio in a continuous-time, finite horizon pure exchange economy. By characterizing the equilibrium conditions as nonlinear partial differential equations, closed-form solutions can be obtained. It is shown that in a stationary economy, the presence of intermediate consumption can have a drastic effect on the possible kinds of time-varying behaviour of the risk premia.

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1 Introduction

The classical *Gordon growth model* (Gordon [11]) is one of the simplest theoretical models in finance. Nevertheless, it provides some basic economic intuitions for the equilibrium relations between prices, returns and dividends. First, the stock prices are high when dividends are expected to grow rapidly or when dividends are discounted at a low rate. Furthermore, the dividend-price ratio has a strong relationship with the prospective stock return: for a given growth rate, the higher the ratio, the higher the expected total return.

The sizable empirical literature on market returns suggest that they exhibit certain degree of predictability for long horizons (see, for example, Poterba and Summers [18], Cecchetti et al. [4], Kandel and Stambaugh [17], and Fama and French [9, 10]). However, the relaxation of the constancy of the discount rate and/or the dividend-price ratio very often introduces nonlinearity in the model and therefore becomes quite difficult to deal with.

One of the most contentious phenomena in the equity markets is the mean reversion in equity prices or returns. Since expected returns are not observable, we cannot be sure that whether the mean reversion is due to market inefficiency, or it should be attributed to the equilibrium asset pricing model. Nonetheless, the work of Cecchetti et al. [4] indicates that mean reversion could be consistent with equilibrium. Other researchers have also tried to explain mean reversion by applying habit formation models, or more general utility functions such as the Epstein-Zin utility.

In the continuous-time setting, many researchers have attempted to deal with this equilibrium problem by employing different approaches and assumptions. For example, the price of risk can be modelled explicitly as following a mean-reverting stochastic process (see Black [2]). However, the way that the equilibrium behaviour is modelled *exogenously* is not quite satisfactory. A more systematic approach to analyze the problem might be to first characterize the equilibrium conditions (mostly as partial differential equations) that the dynamics of the market portfolio must satisfy in equilibrium, and then look for analytic solutions to these PDEs which are consistent with a rational economy. Along this line of research, some progress has been made through the work of Bick [1], He and Leland [13], Hodges and Carverhill [14], and Hodges and Selby [15]. One of the contributions is that by taking advantage of the modern asset pricing theory, the economy could now be better understood in a systematic way. Yet, since these papers assume the representative agent maximizes her expected utility only over the terminal wealth, there is much work to be done.

The main purpose of this paper is to investigate the theoretical constraints on the time variation in the risk premia of the market portfolio. Specifically speaking, the paper follows the above stream of research and extends the analysis one step further to the more complicated case where the representative agent demands intertemporal consumption as well as her terminal wealth at horizon date. The role of dividends comes to play quite naturally within the single representative agent framework. The paper demonstrates how we can construct a model of an exchange economy in which security prices display mean reversion (the representative agent has diminishing relative risk aversion). [1]

The remainder of this paper is organized as follows. Section 2 describes the conventional setting of a single representative agent economy and outlines respectively the formulation for both the case without intermediate consumption and with it. Section 3 provides the characterization of equilibrium asset price processes for both cases. The main part of this paper, section 4, then derives a set of closed form solutions for a subclass time-homogeneous diffusion processes in a Black-Scholes economy. Numerical examples are given in Section 5 to demonstrate the behaviour of both price of risk and dividend yield. Section 6 concludes the article.

2 The formulation

We consider a continuous time, finite horizon, pure exchange economy of Lucas [16]. The financial markets are complete and we assume that the econ-

[1] As the remark made by Campbell et al. [3], there is increasing interest in the idea that risk aversion may vary over time with the state of the economy. They also address the prospect that the time-varying risk aversion might be able to explain the large body of evidence that excess returns on stocks and other risky assets are predictable.

omy can be described by a single representative agent. The agent trades and acts as an expected utility maximizer and in equilibrium will optimally hold the market portfolio (representing the aggregate wealth of the whole economy) through time. There are two long-lived financial securities available for trading: a risky asset (the *stock*), and a locally riskless asset (the *bond*). At time t, the *trading price* of the stock is denoted by S_t and the holder of the stock is entitled to its dividends, if the stock is dividend-paying. The bond price is denoted by R_t, and increases at the instantaneous riskless rate of interest r_t. In equilibrium, there is one share of the stock outstanding and the bond is in zero net supply.

In the rest of this section, we shall describe respectively two distinct economic settings. First, an economy where the representative agent is concerned with her terminal wealth only. Second, an economy where the representative agent considers not only the terminal wealth level but also intermediate consumption. We will refer from time to time the former case as the *wealth-only* economy, and the later one as the *consumption* economy. Finally, we discuss two important properties arising from the first-order conditions and emphasize the significance of the path-independence result which leads us to the fundamental PDEs as the necessary equilibrium conditions.

2.1 Without intermediate consumption

Assume that, in a pure exchange economy with no production and with no intertemporal consumption, there exists one risky asset (the stock) and one risklesss asset (the bond). The stock (the market portfolio) pays no dividends and its price S follows the SDE:

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dz_t, \tag{1}$$

where z is the standard Brownian motion under the objective probability measure $\mathbb{P}$ and the drift μ and the diffusion σ are deterministic functions of S and t. Define the price of risk as the instantaneous reward per unit of risk and denote by

$$\alpha(S_t, t) = \frac{\mu(S_t, t) - r(S_t, t)}{\sigma(S_t, t)}, \tag{2}$$

where r is the instantaneous riskless interest rate and is assumed to be a deterministic function of S_t and t as well. Thus, we can then rewrite (1) as

$$\frac{dS_t}{S_t} = [r(S_t, t) + \sigma(S_t, t) \cdot \alpha(S_t, t)]\, dt + \sigma(S_t, t) dz_t. \tag{3}$$

Moreover, the bond price R accumulates at the riskless rate r:

$$\frac{dR_t}{R_t} = r_t dt. \tag{4}$$

Assume that the agent is endowed with a positive initial amount but receives no intermediate income. In addition, consumption only occurs at the agent's investment horizon date T. She then aims to optimally allocate her wealth in the stock and the bond in order to maximize her expected utility over the time-T wealth, W_T:

$$\max \quad \mathbf{E}[U(W_T)], \tag{5}$$

where U is a strictly increasing, state-independent, and continuously differentiable von Neumann-Morgenstern utility function.

Denote by Φ the dollar amount invested in the stock. Then the wealth function W follows the process:

$$dW_t = [W_t r_t + \Phi_t(\mu_t - r_t)]\, dt + \Phi_t \sigma_t dz, \tag{6}$$

with $W_0 = w > 0$ (positive initial wealth) and $W_t \geq 0$ (nonnegative wealth constraint), for $0 < t \leq T$.

Following Harrison and Kreps [12], it is well known that in any arbitrage-free pricing system there exists a risk-neutral probability measure under which the drift of the stock returns is the riskless rate r. Hence we can let $\mathbb{Q}$ denote the risk-neutral probability measure and $\mathcal{M} = d\mathbb{Q}/d\mathbb{P}$ the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ (i.e. the *Radon-Nykodym derivative*). The state-price density (SPD) can then be defined as

$$\xi_t = e^{-\int_0^t r_s ds} \cdot \mathcal{M}_t.$$

Since in equilibrium the representative agent will hold the market portfolio, the agent's marginal utility can be related to the SPD throught the first order condition

$$\frac{\partial U(S_T)}{\partial S_T} = \lambda \cdot \xi_T, \tag{7}$$

where λ is the Lagrange multiplier.

2.2 With intermediate consumption

If, alternatively, we allow the presence of intermediate consumption and assume the stock pays dividends, then the *trading price* of the stock, S, can be formulated as:

$$dS_t = [\mu(S_t, t)S_t - D(S_t, t)]\, dt + \sigma(S_t, t)S_t\, dz_t, \tag{8}$$

where z is the standard Brownian motion under the $\mathbb{P}$-measure and D is the cash dividend paid by the stock and is assumed to be a deterministic function of S and t. Similarly, (8) can be rewritten as

$$dS_t = [(r(S_t, t) + \sigma(S_t, t) \cdot \alpha(S_t, t))S_t - D(S_t, t)]\, dt + \sigma(S_t, t)S_t dz_t, \tag{9}$$

where r, σ, and α are as defined before.

The agent's maximization problem is now formulated as the following:

$$\max \quad \mathbf{E}_0 \left[\int_0^T e^{-\rho t} U_1(C_t)dt + U_2(W_T) \right], \tag{10}$$

where ρ is the rate of time preference and U_1 and U_2 are strictly increasing, time additive, and state-independent von Neumann-Morgenstern utility functions and are continuously differentiable where applicable.

Again let Φ_t denote the dollar amount invested in the stock at the beginning of time t-period and C_t the amount being consumed during time t-period. Thus the investor's indirect utility (or value function) can be defined as:

$$J(W_t, S_t, t) = \max_{\Phi, C} \quad \mathbf{E}_t \left[\int_t^T e^{-\rho(s-t)} U_1(C_s)ds + U_2(W_T) \right], \tag{11}$$

where $J(W_T, S_T, T) = U_2(W_T)$ is the boundary condition. The wealth process follows:

$$dW_t = \left[W_t r_t + \Phi_t(\mu_t - r_t) - \left(\frac{\Phi_t}{S_t} \right) C_t \right] dt + \Phi_t \sigma_t dz, \tag{12}$$

where $W_0 = w > 0$ (positive initial wealth) and $W_t \geq 0$ (nonnegative wealth constraint), for $0 < t \leq T$.

Recall that in equilibrium the representative agent's optimal strategy is to hold the market portfolio and consume all the dividends received from the stock investment. Hence, the first order condition of optimality for the Hamilton-Jacobi-Bellman (HJB) equation yields:

$$e^{-\rho t}\frac{\partial U_1(C_t)}{\partial C_t} = \frac{\partial J(S_t, S_t, t)}{\partial S_t} = \lambda \cdot \xi_t, \tag{13}$$

where λ is the Lagrange multiplier; $C_t = D_t$ for $t < T$ and $C_T = S_T$ for $t = T$.

2.3 Monotonicity property and the path independence result

By inspection of the first-order conditions (7) and (13), two important properties emerge. First, by the assumption of increasing utility functions, wealth should be monotonically and inversely related to the marginal utility (or the SPD). This property applies to both cases as it can be seen that from (7), $W_T = V(\lambda \cdot \xi_T)$, where V is the inverse function of the marginal utility $U^{'}$, and from (13), $W_t = I(\lambda \cdot \xi_t)$, where I is the inverse function of the marginal indirect utility $J^{'}$. Thus a portfolio strategy which creates state-dependent wealth will be called an *efficient* strategy only if the monotonicity property is satisfied (see Dybvig [7, 8]).

In the context of equilibrium, since the stock price S_t represents the agent's wealth and the monotonicity property should apply, thus at each point of time $t \in [0, T]$, the process of the SPD ξ_t must be path-independent, regardless the stock price history (see Cox and Leland [5, 6]). It is this result that provides us a convenient way to analyze the equilibrium asset price dynamics in the economy. [2]

3 Equilibrium conditions of the asset prices dynamics

In this section, we shall provide the characterization of equilibrium price processes for both the wealth-only economy and the consumption economy. In each case, we shall first derive a general partial differential equation for the *intertemporal relative risk aversion* f with respect to S. In order for the problem to be easier to solve, we then assume the constancy of r and σ and

[2] See for example, Hodges and Carverhill [14] and He and Leland [13]. These authors also characterize the equilibrium price processes by exploiting this property.

translate the PDE to an equivalent one in terms of the *price of risk* α with respect to a new transformed variable x, where

$$x_t = \ln S_t - \left(r - \frac{\sigma^2}{2}\right) t.$$

3.1 Without intermediate consumption

Theorem 1 (Equilibrium conditions: without consumption) *Assume in the economy, there exists one non-dividend-paying risky asset (the stock) and one riskless asset (the bond). The single representative agent allocates her wealth optimally among these two assets continuously according to her objective function (5) subject to the wealth process (6) and consumes her terminal wealth at time T. The necessary condition for the asset price dynamics (1) to be an equilibrium process when r, μ and σ are deterministic functions of S and t is that the coefficients must satisfy the following PDE:*

$$\mathcal{L}f + f_t + r_S S(f-1) + \sigma\sigma_S S(Sf_S + f^2 - f) = 0, \tag{14}$$

where

$$\begin{aligned} f(S_t,t) &= \frac{\mu(S_t,t) - r(S_t,t)}{(\sigma(S_t,t))^2}, \\ \mathcal{L}f &= \frac{1}{2}\sigma^2 S^2 f_{SS} + \mu S f_S, \end{aligned}$$

and the boundary condition is

$$f(S_T,T) = -S_T \frac{U''(S_T)}{U'(S_T)}.$$

Proof: The main idea is to exploit the path-independence property on the state-price density. Recall that the process of ξ is defined as

$$\xi_t = \exp\left(-\int_0^t r_s ds - \int_0^t \left(\frac{\mu_s - r_s}{\sigma_s}\right) dz_s - \frac{1}{2}\int_0^t \left(\frac{\mu_s - r_s}{\sigma_s}\right)^2 ds\right).$$

Now, define a new variable $Z(S_t,t) = \ln \xi(S_t,t)$. We then apply *Itô's* Lemma to derive dZ and equate it with $d(\ln \xi)$. Collecting dt and dz terms respectively yields

$$\begin{aligned} Z_t + \mu S Z_S + \frac{1}{2}\sigma^2 S^2 Z_{SS} &= -r - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 = -r - \frac{1}{2}\sigma^2 f^2, \\ \sigma S Z_S &= -\left(\frac{\mu - r}{\sigma}\right) = -\sigma f. \end{aligned}$$

Note that for notational simplicity, we have suppressed the time index so that it will not be confused with the partial derivatives.

From the second equation, we can derive $Z_{St} = -\frac{f_t}{S}$, $Z_S = -\frac{f}{S}$ and $Z_{SS} = -\frac{f_S}{S} + \frac{f}{S^2}$. Substitute Z_S and Z_{SS} into the first equation and use the fact that $\mu = r + \sigma^2 f$ to obtain

$$Z_t = -r + rf + \frac{1}{2}\sigma^2 f^2 + \frac{1}{2}\sigma^2 S f_S - \frac{1}{2}\sigma^2 f.$$

Differentiate the above equation with respect to S and equate it with Z_{St} to obtain

$$Z_{tS} = r_S(f-1) + (r+\sigma^2 f)f_S + \frac{1}{2}\sigma^2 S f_{SS} + \sigma\sigma_S(Sf_S + f^2 - f) = Z_{St} = -\frac{f_t}{S}.$$

(14) then immediately follows. □

Theorem 1 states the general equilibrium conditions which the *intertemporal relative risk aversion* f must satisfy. We now narrow our attention to some special cases. The first case is a Black-Scholes economy where both the interest rate r and the volatility of the stock return σ are constant. Therefore, by letting $\sigma_S = 0$ and $r_S = 0$, (14) can be simplified as

$$\mathcal{L}f + f_t = 0, \tag{15}$$

The stronge assumption of constancy of r and σ enables us to obtain a nice result known as the *Burgers' equation.* The finance application of this equation seems to first appear in Hodges and Carverhill [14] and in an independent work of He and Leland [14]. For completeness, we recite the result in the following theorem.

Theorem 2 (Burgers' equation) *Assume constant r and σ and define the transformed state variable x as $x_t = \ln S_t - (r - \sigma^2/2)t$. Then the price of risk α in the wealth-only economy must evolve over time according to the PDE:*

$$\alpha_\tau = \frac{1}{2}\sigma^2\alpha_{xx} + \sigma\alpha\alpha_x. \tag{16}$$

Proof: By definition, we have $\mu - r = \sigma^2 f = \sigma\alpha$ and $x = \ln S - \left(r - \frac{\sigma^2}{2}\right)t$. Thus, we can write

$$\alpha(S,t) \equiv \alpha(x,t) = \sigma \cdot f\left(e^{x+\left(r-\frac{\sigma^2}{2}\right)t}, t\right)$$

and its partial derivatives

$$\begin{aligned}
\alpha_x &= \sigma S f_S, \\
\alpha_{xx} &= \sigma(f_S + S f_{SS})S \\
\alpha_t &= \sigma f_S\left(r - \frac{\sigma^2}{2}\right)S + \sigma f_t.
\end{aligned}$$

Rearranging the above equations to obtain f_S, f_{SS} and f_t and substituting them into (15) yields

$$\alpha_t + \sigma\alpha\alpha_x + \frac{1}{2}\sigma^2\alpha_{xx} = 0.$$

This immediately gives (16). □

Another interesting case is to assume that f is time-invariant in the sense that $f_t = 0$. Then we have the following proposition:

Proposition 1 *For the case of $f_t = 0$, the equilibrium condition for the economy stated in Theorem 1 is that the following equation must be satisfied:*

$$\frac{\partial}{\partial S}[\sigma^2(Sf_S + f^2 - f) + 2r(f-1)] = 0. \tag{17}$$

Or equivalently, there exists a constant K such that

$$\sigma^2(Sf_S + f^2 - f) + 2r(f-1) = K.$$

Proof: By assumption, $\mu = r + \sigma^2 f$ and $f_t = 0$. After some simple manipulation, (17) can then be easily derived from (14):

$$\begin{aligned}
0 &= \mathcal{L}f + f_t + r_S S(f-1) + \sigma\sigma_S S(Sf_S + f^2 - f) \\
&= +\frac{1}{2}\sigma^2 S^2 f_{SS} + (r + \sigma^2 f)Sf_S + S\frac{\partial}{\partial S}[r(f-1)] - rSf_S \\
&\quad +\frac{1}{2}S(Sf_S + f^2 - f)\frac{\partial\sigma^2}{\partial S} \\
&= \frac{1}{2}S\sigma^2\frac{\partial}{\partial S}(Sf_S + f^2 - f) + S\frac{\partial}{\partial S}[r(f-1)] + \frac{1}{2}S(Sf_S + f^2 - f)\frac{\partial\sigma^2}{\partial S} \\
&= \frac{1}{2}S\frac{\partial}{\partial S}\left[\sigma^2(Sf_S + f^2 - f) + 2r(f-1)\right]. \square
\end{aligned}$$

Remark 1 *In the same setting as ours, except assuming constant interest rate, He and Leland ([13], pp. 603-604) provide a similar necessary condition for the time-invariant case, namely* $\sigma^2(f^2 + Sf_S - f) = K$. *Therefore, it must be pointed out that their condition strictly only holds when* $r = 0$.

Their result can be justified if we define S as the ***relative*** *asset price (that is, the risky asset price normalised by the bond price). The drift term μ in PDE (14) should then be interpreted as the* ***risk premium****, provided that the risk premium is a deterministic function of the relative price and time. This is in fact the setup in Bick [1].*

Without further assumptions, the PDE (14) is difficult to solve in general. To date, several functional form solutions to the time-invariant case $f_t = 0$ (with constant σ and $r = 0$) can be found as examples in Bick [1] and in He and Leland [13]. For a more general definition on time independence of the diffusion processes, Hodges and Selby [15] carried out a time-homogeneous analysis for the case with constant volatility and constant interest rate. They seek to find steady-state solutions to the Burgers' equation (16) by constraining the risk premium to vary depending on the level of the market in such a way that the functional form does not depend on time. Interestingly, they conclude there are only two possible viable solutions and one non-viable one for the steady state: the price of risk can be constant or increasing in aggregate wealth, but the only steady state solution with decreasing price of risk admits arbitrage (and is not viable).

The finding of an increasing price of risk is of course disappointing. Nevertheless, it is conceivable that the presence of intermediate consumption might be sufficient to modify this behaviour. As we shall illustrate in the next section, it is indeed the case: with large enough intermediate consumption, there exists a decreasing price of risk in the steady state which stems from decreasing relative risk aversion of the representative agent.

3.2 With intermediate consumption

Now we are to characterize the equilibrium conditions for the economy with intermediate consumption. For notational convenience, we shall define δ as the dividend yield by $\delta(S_t, t) = D(S_t, t)/S_t$. The next two theorems generalize on equations (14) and (16) to include dividends. The approach is similar to that used before.

Theorem 3 (Equilibrium conditions: with consumption) *Assume in the economy, there exists one dividend-paying stock and one riskless bond. The representative agent allocates her wealth optimally among the two assets continuously according to her objective function (10) subject to the wealth process (12) and consumes the dividends paid by the stock investment. The necessary condition for the asset price dynamics (8) to be an equilibrium process when r, μ, δ and σ are deterministic functions of S and t is that the coefficients must satisfy the following PDE:*

$$\mathcal{L}f + f_t + \delta_S S f + r_S S(f-1) + \sigma\sigma_S S(S f_S + f^2 - f) = 0, \qquad (18)$$

where

$$\begin{aligned} f(S_t,t) &= \frac{\mu(S_t,t) - r(S_t,t)}{(\sigma(S_t,t))^2}, \\ \mathcal{L}f &= \frac{1}{2}\sigma^2 S^2 f_{SS} + (\mu-\delta) S f_S, \end{aligned}$$

and the boundary condition

$$f(S_T,T) = -S\frac{U_2''(S_T)}{U_2'(S_T)}.$$

Proof: This is simply a rederivation of Theorem 1 with the presence of intermediate consumption (dividends). Again define a new variable $Z(S_t,t) = \ln\xi(S_t,t)$. We apply *Itô's* Lemma to derive $dZ(S_t,t)$ and equate it with $d(\ln\xi)$. Collecting dt and dz terms respectively yields

$$\begin{aligned} Z_t + (\mu S - D)Z_S + \frac{1}{2}\sigma^2 S^2 Z_{SS} &= -r - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 = -r - \frac{1}{2}\sigma^2 f^2, \\ \sigma S Z_S &= -\left(\frac{\mu-r}{\sigma}\right) = -\sigma f. \end{aligned}$$

Following the same technique in the proof of Theorem 1 and apply $\delta = \frac{D}{S}$, (18) can then be easily obtained. □

Theorem 4 *Assume r and σ are constant and define the transformed state variable x as $x_t = \ln S_t - (r - \sigma^2/2)t$. Then the price of risk α in the consumption economy must evolve over time according to the PDE:*

$$\alpha_\tau = \frac{1}{2}\sigma^2 \alpha_{xx} + \sigma\alpha\alpha_x - \delta\alpha_x + \delta_x\alpha. \tag{19}$$

Proof: Apply $r_S = 0$ and $\sigma_S = 0$ to (18) and rearrange to obtain

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + \mu S f_S - \delta S f_S + \delta_S S f = 0. \tag{20}$$

By definition, $\sigma f = \alpha$ and $x = \ln S - \left(r - \frac{1}{2}\sigma^2\right)t$. Thus, (19) immediately follows by substituting $\sigma S f_S = \alpha_x$, $\delta_S S = \delta_x$ and

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + \mu S f_S = \frac{1}{\sigma}\left(\alpha_t + \sigma\alpha\alpha_x + \frac{1}{2}\sigma^2\alpha_{xx}\right)$$

into (20). □

4 The time-homogeneous case in a Black-Scholes economy

In this section, we study a particularly important case of the time homogeneous economy. We seek to find the steady-state solutions to the PDE (19) in order to see how the price of risk and the dividend (consumption) vary depending on the market level in such a way that the functional forms are independent of time.

We start with two functions, the price of risk α and the dividend yield δ, which together satisfy the PDE (19). By homogeneity in time, we mean that α and δ can be specified as the following forms

$$\alpha(x,\tau) \equiv y(u), \tag{21}$$

$$\delta(x,\tau) \equiv g(u), \quad \text{where } u = x + \theta\tau, \tag{22}$$

for some functions y and g and constant θ.

Performing the partial differentiations to obtain $\alpha_t = -\theta y'$, $\alpha_x = y'$, $\alpha_{xx} = y''$, and $\delta_x = y'$ and then substituting them together with (21) and (22) into (19), the PDE will then be reduced to an ODE:

$$-\theta y' + \sigma y y' + \frac{1}{2}\sigma^2 y'' - g y' + g' y = 0. \tag{23}$$

(23) is not easy to solve in general as it involves two functions y and g but only one state variable x (i.e. the economy is underspecified). Of course we can view (23) as an ODE in y given a g function, but then the question will be how we should choose g. Since g could be a function of a rather arbitrary form, a more plausible question might be to ask whether there are cases which permit closed form solutions.

One simple way to deal with the problem is to assume that the joint behaviour of y and g (through $-gy' + g'y$ terms) will only change the coefficients of the y' and yy' terms. In other words, we assume

$$-gy' + g'y = -p_0 y' + p_1 \sigma y y' \tag{24}$$

for some constants p_0 and p_1. This assumption conveniently leads (23) to a simplified form:

$$-(\theta + p_0)y' + \sigma(1 + p_1)yy' + \frac{1}{2}\sigma^2 y'' = 0. \tag{25}$$

Note that the presence of the diffusion coefficient σ in (24) is necessary in order for the equation (23) to be independent of the time metric. [3]

Consequently, we can rewrite (24) as

$$\left(\frac{g}{y}\right)' = -p_0 \frac{y'}{y^2} + p_1 \sigma \frac{y'}{y}. \tag{26}$$

Integrating (26) once gives

$$g = p_0 + p_1 \sigma y \ln y + p_2 y, \tag{27}$$

for some constants p_0, p_1 and p_2.

We are aware that (27) is a rather arbitrary subclass of the infinitely many possible dividend yield functions. Nevertheless, this function possesses some nice properties. In spite of being specified under strong assumptions, it offers good analytical tractability and seems rich enough to allow various interesting types of behaviour to arise in the economy. It will enable us to gain some insights into the possible interplay between the behaviour of the price of risk and consumption in a stationary economy.

[3] See Hodges and Carverhill [14] for more details.

Returning to the problem of solving y, we first integrate (25) once to obtain

$$\frac{1}{2}\sigma^2 y' = (\theta + p_0)y - \frac{\sigma}{2}(1+p_1)y^2 + \text{ constant} \tag{28}$$

and then discuss the solutions case by case. Since some of the derivations are rather lengthy and complex, we shall provide the functional form solution immediately for each case. Readers who would like to have a quick overview may then skip the detailed derivations and refer to Figure 1 for the graphical illustration of those functions accordingly.

Case 1 *When $\theta = -p_0$ and $p_1 = -1$, the price of risk has a linear form solution:*

$$\alpha(x, \tau) = k_1(x + \theta\tau) + k_2, \tag{29}$$

where k_1 and k_2 are constants. Provided $k_2 > 0$, α can be constant if $k_1 = 0$, or decreasing (increasing) if $k_1 < 0$ ($k > 0$).

Proof: In this case, (28) reduces to

$$\frac{1}{2}\sigma^2 y' = \text{ constant.}$$

Rearrange to obtain

$$\frac{1}{2}\sigma^2 \int \frac{dy}{\text{constant}} = \int du.$$

It integrates to a linear form solution

$$y = k_1 u + k_2,$$

for some constants k_1 and k_2. This in turn gives (29). An unfortunate limitation of this case is that except for $k_1 = 0$, α may go negative for large or small x values (depending on the sign of k_1). □

Case 2 *When $\theta \neq -p_0$ and $p_1 \neq -1$, provided c_2 is real, the price of risk y has a general solution of the form*

$$y = \begin{cases} \frac{(c_2+c_1)e^{s(u)}+c_1-c_2}{e^{s(u)}+1} & for \quad y \in (c_1 - c_2, c_1 + c_2) \\ \\ \frac{(c_2+c_1)e^{s(u)}+c_2-c_1}{e^{s(u)}-1} & for \quad y \notin (c_1 - c_2, c_1 + c_2) \end{cases} \tag{30}$$

where

$$c_1 = \frac{\theta + p_0}{\sigma(1+p_1)}, \qquad c_2 = \pm\sqrt{d + c_1^2}, \tag{31}$$

$s(u) = k + \frac{2(1+p_1)c_2}{\sigma}u$, *and* k *is a constant. For* y *to be positive,* c_1 *must be positive and greater than* c_2 *if we take the positive value of* c_2*. When* $c_2 = 0$*,* y *is constant and equal to* c_1*. Otherwise,* y *has a travelling wave solution from the first equation of (30) which is viable, and has a hyperbolic solution of the form*

$$\alpha(x,\tau) = c_1 + \frac{\sigma}{(1+p_1)(x+\theta\tau)}, \tag{32}$$

from the second equation of (30) which admits arbitrage:

Proof: In this case, we can rearrange (28) to obtain

$$\frac{\sigma}{1+p_1}\int \frac{dy}{d + \frac{2(\theta+p_0)}{\sigma(1+p_1)}y - y^2} = \int du, \tag{33}$$

where d is a constant. Integrating (33) yields

$$\frac{\sigma}{2(1+p_1)c_2}\ln\left[\left|\frac{y+c_2-c_1}{y-c_2-c_1}\right|\right] = u + \text{ constant},$$

where

$$c_1 = \frac{\theta + p_0}{\sigma(1+p_1)}, \quad \text{and } c_2 = \pm\sqrt{d + c_1^2}.$$

Thus, provided c_2 is real and not complex, we obtain a general solution of the form

$$y = \begin{cases} \frac{(c_2+c_1)e^{s(u)}+c_1-c_2}{e^{s(u)}+1} & for \quad y \in (c_1 - c_2, c_1 + c_2) \\ \\ \frac{(c_2+c_1)e^{s(u)}+c_2-c_1}{e^{s(u)}-1} & for \quad y \notin (c_1 - c_2, c_1 + c_2) \end{cases} \tag{34}$$

where $s(u) = k + \frac{2(1+p_1)c_2}{\sigma}u$ and k is a constant. While the first equation in (34) may result in hyperbolic tangent functions consistent with equilibrium, the second equation in general entails trigonometric functions which could not possibly be supported by any reasonable utility function of an economic agent. More specificallly, the price of risk y has an unacceptable sigularity at $s(u) = 0$ for the second equation, except when $c_2 = 0$ but $k \neq 0$ which gives the trivial solution of a constant price of risk

$$y = c_1 = \frac{\theta + p_0}{\sigma(1+p_1)}. \tag{35}$$

In other words, when $c_2 \neq 0$, the second equation of (34) prevents the state variable x from reaching the point

$$x = -\frac{k\sigma}{2(1+p_1)c_2} - \theta\tau.$$

Another special case is to let $k = 0$ and take the limit as c_2 tends to zero. The solution will then be

$$y = c_1 + \frac{\sigma}{(1+p_1)u},$$

which gives

$$\alpha(x, \tau) = c_1 + \frac{\sigma}{(1+p_1)(x+\theta\tau)}, \tag{36}$$

It is obvious to see that α is decreasing (increasing) in x when $p_1 > -1$ ($p_1 < -1$). Unfortunately, (36) has a singularity at $p_1 = -1$ and/or $x = -\theta\tau$. In other words, this model permits arbitrage in the economy.

The more interesting case is the stable travelling wave solution obtained from the first equation. By inspection of (35), it can be seen that if we assume a positive risk premium, then $c_1 > 0$ would be required. That is, we have rulled out the possibility of $\theta = -p_0$ so

$$\text{sign}(\theta + p_0) = \text{sign}(1 + p_1).$$

Provided that c_2 is positive, c_1 must be greater than c_2 and we obtain two alternative scenarios: [4]

1. ($\theta > -p_0$ **and** $p_1 > -1$) The solution y is increasing and is bounded below and above

$$\begin{cases} y \to c_1 - c_2, & \textit{when} \quad x \to -\infty \\ y \to c_1 + c_2, & \textit{when} \quad x \to +\infty \end{cases} \tag{37}$$

   It is worthnoting that the travelling wave solution (increasing in x) Hodges and Selby [15] obtained is a special case in this scenario with $p_0 = p_1 = 0$ (i.e. no consumption).

[4]Note that taking a negative c_2 will not change the properties of the solutions.

2. ($\theta < -p_0$ **and** $p_1 < -1$) The solution y is decreasing and is bounded below and above

$$\begin{cases} y \to c_1 + c_2, & when \quad x \to -\infty \\ y \to c_1 - c_2, & when \quad x \to +\infty \end{cases} \tag{38}$$

Note that this is the case we are most interested in, and it can not be obtained without intermediate consumption. [5]

Case 3 *When $\theta \neq -p_0$ and $p_1 = -1$, the price of risk y has a solution of an exponential form:*

$$y = e^{m(u)} + L, \tag{39}$$

where

$$m(u) = j + \frac{2(\theta + p_0)}{\sigma^2} u,$$

and j and L are constants and $L \geq 0$. y is decreasing (increasing) in x if $\theta < -p_0$ $(\theta > -p_0)$.

Proof: In this case, (28) reduces to

$$\frac{1}{2}\sigma^2 y' = (\theta + p_0)y + \text{ constant}.$$

Rearrange to obtain

$$\frac{\sigma^2}{2(\theta + p_0)} \int \frac{dy}{y + \text{ constant}} = \int du.$$

This integrates to

$$\frac{\sigma^2}{2(\theta + p_0)} \ln(y + r_1) = u + r_2, \qquad \text{for } y \in (-r_1, +\infty) \tag{40}$$

where r_1 and r_2 are constants. The solution is then an exponential one

$$y = e^{m(u)} + L, \tag{41}$$

[5] It is also interesting to derive a y function which depends on S only. This can be achieved by letting $\theta = -\left(r - \frac{\sigma^2}{2}\right)$ so that the resulting y can be expressed as

$$y = \frac{(c_2 + c_1)B \cdot S^A + c_1 - c_2}{B \cdot S^A + 1}, \text{ for } y \in (c_1 - c_2, c_1 + c_2),$$

where $A = \frac{2(1+p_1)c_2}{\sigma}$ and B is a positive real. Again provided $\theta < -p_0$ and $p_1 < -1$, we can obtain a decreasing y.

where

$$m(u) = j + \frac{2(\theta + p_0)}{\sigma^2} u,$$

and j and L are constants. Obviously, (41) is unbounded above and has a lower bound L. Thus, a positive price of risk requires $L \geq 0$ in (41) and $r_1 \leq 0$ in (40). Finally, y is decreasing (increasing) in x if $\theta < -p_0$ ($\theta > -p_0$) as $y_x = \frac{2(\theta+p_0)}{\sigma^2} e^{m(u)}$.

5 Numerical examples

In this section, we illustrate the analytical findings presented in Sections 4 using an empirically plausible parameterization of the process for stock indices and the process for dividend yields. Section 5.1 first demonstrates and also summarizes the attainable patterns for the price of risk, and then Section 5.2 presents the numerical results for the chosen model.

5.1 The behaviour of the price of risk

Figure 1 gives typical plots for the cases analyzed in Section 4. Panel I shows the linear form solutions, Panel II shows the travelling wave solutions, Panel III shows the exponential form solutions, and finally Panel IV shows the hyperbolic form solutions with singularity. The solid lines represent the patterns which can be obtained only from the consumption economy, while the dashed lines represent the patterns which can also be achieved from the wealth-only economy. Note the added flexibility which comes from including intermediate consumption. Additionally, the Panel IV case should be excluded since it admits arbitrage.

5.2 The behaviour of dividend yield

As is well known that in the exchange economy of Lucas, consumption equals dividend. Therefore, it seems desirable on empirical ground that the dividend yields should be (1) decreasing in the state of economy, (2) increasing in the price of risk, and (3) decreasing in the asset price (supposedly sticky). In addition, an increase in dividends should imply an increase in prices.

Panel I

Panel II

Panel III

Panel IV

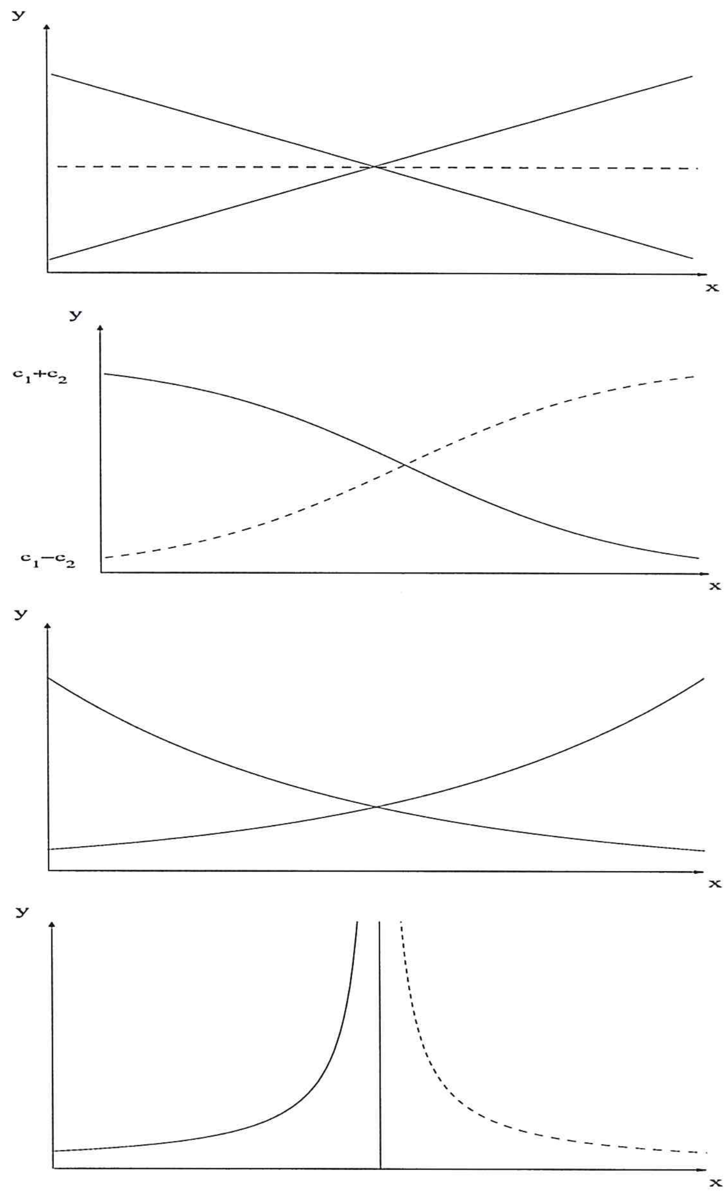

Figure 1: Behaviour of the price of risk: Illustrative plots. These graphs are plotted based on the analytical solutions of the homogeneous case in Section 4. The solid lines represent the patterns which can be obtained only from the economy with intermediate consumption, while the dashed lines represent the patterns which can also be achieved from the terminal wealth economy.

We can show that our model can successfully generate the above properties. For instance, taking the decreasing travelling wave solution for the price of risk, we can postulate the dividend yield δ so that it is increasing in the price of risk y, for $y \in (c_1 - c_2, c_1 + c_2)$ and is bounded below and above:

$$p_0 + p_1\sigma(c_1 - c_2)\left[\ln\frac{c_1 - c_2}{c_1 + c_2} - 1\right] \leq \delta \leq p_0 - p_1\sigma(c_1 + c_2).$$

The restrictions imposed on the parameters p_0, p_1 and p_2 are that

$$\begin{aligned} p_0 &= \delta_{min} - p_1\sigma(c_1 - c_2)\left[\ln\frac{c_1 - c_2}{c_1 + c_2} - 1\right], && (42)\\ p_1 &< -1, && (43)\\ p_2 &= -p_1\sigma\ln(c_1 + c_2) - p_1\sigma, && (44)\end{aligned}$$

where $\delta_{min} > 0$ is the minimum value of the dividend yield. Recall that when $p_1 > -1$, the price of risk (30) is in an increasing form (37). Thus, the interpretation behind the restriction on p_1 is that the dividend must be large enough to be able to flip the increasing pattern to a decreasing one.

We have explored the behaviour of the model using plausible parameter values for the case in which the price of risk takes the decreasing travelling wave form. The parameter values are summarized in Table 2. Figure 2 shows how the price of risk varies over time with respect to the state of economy x, for an investment horizon of 50 years. Figure 3 demonstrates the behaviour of the dividend yield with respect to the price of risk y, the state of economy x, and the stock index S, respectively. Note that the dividend yield stays within a sensible range from 1.5 to 5.85%, and that it is inversely related to the market level.

6 Concluding remarks

There seems a general agreement among financial economists that there is some predictability in stock market index returns. It remains, however, something of a puzzle as to whether it is to do with pricing anomalies or whether it reflects the nature of the risk premia within the underlying economy.

Table 1: Summary of the parameterization. These parameter values are used to calculate the travelling wave price of risk and the corresponding dividend yield for a time horizon of 50 years.

Notation	Value
r	0.06
σ	0.15
S_0	10
x_0	2.3026
k	0
θ	0.0465
c_1	0.425
c_2	0.25
δ_{min}	0.015
p_0	-0.0528
p_1	-1.0995
p_2	0.1001

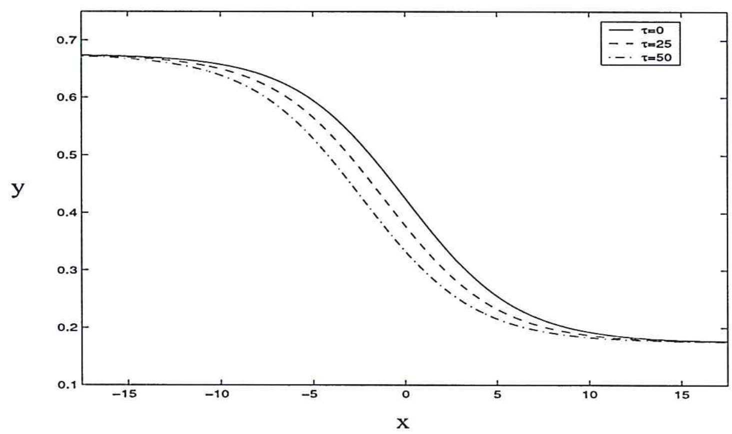

Figure 2: The price of risk: An example. These plots are calculated from the travelling wave solution using the parameter values listed in Table 1. The total investment horizon is 50 years. τ is the remaining time to terminal date.

Panel I

Panel II

Panel III

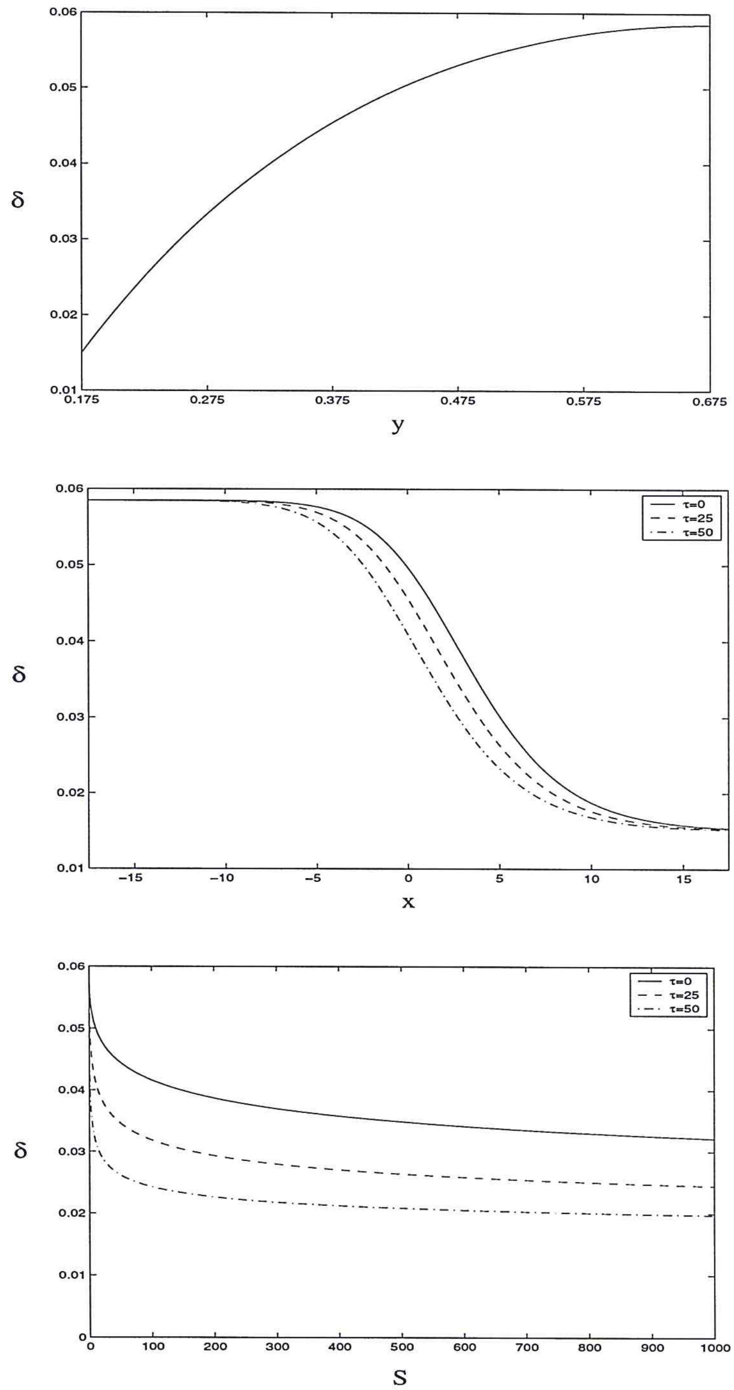

Figure 3: The dividend yield: An example. These plots are calculated based on the travelling wave solution for the price of risk and the corresponding dividend yield function using the parameter values listed in Table 1. The total investment horizon is 50 years22τ is the remaining time to terminal date.

Our paper can be viewed as an attempt to approach this puzzle by offering an alternative model of the asset price dynamics and showing that the time variation in asset expected returns (often postulated by some empiricists) could be consistent with equilibrium asset pricing models. As demonstrated in the previous section, we have explored the possibility of a decreasing price of risk in the state of economy. It can be shown that this amounts to some degree of mean reversion in the expected returns and the agent displays a decreasing relative risk aversion. The model therefore indicates the potential that the resulting time-varying price of risk and time-varying risk aversion can better explain the time-varying equity risk premium.

In addition, even though it has not been the focus of this paper, it is not difficult to show how the representative agent's utility can be backed out in our model. [6] Needless to say, the assumptions of constant interest rate and constant volatility which our time-homogeneous solutions were based on are quite strong. It would seem natural to relax these assumptions and extend to models which can handle stochastic volatility and/or stochastic interest rate. Nevertheless, it is important and instructive to analyze the nature of the behaviour which is possible within this framework and a representative agent equilibrium.

[6] To see this, we only need to write down the definition for the intertemporal relative risk aversion as

$$f(S_t,t) = \frac{\alpha(S_t,t)}{\sigma} = -S_t\frac{J_{SS}(S_t,S_t,t)}{J_S(S_t,S_t,t)}. \tag{45}$$

Or, in terms of x,

$$f(x_t,t) = 1 - \frac{J_{xx}(x_t,t)}{J_x} = -\frac{\mathcal{M}_x(x_t,t)}{\mathcal{M}(x_t,t)} = \frac{\alpha(x_t,t)}{\sigma}. \tag{46}$$

For any given time $t \leq T$, it is clear that (46) is simply an expression of ordinary differential equations. Thus, provided the functional form of the price of risk is known, we can integrate over x to recover the supporting utility J_x. Or equivalently we can back out $\mathcal{M}$ which in turn gives the state-price density function of the economy. More specifically, the last equality of (46) is equivalent to

$$\frac{\partial}{\partial x_t}\ln[\mathcal{M}(x_t,t)] = -\frac{\alpha(x_t,t)}{\sigma}. \tag{47}$$

Therefore, by integration, it follows that $\mathcal{M}$ can be written as a function up to some constant A:

$$\mathcal{M}(x_t,t) = A\cdot\exp\left(-\frac{1}{\sigma}\int_{x_0}^{x_t}\alpha(\eta,t)d\eta\right).$$

Finally, although the empirical issues are beyond the scope of this paper, it would be nice to see how we can devise some kind of procedures so that the model can be empirically estimated/tested. Both areas will be challenge resourcefulness of future research.

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