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September 2001

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FORC Preprint: 2001/119

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First Version: December 2000. This Version: September 2001.

Abstract

We propose a numerical method for barrier options valuation assuming that the trigger is checked at fixed times. Our method can deal with the interesting cases of time-varying barriers and non equally spaced monitoring dates. The convergence of the method is proved and its computational cost is found to be linear in the number of monitoring dates and quadratic in the spatial discretization. We also discuss extensively the empirical performance of the method, including the calculation of the delta and gamma coefficients, under the GBM and CEV stochastic specifications.

1. Introduction

In this paper we propose a simple and accurate valuation method for discrete barrier options assuming that the transition density of the underlying process is known analytically.. In particular, we discuss the cases of lognormal and constant elasticity of variance price dynamics. Barrier options are a common, extensively traded type of exotic derivatives. Barrier options are activated (knock-ins) or terminated (knock-outs) if a specific trigger is reached within the expiry date.

There are now several papers dealing with the pricing of barrier options and a great number of valuation techniques have been proposed. However, in practice, barrier options differ from those studied in the academic literature under many respects. One of the most important is the monitoring frequency of the underlying assets. In some cases, the discrepancy between option prices under continuous and discrete monitoring can be huge, as shown by Heynen and Kat [29] for lookback options and by Kat and Verdonk [33] for barrier options. With discrete monitoring the trigger is checked at fixed times and a knock-out (knock-in) becomes more (less) expensive as the number of monitoring dates increases. In case of no monitoring a knock-out is equivalent to a plain vanilla option. Moreover, the convergence of the discretely monitored option price to the continuous case is known to be extremely slow. This is indeed the main motivation for our study. In fact, many approximations, such as trees or the continuity formula correction proposed in Broadie et al. [13], can be easily implemented only when the underlying price follows a lognormal process and the barriers are constant or an exponential function of time. Therefore, there is a need of alternative numerical methods general enough to deal with different processes and time varying barriers.

In this paper we exploit the fact that the price of a discrete barrier option can be obtained evaluating N one dimensional nested integrals. The iterative scheme we propose

*Aknowledgments: A special thanks goes to S. Hodges (FORC, University of Warwick) and N. Doninelli (Mediobanca DueEMME, Italia) for helpful comments and suggestions.

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is based on a linear polynomial interpolation technology and suitable Newton-Cotes formulae, namely, the composite trapezoidal and Simpson rules. The iterative scheme we construct is proved to converge (see Theorem 4.1) and provides a good approximation of the option price for each maturity and position of the underlying asset price relative to the barriers.

To summarize, there are three main characteristics of our method. First, its generality, given that it can be applied with little assumptions, i.e. when the *conditional* transition density of the underlying process is known analytically[1]. The use of the conditional transition density has several implications: i) we do not incur in a time discretization error, that is encountered when using finite difference or finite element methods, ii) we do not need to solve linear systems as required when using implicit finite difference schemes, iii) we can apply the method straightforwardly to fairly "non-standard" processes and time-varying barriers. As an example, we provide a full discussion of the CEV case and report significant differences with respect to the GBM case. We recall that the continuous formula as well as the approximations which can be found in the literature assume a lognormal process for the underlying. iv) The degree of accuracy of our method doesn't depend on the positioning of the barriers with respect to the grid nodes. As a consequence, we do not require an increase in the number of nodes near the barriers or their clever positioning, as happens, for example, when using trees. Second, there is a clear advantage when using piece-wise linear approximations instead of global interpolations, such as Hermite or Laguerre polynomials as the computational burden of calculating the solution inside the barriers is greatly reduced. Third, we establish the convergence of the solution and its first derivative in the supremum norm.

The structure of the paper is as follows. In the first section we review some of the literature on the pricing of discrete barrier options. In the second section we introduce the model and in section three we present the numerical scheme and its main properties. In section four we provide some numerical results for the GBM and CEV processes.

2. Review of the literature

As we remarked in the introduction, the pricing of barrier options have been almost always discussed in the case of continuous monitoring of the underlying asset. A listing of pricing formulae for different types of barrier options can be found in Rubinstein and Reiner [42]. A mathematically oriented discussion of the barrier option pricing problem is contained in Rich [40]. In a nutshell, there are several approaches to barrier option pricing: a) the probabilistic method, see Kunitomo and Ikeda [34]; b) the Laplace Transform technique, see Pelsser [38], Sbuelz [43], Jamshidian [32], Geman and Yor [25]; c) the Black-Scholes PDE, which can be solved using separation of variables, see Hui [30] and Hui et al. [31] or finite difference schemes, see Boyle and Tian [10] or Zvan et al. [51], e) binomial and trinomial trees see Boyle and Lau [9], Ritchken [41], Heynen and Kat [29], Tian [48], f) Monte Carlo simulations with various enhancements, see Andersen and Brotherton-Ratcliffe [2], Baldi et al. [5], Beaglehole et al. [6].

Much of the work related to the pricing of discretely monitored exotic options is based on binomial or trinomial trees, assuming a lognormal specification for the underlying and constant or exponentially time varying barriers. Some authors have studied the pricing problem using approximations, numerical solutions of PDEs or numerical integration. In the following subsections we briefly review these approaches.

[1] In general, when the conditional transition density is not known analitically and the distance between the monitoring dates is not too large, an approximate solution can be obtained replacing the unknown density with a normal density having the same first two conditional moments.

Binomial and Trinomial Trees

In general, the binomial tree framework can be easily adapted to deal with the barrier option case. For example, when pricing down-out options, we just set all the option values equal to the rebate if the stock price is lower than the out barrier. However, convergence could be very slow and a high number of time steps is required before the pricing error becomes acceptable. Moreover, as Boyle and Lau [9] correctly point out, an increase in the number of time steps causes the relative error to decline towards zero before jumping to a new peak (the so called "saw-tooth pattern"). In particular, these authors remark that the positioning of the nodes with respect to the barrier may have a considerable effect on option prices. In case of continuous monitoring, the best outcome is obtained when a layer of nodes falls exactly on (or just below) the barrier. In case of discrete monitoring, the best results are obtained when the lattice is constructed so that the barrier falls exactly in the middle of two nodes[2].

An explanation of the reasons of the bad performance of the binomial scheme can be found in Figlewski and Gao [23] pag. 317, and in Steiner et al. [45], pag. 76. These authors contend that it is difficult to assign the *correct* probabilities at the rebate nodes. Depending on the position of the barrier, the binomial distribution assigns a probability than is either too high or too low. Then, option prices result to be underestimated or overestimated. In fact, a small increase in the number of time steps can cause a significant change in the probability of the option being knocked out, that is, a jump in the option price. For this reason, Steiner et al. [45] construct a binomial tree where the option price near the barrier is obtained as a weighted average of the rebate value and the discounted expected value. The weight is chosen in order to get the correct probability at the rebate nodes. Similarly, Kat and Verdonk [33] propose an approximation given by the average of the binomial price just before and just after a jump.

Considering trinomial trees instead of binomial trees should improve the results, although the increased accuracy could be offset by increased computational time, as reported in Broadie and Detemple [12] . Moreover, also in this case, a direct application does not eliminate the characteristic saw-tooth pattern. Ritchken [41] observes that in the Boyle-Lau binomial lattice the number of time steps must assume specific values. A high number of nodes is required in this case and there are not enough degrees of freedom to accommodate for non-constant barriers. This author suggests the use of a trinomial tree where the number of steps is adjusted appropriately. An additional stretch parameter is modified until a layer of nodes coincides with the barrier. However, the trinomial tree can still show errors as large as 3.7 cents using 8000 steps, as reported in [14]. As with the binomial tree, a very large numbers of time steps are required when the initial stock price is close to a barrier. Moreover, this method cannot deal with non-constant or non-exponentially time varying barriers.

Cheuk and Vorst [15] propose a shifted trinomial tree. In this case, at each monitoring date, the barrier falls exactly in the middle of a layer of nodes. With daily monitoring this method can become computationally intensive, since more than 50 trinomial steps between successive monitoring dates are required in order to achieve reasonable accuracy.

Broadie et al. [14] suggest several improvements to the Ritchken [41] trinomial tree and obtain a penny accuracy in the difficult case $S_0 = K = 100$, $\sigma = 0.6$, $r = 0.1$, $T = 0.2$, $N = 4$, $l = 95$ using 256 time steps (where N is the number of monitoring dates and l is the barrier). This enhanced trinomial method performs well for small N or when the number of time steps between the monitoring points increase.

Figlewski and Gao [23] propose an adaptive mesh model. Their trinomial lattice is finer next to the barrier so that the valuation information is transmitted properly. Again, as

[2] Tavella and Randall [47], pagg. 194-6, give a detailed discussion of this problem.

the asset price moves near the barrier the number of time steps goes to infinity. However, this occurs only near the boundary in the time-price space.

Approximations

Broadie et al. (BGK) [13], [14] develop a method to relate the prices of discrete and continuous time options assuming lognormal price dynamics and constant or exponentially time varying barriers. They obtain a simple correction to the continuous time formula. The barrier in the continuous formula is shifted in order to correct for the fact that when the discretely monitored process breaches the barrier, it overshoots it. This method incurs in large pricing errors when the barrier is close to the current stock price. A similar approach is followed by Wei [49] that suggests a procedure based on a polynomial interpolation. Wei's method can deal with discrete barrier options with either a flat or exponential barrier. Compared to the correction formula proposed by BGK, this interpolation method is to be preferred when the stock price is close to the barrier or when the monitoring frequency is not high (e.g. weekly). On the contrary, when the barrier is not close to the current stock price the BGK method dominates Wei's method.

Levy and Mantion [35] use a heuristic approach. They observe that the price difference between the continuous and discrete approximation is monotonically increasing in the ratio of the sampling interval to the time to expiry. This error is modelled as a function of the square root of the sampling interval. The price difference is obtained using a second-order Taylor expansion with respect to the square root of the sampling interval and the free parameters are chosen in order to correctly price barrier options with one and two monitoring dates, i.e. when closed form solutions are available. They report encouraging numerical examples when the barrier is very far away from the spot price.

Finite Difference and Finite Element Methods

A detailed treatment of discretely sampled options using a PDE approach is given in Wilmott et al. [50] and in Tavella and Randall [47]. We remark that finite difference or finite element methods, that do not exploit the information of the conditional transition density when available, are always prone to time discretization error. Moreover, implicit and semi-implicit difference schemes require the solution of a tridiagonal linear system at every time step. Another important problem is that some care has to be taken in using a scheme such as the Crank-Nicholson. This scheme is highly accurate, however, as shown in Smith [44], pag. 122-124, "numerical studies indicate that very slowly decaying finite oscillations can occur with the Crank-Nicholson method in the neighborhood of discontinuities in the initial values"[3]. These oscillations are due to the very sharp gradient occurring when the solution is periodically set to zero at the barrier and do not decay fast enough. Although the oscillations are localized around the barrier, the resulting option value is everywhere slightly too large. Absence of spurious obscillations in the Crank-Nicholson scheme can require a demanding restriction on the time step size as a function of the stock grid spacing[4]. Moreover, it is not always easy to find a simple expression for the restriction on the time step when dealing with processes different from the GBM. Therefore, we should be careful in using naive implementations of finite difference schemes. For this reason, Zvan et al. [51] propose an accurate first-order implicit method [5] based on a point-distributed finite volume scheme used in computational fluid dynamics which can be used for PDE models with general algebraic constraints on the solution. Unfortunately, this method is computationally demanding given that a reasonably fine grid is required in order to obtain accurate prices.

[3] For details see Zvan and al. [51] and Fusai and Tagliani [24].

[4] This restriction is a function of the volatility parameter of the underlying process.

[5] A first-order implicit upwind scheme is proposed in Fusai and Tagliani [24] for discretely sampled occupation time derivatives.

Boyle and Tian [10] develop a modified explicit finite difference method for discrete barrier options pricing.

Duan et al. [20] use a time homogeneous Markov Chain to approximate the underlying asset price process. Matching the discrete time step of the Markov chain with the monitoring frequency of the barrier, this approach can easily deal with very general discrete barriers. Moreover, models with time-varying volatility such as the GARCH can be easily treated under this framework. This approach has several other advantages: a) the spacing of the price grid can be made independent of the number of time steps b) the transition matrix associated with the Markov chain is usually sparse, so that storage and computation costs are reduced.

Numerical Integration

Ait-Sahalia and Lai [4] compute the price of discretely monitored barriers evaluating one-dimensional integrals from one monitoring date to the next. In [3] the same authors show how to deal with the case of discrete lookbacks, exploiting the duality property of random walk. Just as in AitSahlia and Lai [3], [4], we reduce the pricing problem to a recursion in N-steps, where N is the number of monitoring dates.

Sullivan [46] has also proposed a scheme combining numerical integration and function approximation. He approximates the solution at the monitoring dates using Chebyshev polynomials and computes the integrals using Gauss-Legendre quadrature. Sullivan obtains highly accurate results, but does not formally prove the convergence to the desired multidimensional integration. Our approach is based on a similar idea. At the monitoring dates we approximate the solution using a linear combination of hat functions and then we make use of numerical integration methods to relate the coefficients of this linear combination at different dates.

3. The model

Consider the following unidimensional risk-neutral dynamics:

$$dx = rxdt + \sigma(x,t)\, dW_t$$

where r is the constant risk-free rate and $\sigma(x,t)$ is the instantaneous volatility function and assume that the process is markovian and the conditional transition density of the process is known analytically..

We want to price discrete barrier options in general, but, in order to make our analysis concrete we concentrate on a double barrier knock-out option, i.e. a call option that expires worthless if one of the two barriers has been hit at a monitoring date. Let $0 = t_0 < t_1 < ... < t_n < ... < t_N = \Omega$ be the monitoring dates and $l_n = l$ be the lower barrier and $u_n = u$ the upper barrier active at time t_n[6]. Again, we are assuming constant barriers only for aim of simplicity and notational convenience.

Denote by $v(x,t,n) \equiv v(x,t,n;l,u)$ the price of the barrier option when $t > t_n$. Then, $v(x,t,n)$ satisfies the well known Black-Scholes PDE:

$$-\frac{\partial v(x,t,n)}{\partial t} + rx\frac{\partial v(x,t,n)}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 v(x,t,n)}{\partial x^2} = rv(x,t,n)$$

Given that the trigger condition is checked only at fixed times, we need to update the initial condition at the monitoring dates $0 = t_0 < t_1 < ... < t_N$:

$$\begin{aligned} v(x,t_n,n) &= v(x,t_n,n-1)\,\mathbf{1}_{l,u}(x) \\ v(x,t_0,0) &= (x-K)^+\,\mathbf{1}_{l,u}(x) \end{aligned}$$

[6] Eventually, we have $l = 0$ or $u = +\infty$ in the case of single barrier options.

where $\mathbf{1}_{l,u}(x)$ is the indicator function:

$$\mathbf{1}_{l,u}(x) = \begin{cases} 1 & if \quad l \leq x \leq u \\ \\ 0 & if \quad x \notin [l\,,u] \end{cases}$$

Let $p(y,\tau;x,n)$ be the transition density of the process to y at time $t_n+\tau$ starting from x at time t_n. Again, without loss of generality, consider a time-homogenous process:

$$p(y,t_n+\tau;x,t_n) = p(y,\tau;x,0) \equiv w(y,\tau;x)$$

Then, the solution of the above PDE is obtained recursively:

$$v(x,t_n+\tau,n) = \mathbf{1}_{l,u}(x)\int_{-\infty}^{+\infty} w(\xi,\tau;x)\,v(\xi,t_n,n-1)\,d\xi \tag{3.1}$$

Using the fact that $v(\xi,t_n,n-1)$ is zero outside $[l,\,u]$, we have:

$$v(x,t_n+\tau,n) = \int_l^u w(\xi,\tau;x)\,v(\xi,t_n,n-1)\,d\xi,\;\; l<x<u \tag{3.2}$$

The solution is given by recursive univariate integration: one integral for every monitoring date. In order to exploit the recursive structure of the problem, we compute the option price only at the monitoring dates and then at the intermediate date $t_n+\tau$. In other words, we want to compute $v(x,t_n,n)$, $n\geq 0$. In the next section we develop a numerical scheme to solve this recursive integration and detail its main properties, including convergence and computational cost. Then, we apply our scheme to the Geometric Brownian motion and the CEV case.

4. The numerical method

In this section we discuss five issues:

a) the approximation of the solution (option price at each monitoring date) by a linear combination of hat functions. The results in this section can be easily generalised to several other approximating functions;

b) the calculation of the coefficients of the linear combination which requires the recursive computation of an integral, efficiently approximated using the Trapezoidal or the Simpson rule (equations (4.19), (4.20), (4.21) and (4.22));

c) the convergence of the numerical solution and its first derivative in the supremum norm (Theorem 4.1);

d) the calculation of the *Delta* coefficient, i.e. the sensitivity of the option value to variations in the price of the underlying asset,

e) the convergence of the discrete option price to the continuous case. We show formally, in the cases of GBM and CEV price dynamics (see Corollaries 1 and 2), that the convergence is rather slow as the existing empirical evidence has already suggested.

Let us start with the basic recursive equation:

$$v(x,t_n+\tau,n) = \int_l^u w(\xi,\tau_n;x)\,v(\xi,t_n,n-1)\,d\xi. \tag{4.1}$$

with $\tau_n = t_{n+1}-t_n$, $x\in[l,u]$, $n=0,1,\ldots$. In case of equally spaced monitoring dates we have $\tau_n=\tau$. Hereinafter we set $v_n(x) = v(x,t_n,n)$.

Divide $[l,\, u]$ into $m-1$ subintervals $I_k = [x_k, x_{k+1}]$, $k = 1, 2, ..., m-1$ with end points given by:

$$l = x_{1,m} < x_{2,m} < ... < x_{m-1,m} < x_{m,m} = u \tag{4.2}$$

and let h_m be the mesh of the partition $\{I_k\}_{k=1,2,...,m-1}$. We assume that there is a C such that:

$$h_m = \max\{|x_{k,m} - x_{k-1,m}| : 2 \leq k \leq m\} \leq \frac{C}{m} \tag{4.3}$$

Note that when we choose equally spaced points in (4.2), i.e. $h_m = (u-l)\,(m-1)$, then $C = 2(u-l)$.

Let $L_i\,(x)$, $i = 1, 2, ..., m$, be the hat functions given by:

$$L_{i,m}\,(x) = L_i\,(x) = \begin{cases} \frac{x_{i+1,m}-x}{x_{i+1,m}-x_{i,m}} & x \in I_i, i < m \\ \\ \frac{x-x_{i-1,m}}{x_{i,m}-x_{i-1,m}} & x \in I_{i-1}, i > 1 \\ \\ 0 & otherwise \end{cases} \tag{4.4}$$

and denote with $v_n^m\,(x)$, $x \in [l, u]$, $m = 1, 2, ..., n = 1, 2, ...$ the linear interpolation:

$$v_n^m\,(x) = \sum_{j=1}^{m} \alpha_{j,m}^n L_j\,(x) \tag{4.5}$$

and

$$v_0^m\,(x) = \left(\prod\nolimits_m v_0\right)(x) = \sum_{j=1}^{m} \alpha_{j,0}^n L_j\,(x) \tag{4.6}$$

where $\prod_m$ is the interpolant operator, that is:

$$\alpha_{j,m}^0 = v_0\,(x_{j,m})\,,\; j = 1, 2, ..., m. \tag{4.7}$$

By construction, the interpolating function equals the initial condition at the grid points $x_{j,m}$, whilst between two grid points the initial condition comes from the linear combination of hat functions given in equation (4.6). We remark that the hat function can be interpreted financially as a butterfly position. The financial intuition behind equation (4.5) is straightforward. We need to find, for every monitoring date, the appropriate number of expiring butterflies allowing to build up the value of the barrier option. The coefficients of the linear combinations can be then be interpreted as *hedging coefficients* of the portfolio of butterflies. However, this method does not provide a static hedge for the contract (the replicating portfolio has to be rebalanced at every monitoring date since the coefficients in the linear combination depend on j).

Note that the results presented in this Section also hold when we choose $L_i(x)$, $i = 1, 2, \ldots, m$ to be smooth functions satisfying the following conditions:

$$L_i(x_{j,m}) = \delta_{i,j},\; L_i(x) \geq 0,\; \sum_{i=1}^{m} L_i(x) = 1,\; \forall x \in\; [l, u], \tag{4.8}$$

where $\delta_{i,j} = 1$ when $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. Among these interpolating functions we have, for instance, the B-splines of any order. In any case for computational purposes and to maintain the financial intuition we have preferred to use hat functions. In fact, the computation of the hat functions is well posed and the computational cost is low.

Let us consider the following recursive relation for v_n^m, that is:

$$v_{n+1}^m(x) = \int_l^u w(y, \tau_n; x)\, v_n^m(y)\, dy \tag{4.9}$$

Using equation (4.5) we get:

$$\sum_{j=1}^m \alpha_{j,m}^{n+1} L_j(x) = \sum_{j=1}^m \alpha_{j,m}^n \int_l^u w(y, \tau_n; x)\, L_j(y)\, dy \tag{4.10}$$

where the initial guess of the recursive formula is given by:

$$\alpha_{j,m}^0 = v_0(x_j), \ j = 1, 2, \ldots .n. \tag{4.11}$$

Lemmas 2, 3 and 4 in Appendix A show how to determine a bound to the norm of the difference between the true value v_n and its approximation v_n^m. Let us define

$$T_n = \sup_{l \le x \le u} \int_l^u |w(y, \tau_n; x)|\, dy \le T, \ n = 0, 1, \ldots,$$

then we have (see Lemma 4):

$$\|v_n - v_n^m\|_\infty \le \prod_{p=1}^n T_p \frac{C_{v0}}{m} \le T^n \frac{C_{v0}}{m}. \tag{4.12}$$

where C_{v_0} is a suitable constant depending on v_0. Given the time to maturity and the m-grid, this result shows how important is the constant T in determining the convergence of the method as the number of monitoring dates increases.

At this stage, the main difficulty with the approximation v_n^m is that it still requires: a) the computation of the coefficients $\alpha_{j,m}^n$ and b) the computation of a sequence of integrals, one for each monitoring date. The next step is then to approximate the integral appearing in (4.10) using a simple quadrature rule and to exploit the functional form of the approximation in order to obtain a computable sequential procedure for the coefficients $\alpha_{j,m}^n$ and for v_n^m.

Let us consider the problem of computing the function v_n^m. Since $L_j(x_i) = 0,\ i \ne j$, $L_j(x_i) = 1,\ i = j$ and evaluating equation (4.10) at $x = x_{i,m}, i = 1, 2, \ldots, m$, we obtain a linear transformation, giving the values of $\alpha_{i,m}^{n+1}$ in terms of the coefficients at the previous monitoring date n, $\alpha_{j,m}^n$:

$$\alpha_{i,m}^{n+1} = \sum_{j=1}^m \alpha_{j,m}^n \int_l^u w(y, \tau_n; x_{i,m})\, L_j(y)\, dy\,,\ i = 1, 2, \ldots m. \tag{4.13}$$

The computation of $\alpha_{i,m}^n$ requires a numerical integration. We accomplish this task using simple schemes (the Trapezoidal and Simpson rule). This procedure allows to obtain a simply computable expression for our approximating function v_n^m.

Let $H_{n,m} \in R^{m \times m}$ be the following matrix:

$$(H_{n,m})_{i,j} = \int_l^u w(y, \tau_n; x_{i,m})\, L_j(y)\, dy \tag{4.14}$$

$i = 1, 2, \ldots m$, and $j = 1, 2, \ldots m$ and $\mathbf{a}_{n,m} \in R^m$ be a vector with components:

$$a_{n,m,i} = \alpha_{i,m}^n, i = 1, 2, \ldots, m \tag{4.15}$$

We can rewrite equation (4.13) as follows:

$$\mathbf{a}_{n+1,m} = H_{n,m}\mathbf{a}_{n,m},\, n = 0, 1, .. \tag{4.16}$$

Note that when the kernel $w(y, \tau_n; x)$ is independent of n, (this occurs if the monitoring dates are equally spaced or if the transition density is time-homogenous) the matrix $H_{n,m}$ is independent of n, that is $H_{n,m} = H_m$, and it can be computed once. We approximate the integrals appearing in (4.14) using the trapezoidal rule and the Simpson rule. Assuming, for simplicity, that the points in formula (4.2) are equally spaced, that is $h_m = h = (u-l)/(m-1)$, consider the following matrices $H^T_{n,m} \in R^{m\times m}$, $H^S_{n,m} \in R^{m\times m}$ associated to the trapezoidal and the Simpson formula:

$$\begin{cases} \left(H^T_{n,m}\right)_{i,j} = w\left(x_{j,m}, \tau_n; x_{i,m}\right) h & \begin{array}{l} i = 1, ..., m \\ j = 2, ..., m-1 \end{array} \\ \left(H^T_{n,m}\right)_{i,1} = w\left(x_{1,m}, \tau_n; x_{i,m}\right) \frac{h}{2} & i = 1, 2, ..., m \\ \left(H^T_{n,m}\right)_{i,m} = w\left(x_{m,m}, \tau_n; x_{i,m}\right) \frac{h}{2} & i = 1, 2, ..., m \end{cases} \tag{4.17}$$

and when m is even ($m = 2\nu$, ν integer):

$$\begin{cases} \left(H^S_{n,m}\right)_{i,2j+1} = w\left(x_{2j+1,m}, \tau_n; x_{i,m}\right) \frac{4}{3}h & \begin{array}{l} i = 1, ..., m \\ j = 1, ..., \nu-1 \end{array} \\ \left(H^S_{n,m}\right)_{i,2j} = w\left(x_{2j,m}, \tau_n; x_{i,m}\right) \frac{2}{3}h & \begin{array}{l} i = 1, ..., m \\ j = 1, ..., \nu-1 \end{array} \\ \left(H^S_{n,m}\right)_{i,1} = w\left(x_{1,m}, \tau_n; x_{i,m}\right) \frac{h}{3} & i = 1, ..., m \\ \left(H^S_{n,m}\right)_{i,m} = w\left(x_{m,m}, \tau_n; x_{i,m}\right) \frac{h}{3} & i = 1, ..., m \end{cases} \tag{4.18}$$

Now we have derived a computable expression for the coefficients of the linear combination and for our approximating functions:

$$\mathbf{a}^T_{n+1,m} = H^T_{n,m}\mathbf{a}^T_{n,m},\, n = 0, 1, ... \tag{4.19}$$

$$\mathbf{a}^S_{n+1,m} = H^S_{n,m}\mathbf{a}^S_{n,m},\, n = 0, 1, ... \tag{4.20}$$

$$\mathbf{a}^T_{0,m} = \mathbf{a}^S_{0,m} = \mathbf{a}_{0,m} \tag{4.21}$$

$$v^{m,T}_n = \sum_{j=1}^{m} a^T_{n,m,j} L_j(x) \tag{4.22}$$

$$v^{m,S}_n = \sum_{j=1}^{m} a^S_{n,m,j} L_j(x) \tag{4.23}$$

Using the Lemmas proved in Appendix B, we can now prove our main result, the convergence of the method as $m \to \infty$.

Theorem 4.1. *Let v_n be the function given by formula (4.1), and $v^{m,T}_n$, $v^{m,S}_n$ be given by formulae (4.22) and (4.23) respectively. We have*

$$\lim_{m\to+\infty} \|v_n - v^{m,T}_n\|_\infty = 0\,, \tag{4.24}$$

$$\lim_{m\to+\infty} \|v_n - v^{m,S}_n\|_\infty = 0. \tag{4.25}$$

Proof: We prove equation (4.24). Equation (4.25) can be proved similarly. Let v_n^m be given by (4.9). Applying the triangle inequality in the sup norm we have:

$$\|v_n - v_n^{m,T}\|_\infty \leq \|v_n - v_n^m\|_\infty + \|v_n^m - v_n^{m,T}\|_\infty. \tag{4.26}$$

Using the properties of the function L_j, $j = 1, 2, \ldots$, given in (4.8) we obtain

$$\begin{aligned} \|v_n^m - v_n^{m,T}\|_\infty &= \max_{l \leq x \leq u} \left| \sum_{j=1}^m (\alpha_n - a_{n,m,j}^T) L_j(x) \right| \\ &\leq \sup_{1 \leq j \leq m} \left| \alpha_{j,m}^n - a_{n,m,j}^T \right| \sum_{j=1}^m L_j(x) \\ &\leq \|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^T\|_\infty. \end{aligned} \tag{4.27}$$

An application of Lemma 4 in Appendix A and Lemma 9 in Appendix B concludes the proof. ♦

Finally, we give a formula to approximate the Δ coefficient, i.e. the sensitivity of the option premium to variations in the underlying asset price.

Lemma 1. *We have:*

$$\left(\frac{\partial v_n}{\partial x}\right)_{x = x_{i,m}} = \frac{a_{n,m,i+1}^T - a_{n,m,i-1}^T}{h} + O(h),\ h \to 0, \tag{4.28}$$

and

$$\left(\frac{\partial v_n}{\partial x}\right)_{x = x_{i,m}} = \frac{a_{n,m,i+1}^S - a_{n,m,i-1}^S}{h} + O(h),\ h \to 0, \tag{4.29}$$

where $O(\cdot)$ is the Landau symbol.

Proof: The thesis follows taking finite central differences and using Lemma 2, Lemma 3, Lemma 5 and Lemma 6 proved the Appendices.♦

In the remainder of this section we examine the convergence of the scheme as the number of the monitoring date increases in two concrete examples: the Gaussian case, arising when the stock price evolves according to a GBM process and the CEV process case.

In the Gaussian case the transition density is, see Section 5.1:

$$w(y, \tau_n; x) = \frac{1}{\sqrt{4\pi c^2 \tau_n}} e^{-(x-y)^2/(4c^2\tau_n)}, x \in [0,1], y \in [0,1]. \tag{4.30}$$

In this case we can compute the constant T_n which is crucial in understanding the convergence of the pricing formula to the continuous case as $n \to +\infty$ (see equation 4.12 and Lemma 3). In particular, Corollary 1 states that as $n \to +\infty$, $T_n \to 1$, so that, although convergence is guaranteed, it is slowed down. Numerical examples of this result are provided in Table 2.

Corollary 1. *Let $w(y; \tau_n; x)$ be given by (4.30) then we have*

$$T_n = 1 - 2\frac{1}{\sqrt{\pi}} \int_{\frac{1}{2c\sqrt{\tau_n}}}^{+\infty} e^{-y^2} dy < T = 1, \lim_{n \to +\infty} T_n = 1. \tag{4.31}$$

Proof.

$$
\begin{aligned}
T_n &= \sup_{0\le x\le 1} \tfrac{1}{2c\sqrt{\pi\tau_n}} \int_0^1 e^{-(x-y)^2/(4c^2\tau_n)}dy \\
&= \tfrac{1}{2c\sqrt{\pi\tau_n}} \sup_{0\le x\le 1} \left\{ \int_{-\infty}^{+\infty} e^{-(x-y)^2/(4c^2\tau_n)}dy - \int_{-\infty}^{0} e^{-(x-y)^2/(4c^2\tau_n)}dy - \int_{1}^{+\infty} e^{-(x-y)^2/(4c^2\tau_n)}dy \right\} \\
&\le \tfrac{1}{2c\sqrt{\pi\tau_n}} \sup_{0\le x\le 1} \left\{ \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2}dp - \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{x}{2c\sqrt{\pi\tau_n}}} e^{-p^2}dp - \tfrac{1}{\sqrt{\pi}} \int_{\frac{1-x}{2c\sqrt{\pi\tau_n}}}^{+\infty} e^{-p^2}dp \right\} \\
&= 1 - \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{1}{2c\sqrt{\pi\tau_n}}} e^{-p^2}dp - \tfrac{1}{\sqrt{\pi}} \int_{\frac{1}{2c\sqrt{\pi\tau_n}}}^{+\infty} e^{-p^2}dp \\
&= 1 - \tfrac{2}{\sqrt{\pi}} \int_{\frac{1}{2c\sqrt{\pi\tau_n}}}^{+\infty} e^{-p^2}dp,
\end{aligned}
\tag{4.32}
$$

and this concludes the proof. ♦

Now consider the following Square Root kernel (see Section 5.2):

$$
w(y,\tau_n;x) = \sqrt{\frac{x\,e^{r\tau_n}}{y}}\gamma_n\, e^{-\gamma_n(xe^{r\tau_n}+y)} I_1\left(2\gamma_n\sqrt{xe^{r\tau_n}y}\right),\ y\in[l,u],\ x\in[l,u], \tag{4.33}
$$

where $\gamma_n = 2r/\left(\sigma^2\left(e^{r\tau_n}-1\right)\right)$ and $I_1(z)$ is the modified Bessel function (see [1] pag. 374-8).

Note that γ_n goes to infinity as τ_n goes to zero. From [1] pag. 377 formula 9.7.1 we have:

$$
I_1(z) = \frac{1}{\sqrt{2\pi\, z}} e^z \left(1 + O\left(\frac{1}{z}\right)\right),\ z\to\infty \tag{4.34}
$$

From (4.33) and (4.34) we obtain the following expansion:

$$
w(y,\tau_n;x) = \sqrt[4]{\frac{xe^{r\tau_n}}{y^3}}\frac{\sqrt{\gamma_n}}{2\sqrt{\pi}}e^{-\gamma_n\left(\sqrt{xe^{r\tau_n}}-\sqrt{y}\right)^2} + O\left(\frac{1}{\gamma_n}\right),\ \gamma_n\to\infty \tag{4.35}
$$

Then, using the transformation:

$$
z = \sqrt{2\gamma_n}\left(\sqrt{xe^{r\tau_n}} - \sqrt{y}\right)
$$

the transition density for the variable z is given by a standard normal density. This fact formally confirms, at least for this process, that the normal density is a reliable approximation when $n\to\infty$. Then, Corollary 2 shows two estimates for T_n. The first, (4.36), is relevant when n is fixed and m goes to infinity. The second, (4.37), is important when we consider the convergence as $n\to\infty$ for a fixed time to maturity. Note that $T_n\to 1$ in this case and that the convergence is slowed down as in the previous case.

Corollary 2. *Let $w(y;\tau_n,x)$, $y\in[l,u]$, $x\in[l,u]$ be given by (4.33) then we have*

$$
T_n \le \gamma_n^2 u e^{r\tau_n}(u-l), \tag{4.36}
$$

and when $\tau_n\to 0$ ($\gamma_n\to 0$):

$$
\begin{aligned}
T_n \;=\; & 1 - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\gamma_n}\left(\sqrt{xe^{r\tau_n}}-\sqrt{u}\right)} e^{-z^2}dz - \frac{1}{\sqrt{\pi}}\int_{\sqrt{\gamma_n}\left(\sqrt{xe^{r\tau_n}}-\sqrt{l}\right)}^{+\infty} e^{-z^2}dz \\
& + \frac{1}{\sqrt{\gamma_n}}\frac{1}{\sqrt{2xe^{r\tau_n}}}\frac{1}{\sqrt{\pi}}\int_{\sqrt{\gamma_n}\left(\sqrt{xe^{r\tau_n}}-\sqrt{l}\right)}^{\sqrt{\gamma_n}\left(\sqrt{xe^{r\tau_n}}-\sqrt{l}\right)} e^{-z^2}dz
\end{aligned}
\tag{4.37}
$$

Proof. The proof of equation (4.36) is a consequence of definition (A.3) and the following property of the modified Bessel function of the first kind (see [1] pag. 375 formula 9.6.3 and pag. 362 formula 9.1.62):

$$\left|I_1\left(2\gamma_n\sqrt{xe^{rT_n}y}\right)\right| = \left|e^{-i\frac{\pi}{2}}J_1\left(2\gamma_n\sqrt{xe^{rT_n}y}e^{i\frac{\pi}{2}}\right)\right| \leq \left|e^{-i\frac{\pi}{2}}\gamma_n\sqrt{xe^{rT_n}y}e^{2\gamma_n\sqrt{xe^{rT_n}y}}\right| \quad (4.38)$$

where i is the imaginary unit and $J_1(z)$ is the Bessel function of the first kind. Using (4.38) and arguing as in Corollary 1, we obtain (4.37).

5. Numerical results

In this section we illustrate numerically some of properties discussed in the previous section. In particular, we show: a) the convergence of the method when the number m of grid points increases, b) the behavior of the scheme versus the challenging problem presented in Broadie et al. [13], c) an estimate of the rate of convergence and of the computational cost of the method, d) the behavior of the delta and gamma coefficients of the discrete barrier option with respect to the continuous time case and their convergence as the grid spacing increases, e) a comparison with other numerical schemes proposed in the literature in the case of GBM, f) a comparison between option barrier prices under the GBM and CEV specifications.

5.1. Geometric Brownian Motion

Assume that the underlying asset price is driven by a GBM process ($dx_t = rx_tdt + \sigma x_tdW$, with x_0 the initial stock price) and denote by $v(x,t,n)$ the price of the option for $t > t_n$. Given $x_0 = x$, we can use the standard transformation (see Wilmott et al. [50] pag. 98):

$$v(x,t,n) = e^{\alpha z+\beta t}g(z,t,n)$$

where:

$$z = \frac{\ln(x/l)}{\ln(u/l)};\bar{k}=\frac{\ln(k/l)}{\ln(u/l)};\mu = r-\sigma^2/2$$

$$\alpha = -\frac{\mu}{\sigma^2}\ln(u/l);c^2 = \frac{\sigma^2}{\ln^2(u/l)};\beta = \alpha\frac{\mu}{\ln(u/l)}+\frac{\alpha^2c^2}{2}-r;$$

and obtain the following recursive representation for the function $g(z,t,n)$:

$$g(z,t,n) = \begin{cases} \int_0^1 p(z-\xi,t-t_n;0,0)\,g(\xi,t_n,n-1)\,d\xi & n=1,2... \\ \int_0^1 p(z-\xi,t-t_n;0,0)\,e^{-\alpha\xi}\left(le^{\ln(u/l)\xi}-le^{\ln(u/l)k}\right)^+d\xi & n=0 \end{cases} \quad (5.1)$$

where $p(z,t) = p(z,t;0,0)$ is the heat (Gaussian) kernel:

$$p(z,t) = \frac{1}{\sqrt{2\pi c^2t}}e^{-z^2/(2c^2t)}$$

The single barrier case can be treated taking a very large u (in the case of upper barrier) or l near zero.

First of all, we show the convergence of the method when the number m of the grid points (see equation (4.2)) increases. Figures 1a) and 1b) show the price of a double knock-out call versus the number m of grid points. The current price of the underlying asset is set at 100, 109, 109.9, 109.99, the time to maturity is 0.1 years (five weeks), $N = 125$, the risk-free rate r is 0.1 per annum, the instantaneous volatility σ is 0.2, the strike price is $K = 100$, the lower barrier is $l = 95$ and the upper barrier is $u = 110$. Figures 1c) and 1d) show the price of a down and out call option when the spot price is set at $100, 96, 95.1, 95.01$, $T = 0.2$, $n = 50$, $r = 0.1$, $\sigma = 0.6$, $l = 95$, $u = +\infty$. Pricing this call is a challenging problem due to the value of the volatility coefficient and the position of the spot price with respect to the barrier. Figure 1d shows that the convergence of the method is not affected by the relative position of the barrier. Moreover, we observe that the method can be applied to price single barrier down and out or up and out calls simply choosing u sufficiently large or l sufficiently small (in Figures 1c and 1d we have set $u = 1000$). Note that the method gives an accurate approximation with $m \geq 500$ for both quadrature rules (Simpson and Trapezoidal rule). This is also confirmed in Table 1 where, using the Simpson quadrature, we obtain a 5 digits accuracy with $m = 200$.

We have also investigated how the accuracy of the method changes using the Simpson or the Trapezoidal rule. For example, Figure 2 shows the results we obtained for the challenging problem described in Section 1, that is with $\sigma = 0.6$ and $x_0 = 100$. Figures 2a), 2b) show the behavior of the method when we use the Trapezoidal rule (solid line) and the Simpson rule (dotted line). We can see that the use of Simpson rule slightly improves the convergence of the method. Figures 2c), 2d) show the difference (absolute: $\|\mathbf{v}^S_{n,m} - \mathbf{v}^T_{n,m}\|_\infty$, relative: $\|\mathbf{v}^S_{n,m} - \mathbf{v}^T_{n,m}\|_\infty / \|\mathbf{v}^S_{n,m}\|_\infty$) between the solutions obtained applying the different quadrature rules versus m. Note that the relative differences is less than 10^{-4}.

Figure 3 gives an estimate of the rate of the convergence of the method when we use the trapezoidal rule (circles) and the Simpson rule (stars). In particular we plot the following ratios[7]:

$$o = \frac{\|\mathbf{v}^T_{n,2m} - \mathbf{v}\|_\infty}{\|\mathbf{v}^T_{n,m} - \mathbf{v}\|_\infty}, * = \frac{\|\mathbf{v}^S_{n,2m} - \mathbf{v}\|_\infty}{\|\mathbf{v}^S_{n,m} - \mathbf{v}\|_\infty}, \tag{5.2}$$

versus the number of grid points m, where $\mathbf{v}$ is the price of a double knock out call with $x_0 = 100$, $T = 0.5$, $r = 0.1$, $\sigma = 0.2$, $K = 100$, $l = 95$, $u = 120$ and $n = 5$. Using the Simpson rule when m is not too large seems to provide better results.

Figure 4 shows the price of a double knock-in call for several monitoring dates with $x_0 = 100$, $T = 0.2$, $r = 0.05$, $\sigma = 0.2$, $K = 100$, $l = 90$, $u = 110$. We use $m = 1200$ to get the results. The figure shows that the price of the continuously monitored barrier option is a poor approximation of the price of the discrete monitoring barrier option even when the number of monitoring dates is large.

Regarding the computational cost of the method, we observe that to get the solution at the *n-th* monitoring date for each value of $x \in [l, u]$ requires $O\left(nm^2\right)$ function evaluations (matrix $H^S_{n,m}$, $H^T_{n,m}$) plus $O\left(nm^2\right)$ elementary operations (see formulae (4.19) and (4.20)). Therefore the total cost is of $O\left(nm^2\right)$ elementary operations when $n \to \infty$, $m \to \infty$. Table 1 shows the computational cost for a barrier option with $x_0 = 0$, $T = 0.5$, $n = 25$, $r = 0.1$, $\sigma = 0.2$, $K = 100$, $l = 95$, $u = 110$. In particular this table shows the time required by the method measured in seconds, when we use the Trapezoidal rule[8]. Note that when m is multiplied by a factor of two the time is multiplied by a factor of four. This behavior confirms the estimate of the computational cost given above. Notice that with the Simpson scheme we obtain a 5 digits accuracy with $m = 200$. This computation requires only 0.16 seconds.

[INSERT TABLE 1]

[7] This ratio appears to be important in estimating the error, as from equation (4.27).

[8] The time is measured running a routine coded in Fortran on a Pentium II 600MH PC.

Table 2 shows the convergence of the discretely monitored barrier option price to the corresponding continuously monitored option price, that is, the convergence of the method as $n \longrightarrow \infty$.

[INSERT TABLE 2]

Table 2 confirms two results. First, we can see that the computational cost increases linearly in the number of monitoring dates. In fact, if we multiply the number of monitoring dates by 2, the time is multiplied by the same factor. Second, the convergence to the continuous time solution is slower as the number of monitoring dates increases. This behavior is justified by Lemma 3 and Corollary 1. In fact the reduction factor T_n goes to one when $n \longrightarrow \infty$, that is $\|v_{n+1} - v_{n+1}^m\|_\infty$ decreases slowly, since we have:

$$\|v_{n+1} - v_{n+1}^m\|_\infty \leq T_n \|v_n - v_n^m\|_\infty \approx \|v_n - v_n^m\|_\infty, \text{ as } n \longrightarrow \infty. \tag{5.3}$$

This result confirms the importance of an accurate procedure for pricing discretely monitored options. Moreover, the significant discrepancies between discrete and continuous formulas are registered also for the Greeks. For example, we have a delta of -0.025 and a gamma of -0.0475 for 10 reset dates and a delta of -0.0871 and a gamma of -0.0408 for 500 reset dates[9]. Clearly it would be dangerous to use a continuous time formula to delta-hedge a position.

In Table 3 we have compared the results obtained using our method with other approaches, such as the Markov Chain (MCh) method of Duan et al. [20], the Monte Carlo (MC) simulation method, the tree method (Tree) of Cheuk and Vorst [15], the trinomial tree (TT) of Broadie et al. [14]. In Table 4 we compare our results with those reported in Ait-Sahalia and Lai [3] and in Broadie et al. [14]. We remark that, in order to achieve a reasonable accuracy, the recursive method proposed in [3] requires a grid spacing of 0.001 or 0.0005 meaning at least 1000 or 2000 function evaluations at every monitoring date. The trinomial tree in [13] requires at least 80000 time steps and about an hour of computation per option price to obtain a four-digits accuracy .

[INSERT TABLE 3]
[INSERT TABLE 4]

In Table 5 we empirically illustrate the convergence of the scheme when computing the delta and the gamma as the grid spacing increases with the underlying price near the barrier and high volatility. A four decimal digits accuracy can be achieved with only 500 grid nodes[10].

5.2. The CEV process

Most of the research on barrier options assumes that the underlying asset follows a simple GBM. However, it is well known that this model has several drawbacks, for example the smile effect. The CEV model, introduced by Cox and Ross [18] and analysed in more detail in Cox [16], is often used among practitioners to mitigate the smile effect. The instantaneous volatility of the underlying asset is linked to the price level, which

[9] Parameters used: spot price = 100, strike = 100, l = 100, u = 110,r = 0.1, σ = 0.1, T = 0.5. The grid spacing is m = 1200.

[10] The convergence of the scheme for single barrier options results to be slower. For example using the same parameters as in Table 5 and setting u = 1000 we have a delta of 1.7573405, 1.7632039 and 1.7646095 for respectively 500, 1000 and 2000 grid points. For the gamma we have: 0.0075949, 0.0067042 and 0.0064945. Then an accetable accuracy can be obtained with 1000 grid points.

is consistent with the empirical leverage effect, a feature that Fisher Black noted as an empirical characteristic of stock prices: stock volatility has a negative correlation with variations in the asset price. In this case the dynamics of the underlying price dynamics is given by:

$$dx = rxdt + \sigma x^{\lambda/2} dW_t$$

where $0 \leq \lambda < 2$. When $\lambda = 2$, we obtain the lognormal model. This process has been recently applied to the pricing of barrier options[11], but only in the case of continuous monitoring. Therefore, as already seen in the GBM case, we can expect consistent price discrepancies with respect to the case of discrete monitoring.

The transition density of the process,which can be expressed in terms of a noncentral chi-squared distribution has been obtained using different procedures in Cox [16], Emanuel and MacBeth [21], Goldenberg [26] and Davydov and Linestky [19]. The pricing of barrier options with continuous monitoring and $\lambda = 1$ has been studied by Lo et al. [36] and is strictly related to the interest rate model of Cox et al. [17] which is based on the process studied in Feller [22].

As in Lo et al., we consider the simplest case, when $\lambda = 1$. In this case the process is known as the square-root process and the (time-homogenous) transition density is given by:

$$p(y,t;x,0) = e^{-\frac{2r}{\sigma^2(e^{rt}-1)}(e^{rt}x+y)} \frac{2r}{\sigma^2(e^{rt}-1)} \sqrt{\frac{xe^{rt}}{y}} I_1\left(4\frac{r}{\sigma^2(e^{rt}-1)}\sqrt{e^{rt}xy}\right)$$

where I_1 is the modified Bessel function of the first kind. If we set:

$$\beta = -\frac{1}{2}; \tau(t) = \frac{(1-e^{-rt})}{r}; v = 1; \gamma = \frac{4x}{\sigma^2 \tau(t)}$$

then the random variable $4e^{-rt}y/\left(\sigma^2\tau(t)\right)$ has a non-central chi-squared density with four degrees of freedom and noncentrality parameter γ.

In Table 6 we compare the prices of the discrete barrier under the GBM and CEV specifications for different monitoring dates and different strike prices. In order to make the prices of the two models comparable, the volatility parameter for the GBM has been set equal to the implied volatility obtained using as prices for the plain vanilla calls those given by the CEV model. This table shows that the relative difference between the prices computed with the two models can be greater than 10% [12]. This significant difference is confirmed when calculating the Delta and the Gamma. Using the same parameters we have obtained a delta (gamma) of 0.5779 (-0.0269) for the CEV process and a delta of 0.4692 (-0.0381) for the BS process (a 23.17% (-29.40%) difference).

[INSERT TABLE 6]

6. Conclusions and suggestions for further research

In this paper we have proposed a computationally simple method to price discrete barrier options. The idea is to approximate the price function by a linear combination of hat functions at every monitoring date. The coefficients of this linear combination can be easily obtained by recursion. Our approach can deal with time-varying barriers and can be used

[11] See for example Boyle and Tian [11] that price barrier options using a trinomial tree, Davydov and Linestky [19] using a Laplace transform approach and Lo and al. [36] using eigenfunction expansion.

[12] Using the same parameters as in Table 6 and increasing the time to maturity to 1 year, the percentage difference becomes superior to the 20%.

when the transition density of the underlying process is known analytically.. The accuracy and the computational cost of the scheme are not sensitive to the distance between the underlying asset price and the barrier. Moreover, the method allows the computation of accurate Greeks.. Finally, we have treated the GBM and CEV case in great detail. We believe that our method could be successfully applied to price Bermudian options and other path dependent exotic securities such as Asians and Lookbacks. Other extensions to multidimensional processes also deserve some attention.

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A. Appendix A

In this Appendix we give some Lemmas in order to establish the convergence of the scheme. In particular Lemma 2 and Lemma 3 state that when we can compute analytically the integrals appearing in (4.10), the convergence of v_n^m to v_n is guaranteed.

Lemma 2. *Let v_0 be a Lipschitz continuous function on [l, u], twice continuously differentiable on [l, u] except at a finite number of points, then we have:*

$$\|v_0 - v_0^m\|_\infty \leq \frac{C_{v_0}}{m} \tag{A.1}$$

and

$$\|v_1 - v_1^m\|_\infty \leq \frac{C_{v_0}}{m^2} \tag{A.2}$$

where C_{v_0} is a positive constant depending on v_0.

Proof. The proof is a consequence of [27] pag. 18, Def 1.4.10 remark 1.4.11. and some properties of the function $L_i(x)$, $i = 1, 2, \ldots$.♦

Lemma 3. *Let $w(y, \tau_n; x)$, $x \in [l, u]$, $y \in [l, u]$ satisfy the following assumption:*

$$T_n = \sup_{l \leq x \leq u} \int_l^u |w(y, \tau_n; x)| dy \leq T, \ n = 0, 1, ..., \tag{A.3}$$

where T is a positive constant, and let v_n be the solution of (4.1) with v_0 assigned and v_n^m be the solution of (4.9) and (4.11). Then we have:

$$\|v_{n+1} - v_{n+1}^m\|_\infty \leq \prod_{s=0}^{n} T_s \|v_0 - v_0^m\|_\infty \leq T^{n+1} \|v_0 - v_0^m\|_\infty \tag{A.4}$$

Proof: Subtracting equation (4.9) from (4.1) we obtain:

$$v_{n+1}(x) - v_{n+1}^m(x) = \int_l^u w(y, \tau_n; x) (v_n(y) - v_n^m(y)) \, dy, x \in [l, u] \tag{A.5}$$

and using the supremum norm in (A.5), we obtain:

$$\|v_{n+1} - v_{n+1}^m\|_\infty \leq \|v_n - v_n^m\|_\infty \sup_{l \leq x \leq u} \int_l^u |w(y, \tau_n; x)| dy \leq T_n \|v_n - v_n^m\|_\infty \tag{A.6}$$

The thesis follows by recursion on n from (A.6).♦

Lemma 4. *Let v_n, v_n^m be as in Lemma 3 and v_0 as in Lemma 2 then we have*

$$\|v_n - v_n^m\|_\infty \leq \prod_{p=1}^{n} T_p \frac{C_{v_0}}{m} \leq T^n \frac{C_{v_0}}{m}. \tag{A.7}$$

Proof: It is a straightforward consequence of Lemmas 3 and 2♦

B. Appendix B

Using expressions of the quadrature formulae, we prove the following Lemmas, where we use the matrix norm induced by the vector infinity norm, i.e., the maximum absolute row sum. The implications of these Lemmas is the result in Theorem 4.1 in the main test.

Lemma 5. *Let*

$$D_i^n = \sup_{l \le x \le u} \sup_{l \le y \le u} \left| \frac{\partial^i w}{\partial y^i} (y, \tau_n; x) \right| \tag{B.1}$$

then we have:

$$\|H_{n,m} - H_{n,m}^T\|_\infty \le (u-l) \frac{h^2}{2} D_2^n \tag{B.2}$$

and

$$\|H_{n,m} - H_{n,m}^S\|_\infty \le (u-l) \frac{h^4}{180} \left(D_4^n + 4D_3^n/h\right) \tag{B.3}$$

Proof: It follows from the quadrature rule applied to $w(\cdot, \tau_n; x) L_j(\cdot)$.♦

Lemma 6. *Let* $\mathbf{a}_{n+1,m}^T$, $\mathbf{a}_{n+1,m}^S$, $\mathbf{a}_{0,m}^T$ *and* $\mathbf{a}_{0,m}^S$ *as in equations (4.19), (4.20), (4.21). Then we have:*

$$\|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^T\|_\infty \le \|\mathbf{a}_{0,m}\|_\infty \left[\sum_{j=1}^{n} \|H_{n-j,m} - H_{n-j,m}^T\|_\infty \prod_{s=j+1}^{n} \|H_{n-s,m}^T\|_\infty \prod_{s=1}^{j-1} \|H_{n-s,m}\|_\infty \right] \tag{B.4}$$

and:

$$\|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^S\|_\infty \le \|\mathbf{a}_{0,m}\|_\infty \left[\sum_{j=1}^{n} \|H_{n-j,m} - H_{n-j,m}^S\|_\infty \prod_{s=j+1}^{n} \|H_{n-s,m}^S\|_\infty \prod_{s=1}^{j-1} \|H_{n-s,m}\|_\infty \right] \tag{B.5}$$

Proof: We have:

$$\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^T = H_{n-1}\mathbf{a}_{n-1,m} - H_{n-1}^T \mathbf{a}_{n-1,m}^T - H_{n-1,m}\mathbf{a}^T + H_{n-1,m}\mathbf{a}_{n-1,m}^T \tag{B.6}$$

and using the vector maximum norm, we have:

$$\|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^T\|_\infty \le \|H_{n-1,m} - H_{n-1,m}^T\|_\infty \|\mathbf{a}_{n-1,m}^T\|_\infty + \|H_{n-1,m}\|_\infty \|\mathbf{a}_{n-1,m} - \mathbf{a}_{n-1,m}^T\|_\infty \tag{B.7}$$

Equation (B.7) implies:

$$\begin{aligned} \|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^T\|_\infty \le\ & \|H_{n-1,m} - H_{n-1,m}^T\|_\infty \|\mathbf{a}_{n-1,m}^T\|_\infty \\ & + \|H_{n-1,m}\|_\infty \|H_{n-2,m}\|_\infty \|\mathbf{a}_{n-2,m} - \mathbf{a}_{n-2,m}\|_\infty \\ & + \|H_{n-1,m}\|_\infty \|H_{n-2,m} - H_{n-2,m}^T\|_\infty \|\mathbf{a}_{n-2,m}^T\|_\infty \end{aligned} \tag{B.8}$$

Using (B.8), (B.7) and (4.19) we have:

$$\|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^T\|_\infty \le \sum_{j=1}^{n} \|H_{n-j,m} - H_{n-j,m}^T\|_\infty \|\mathbf{a}_{n-j,m}^T\|_\infty \prod_{l=1}^{j-1} \|H_{n-l,m}\|_\infty \tag{B.9}$$

where $\prod_{l=1}^{j-1}$ is 1 if $j-1<1$.

Finally, using (4.19) into (B.9) we obtain:

$$\|\mathbf{a}_{n,m} - \mathbf{a}^T_{n,m}\|_\infty \leq \left[\sum_{j=1}^{n} \|H_{n-j,m} - H^T_{n-j,m}\|_\infty \prod_{s=0}^{n-j-1} \|H^T_{s,m}\|_\infty \prod_{s=1}^{j-1} \|H_{n-s,m}\|_\infty\right] \|\mathbf{a}_{0,m}\|_\infty \tag{B.10}$$

Arguing analogously, we obtain (B.5).♦

Corollary 3. *Let* $\tau_n = \tau$, $n = 0, 1, \ldots$ *and* $H_{n,m} = H_m$, $H^T_{n,m} = H^T_m$, $H^S_{n,m} = H^S_m$, $n = 0, 1, \ldots$, *then we have:*

$$\|\mathbf{a}_{n,m} - \mathbf{a}^T_{n,m}\|_\infty \leq \|\mathbf{a}_{0,m}\|_\infty \|H_m - H^T_m\|_\infty \sum_{j=1}^{n} \|H^T_m\|_\infty^{n-j} \|H_m\|_\infty^{n-j} \tag{B.11}$$

and

$$\|\mathbf{a}_{n,m} - \mathbf{a}^S_{n,m}\|_\infty \leq \|\mathbf{a}_{0,m}\|_\infty \|H_m - H^S_m\|_\infty \sum_{j=1}^{n} \|H^S_m\|_\infty^{n-j} \|H_m\|_\infty^{n-j} \tag{B.12}$$

Proof: It is a direct consequence of Lemma 6. ♦

Lemma 7. *Let* T_n *be the constant defined in (A.3) and let* $H_{n,m}$ *be the matrix in (4.14). We have* $\forall\ n = 1, 2, \ldots$ *and* $\forall\ m = 1, 2, \ldots$:

$$\|H_{n,m}\|_\infty \leq T_n \leq T. \tag{B.13}$$

Proof: We have:

$$\|H_{n,m}\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{m} \left| \int_l^u w(y, \tau_n; x_{i,m}) L_j(y)\, dy \right| \leq \tag{B.14}$$

$$\leq \sum_{j=1}^{m} \int_l^u |w(y, \tau_n; x_{i,m}) L_j(y)\, dy| = \int_l^u |w(y, \tau_n; x_{i,m})\, dy| \sum_{j=1}^{m} L_j(y)\, dy$$

Since $\sum_{j=1}^{m} L_j(y)=1$, $L_j(y) \geq 0$ $\forall y \in [0,1]$, $j = 1, \ldots, m$ the thesis follows.♦

Lemma 8. *Let* $w(y, \tau_n; x_{i,m})$*be a positive function (this is always satisfied being* w *a transition density). Let* T_n *and* D^n_2 *be the constant defined in (A.3), (B.1) respectively, let* $H^T_{n,m}$ *and* $H^S_{n,m}$ *be the matrices defined in formulae (4.17) and (4.18), when* $D^n_2 < +\infty$, $n = 1, 2, \ldots$, *we have:*

$$\|H^T_{n,m}\|_\infty \leq T_n,\ n = 1, 2, \ldots \tag{B.15}$$

and

$$\|H^S_{n,m}\|_\infty \leq T_n, n = 1, 2, \ldots \tag{B.16}$$

Proof: We give the proof for $H^T_{n,m}$.

$$\begin{aligned}
\|H^T_{n,m}\|_\infty &= \max_{1 \leq i \leq m} \sum_{j=1}^{m} \left| \left(H^T_{n,m}\right)_{i,j} \right| = \max_{1 \leq i \leq m} \sum_{j=1}^{m} \left(H^T_{n,m}\right)_{i,j} \\
&\leq \lim_{m \to +\infty} \sup_{1 \leq i \leq m} \left\{ \int_l^u |w(y, \tau_n; x_{i,m})|\, dy + \frac{(u-l)}{12} h^2 D^n_2 \right\} \\
&\leq T_n + \frac{(u-l)}{12} D^n_2 \lim_{m \to +\infty} h^2 = T_n
\end{aligned} \tag{B.17}$$

where we have used the fact that $h = (u-l)/(m-1)$. This concludes the proof.♦

Lemma 9. *Let n be fixed. Then:*

$$\lim_{m \to \infty} \|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^{T}\|_\infty = 0 \tag{B.18}$$

and

$$\lim_{m \to \infty} \|\mathbf{a}_{n,m} - \mathbf{a}_{n,m}^{S}\|_\infty = 0 \tag{B.19}$$

Proof. The proof is a consequence of Lemma 6, Lemma 7 and Lemma 8 taking into account that $\|\mathbf{a}_{n,m}\|_\infty \leq \|v_0\|_\infty$ (see (4.11)).♦

Table 1. Computational cost of the trapezium quadrature increasing the number of nodes m on the grid-spacing. The first column gives the number of nodes in the discretization, the second column the computational time (in seconds) required to estimate the price, the third and fourth column the estimated price using Trapezium and Simpson approximation respectively. Parameters: spot price=100, strike =100, l=95, u=110, r=0.1, σ=0.2, T=0.5, n=25.

m	Time	Trapez.	Simpson
100	0" 0469	0.162841	0.162972
200	0"16	0.162950	0.162987
400	0"77	0.162978	0.162987
500	1"32	0.162981	0.162986
1000	5"28	0.162985	0.162987
1200	7"64	0.162986	0.162987
1500	12"08	0.162986	0.1629896
2000	21"80	0.162987	0.162987
2400	31"70	0.162987	0.162987

Table 2. Convergence of the estimated price to the continuous monitoring solution. In the first column we have the number of monitoring dates, in the second the computational time (in seconds) required and in the third and fourth column the estimated prices using Trapezium and Simpson approximation. Parameters: spot price=100, strike =95, l=90, u=110, r=0.05, σ=0.2, T=1, m =1200, continuous monitoring solution= 0.043002

n	Time	Trapez.	Simpson
50	14"23	0.163939	0.163941
100	27"46	0.118936	0.118938
150	40"75	0.101691	0.101692
300	80"52	0.081674	0.081677
600	160"05	0.069166	0.0691695
1200	319"18	0.061126	0.06111
4800	1333"37	0.05226	0.05226
12000	3324"42	0.04924	0.049254

Table 3. Comparisons of the proposed scheme with the Markov Chain method (MCh) using the results reported in [20], Montecarlo simulation (MC), and Tree method using the results reported in [14]. Parameters: spot price=100, strike =100, r=0.10, σ=0.2, T=0.5. The MCh grid has 1001 points. The MC method has been conducted using the exact solution and with 10000 simulations and a time step equal to the distance between monitoring dates. In the Trap and Simpson method the grid spacing has been set using $m = 1200$.

l	u	n	MCh (1001)	MC (10000)	Tree	Trap.	Simpson
95	110	125	0.0756	0.0752	0.0758	0.0757	0.0757
95	125	125	2.4802	2.4822	2.4823	2.4818	2.4818
95	150	125	5.7998	5.7919	5.7999	5.7991	5.7993
95	110	25	0.1630	0.1633	0.1594	0.1630	0.1630
95	125	25	3.0058	3.0160	2.9895	3.0060	3.0061
95	150	25	6.2990	6.2757	6.2849	6.2990	6.2990
95	∞	125	6.1679	6.1662	6.1692	6.1657	6.1687
99.5	∞	125	1.9618	1.9580	1.9624	1.9579	1.9628
99.9	∞	125	1.5138	1.5104	1.5116	1.5082	1.5123
95	∞	25	6.6307	6.6370	6.6181	6.6305	6.6317
99.5	∞	25	3.3552	3.3494	3.3122	3.3588	3.3564
99.9	∞	25	3.0095	3.0118	2.9626	3.0084	3.0098

Table 4. Comparisons of the proposed scheme with the recursive integration method (RI) in [3] where a grid with 2000 nodes has been used, the continuity correction based on shifting the barrier level in [14] and the trinomial tree in [13]. In the table we consider a single barrier down-out call option for different levels of l and different monitoring dates. Parameters: spot price=100, strike =100, u=∞,r=0.10, σ=0.3, T=0.2. The grid spacing for the Trapezium and Simpson method is $m = 1200$.

l	n	RI	Cont. Corr.	Trinom	Trap.	Simpson
99	5	4.4848	4.050	4.489	4.4891	4.4894
97	5	5.1628	5.028	5.167	5.1672	5.1675
95	5	5.6667	5.646	5.671	5.67098	5.6712
89	5	6.2763	6.284	6.281	6.2808	6.2809
99	25	2.8036	2.673	2.812	2.8114	2.8128
97	25	4.1064	4.113	4.115	4.1146	4.11598
95	25	5.0719	5.084	5.081	5.0805	5.0815
89	25	6.2003	6.210	6.21	6.2099	6.2101

Table 5a. Delta and Gamma for discrete Barrier Options in the BS model for different monitoring dates using the Simpson scheme (the Trapezium gives the same results to the fourth digit). Parameters: spot price=100, strike =100, l=100, u=110,r=0.1, σ=0.1, T=0.5. The grid spacing is $m = 1200$.

n	DeltaSP	GammaSP
10	-0.025	-0.0475
50	-0.0678	-0.0438
100	-0.0766	-0.0426
150	-0.0802	-0.042
200	-0.0823	-0.0417
300	-0.0847	-0.0413
400	-0.0862	-0.041
500	-0.0871	-0.0408

Table 5b. Convergence of the scheme in the computation of Delta and Gamma for discrete double Barrier Options in the BS model as the grid spacing increases using the Trapezium and the Simpson scheme. Parameters: strike = 100, spot price = 100, l=95, u=130 r=0.1, σ=0.6, T=0.2, monitoring dates = 50.

m	DeltaTr	GammaTr	DeltaSP	GammaSP
200	0.1226076	-0.0035806	0.1226398	-0.0035830
250	0.1226369	-0.0035796	0.1226574	-0.0035812
500	0.1226375	-0.0035871	0.1226428	-0.0035875
1000	0.1226325	-0.0035904	0.1226338	-0.0035905
1200	0.1226278	-0.0035918	0.1226287	-0.0035919
1500	0.1226300	-0.0035914	0.1226306	-0.0035915
2000	0.1226310	-0.0035909	0.1226313	-0.0035909
2400	0.1226332	-0.0035908	0.1226335	-0.0035908

Table 6. Prices for discrete Barrier Options in the CEV and BS model for different monitoring dates using the Simpson scheme. Parameters: spot price=20, l=15, u=30,r=0.1, σ_{CEV}=$\sqrt{0.12 * 20}$, $T = 0.5$, $m = 1200$. The percentage difference has been calculated using the price in the BS worlds as reference. In order to make comparable the two models, the volatility parameter in the BS model has been set equal to the implied volatility calculated using the BS formula given the prices for plain vanilla call options in the CEV model (some authors set $\sigma_{BS}^2 = \sigma_{CEV}^2/S$. In the present case then it should be $\sigma_{BS} = \sqrt{0.12} = 0.34641$ approximately the volatility we get for at-the-money volatility).

n	Strike	Simpson CEV	σ_{BS}	Simpson BS	% Difference
1	15	4.9812	0.3721	4.579	8.78%
1	17.5	3.1319	0.3583	2.8773	8.85%
1	20	1.7218	0.3466	1.572	9.53%
1	22.5	0.7929	0.3365	0.7149	10.91%
1	25	0.278	0.3277	0.2459	13.05%
10	15	4.67	0.3721	4.1376	12.87%
10	17.5	2.9543	0.3583	2.6338	12.17%
10	20	1.6064	0.3466	1.4238	12.82%
10	22.5	0.718	0.3365	0.6275	14.42%
10	25	0.2372	0.3277	0.2032	16.73%
20	15	4.5705	0.3721	4.0076	14.05%
20	17.5	2.8922	0.3583	2.5567	13.12%
20	20	1.5651	0.3466	1.3754	13.79%
20	22.5	0.6911	0.3365	0.5983	15.51%
20	25	0.2224	0.3277	0.1885	17.98%
50	15	4.4648	0.3721	3.8726	15.29%
50	17.5	2.8255	0.3583	2.4761	14.11%
50	20	1.521	0.3466	1.3251	14.78%
50	22.5	0.6628	0.3365	0.5682	16.65%
50	25	0.2071	0.3277	0.1734	19.43%
100	15	4.4046	0.3721	3.7971	16.00%
100	17.5	2.7875	0.3583	2.4309	14.67%
100	20	1.496	0.3466	1.297	15.34%
100	22.5	0.6469	0.3365	0.5515	17.30%
100	25	0.1985	0.3277	0.1651	20.23%

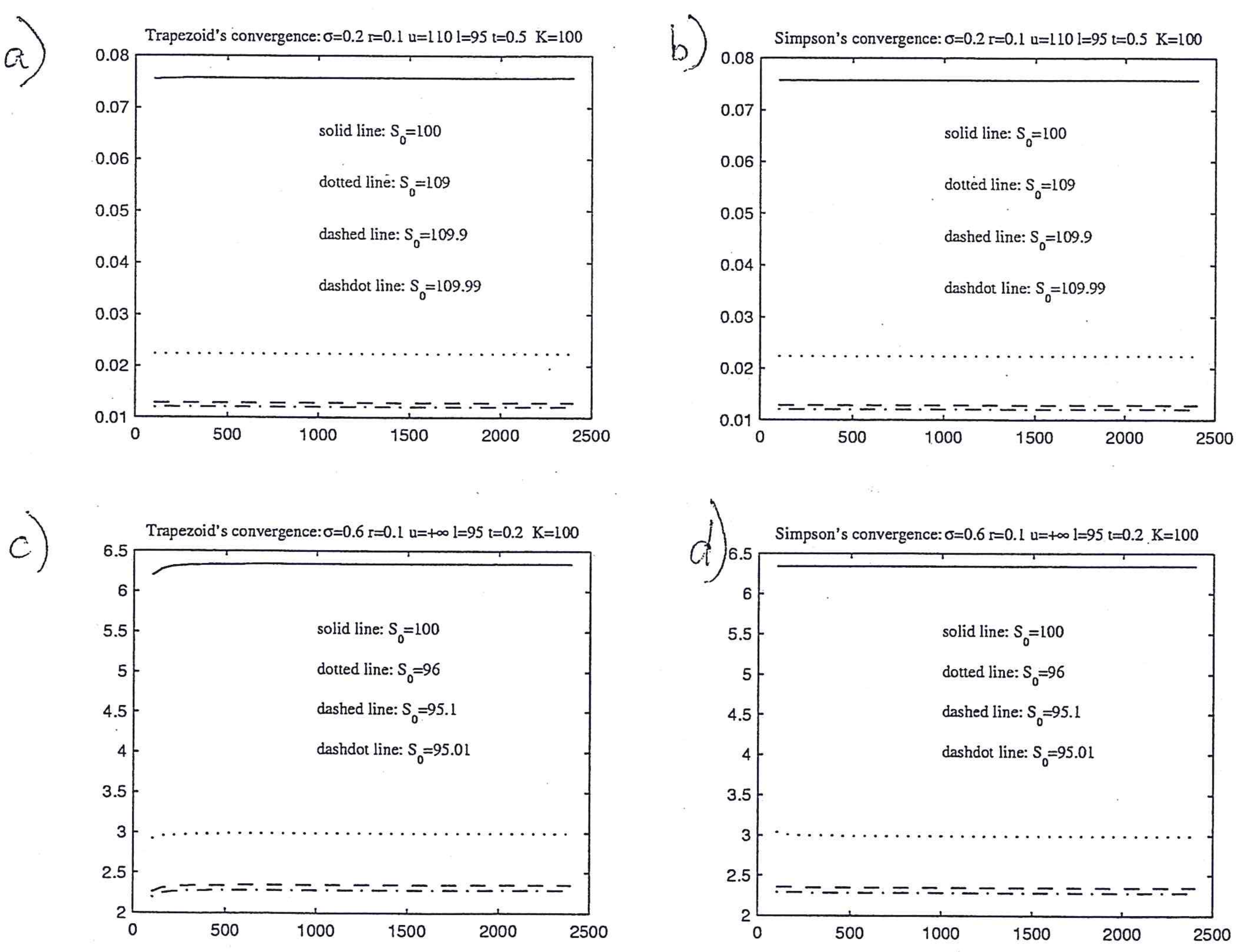


Figure 1: Price of a double knock-out call option versus the number m of grid points and for different stock prices.

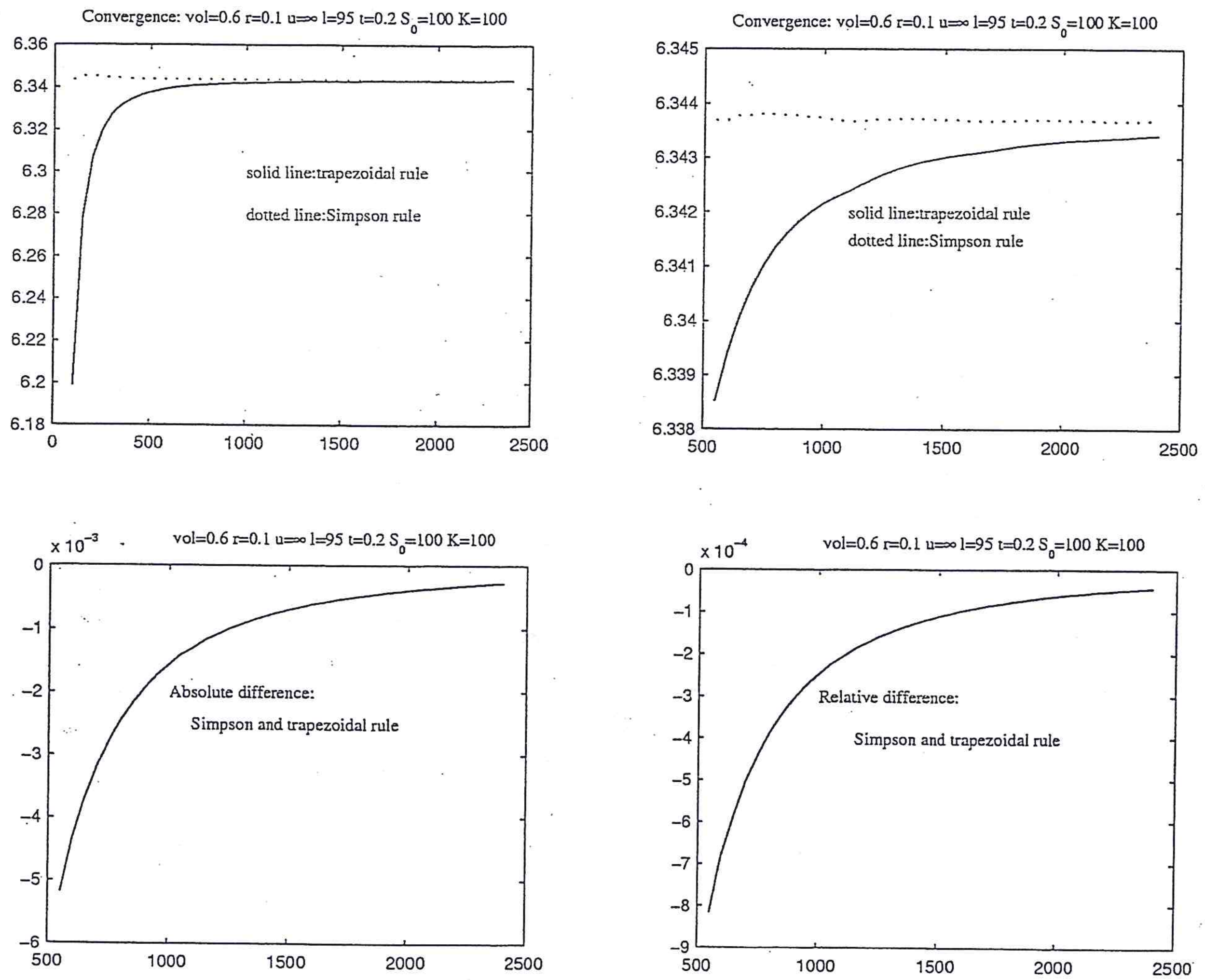


Figure 2: Convergence of the method versus the number m of grid points

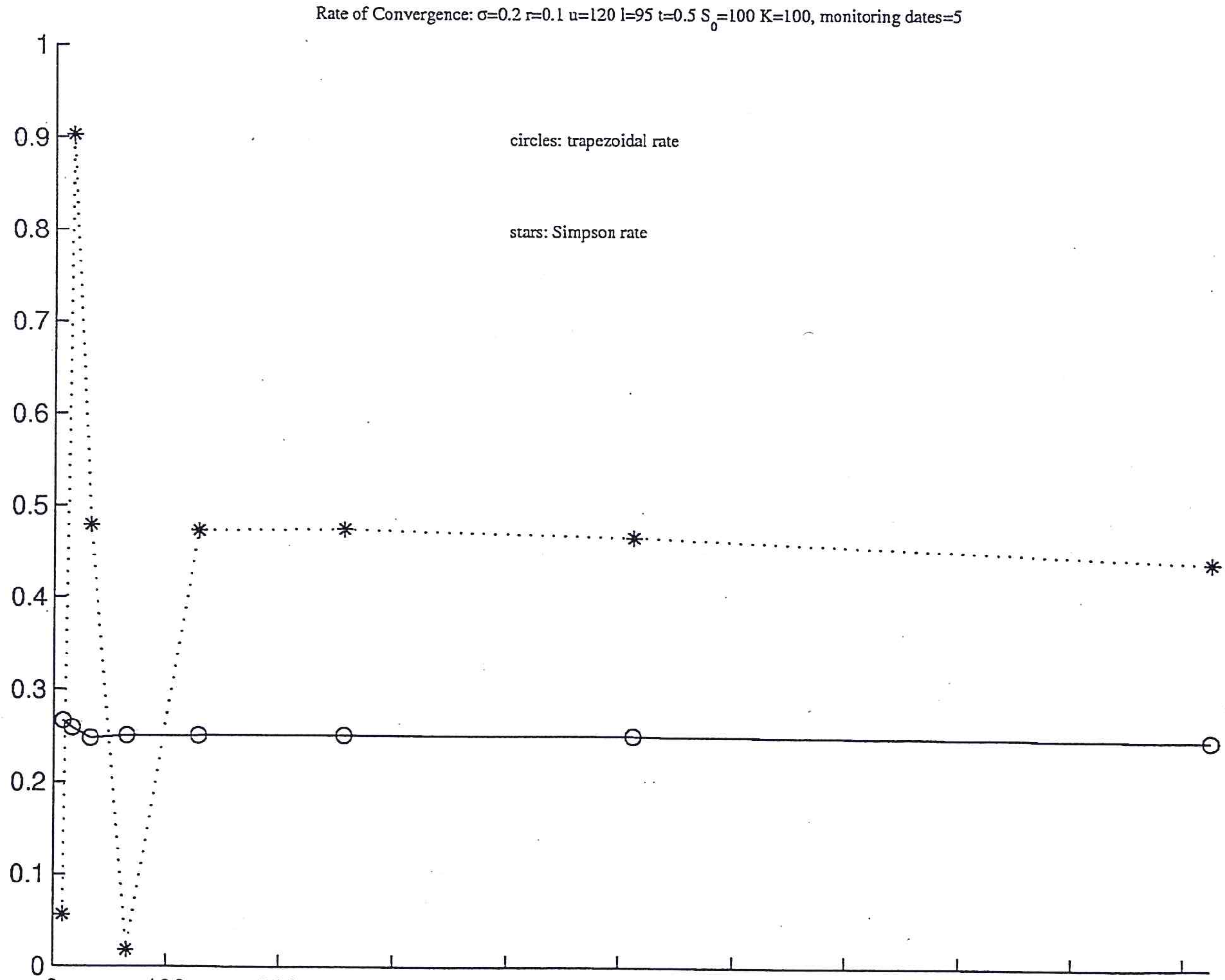
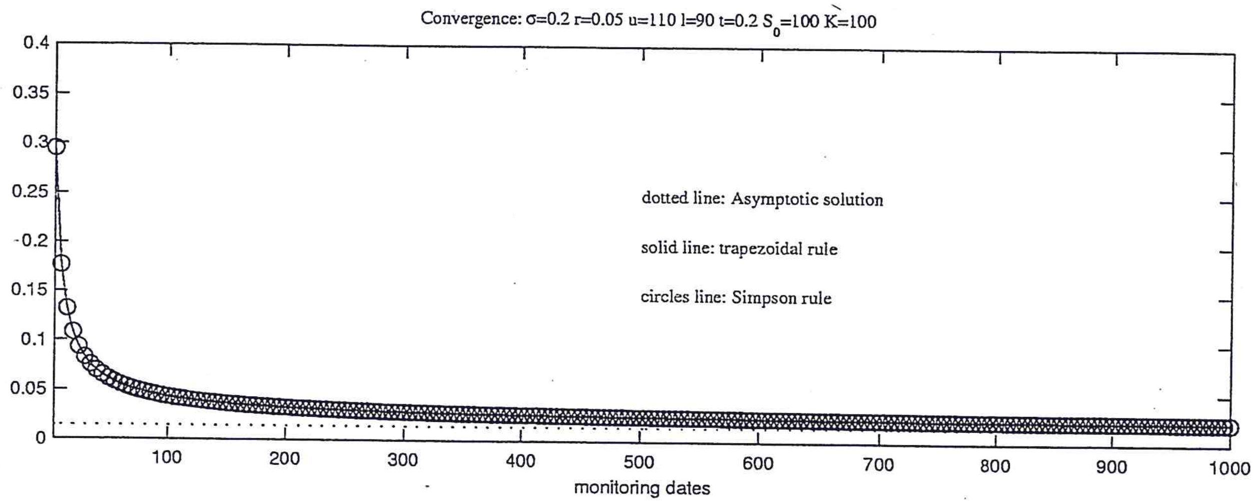


Figure 3: Rate of convergence of the Trapezoidal and Simpson versus the number m of grid points



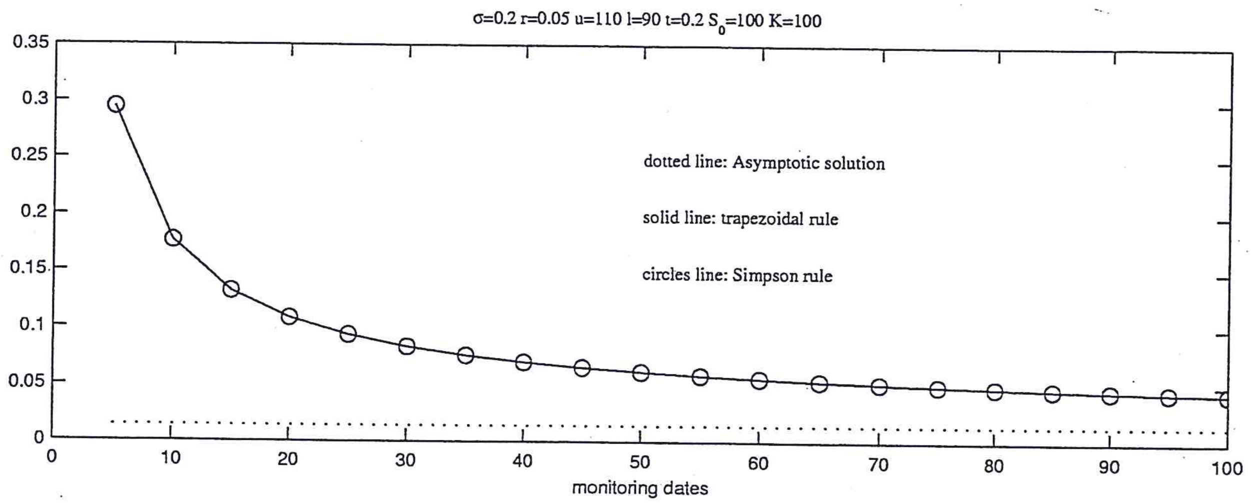


Figure 4: Price of a double knock-out call option versus the number of monitoring dates