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Copula Based Monte Carlo Integration in Financial Problems

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Abstract

A computational technique that transform integrals over $\mathbb{R}^K$, or some of its subsets, into the hypercube $[0,1]^K$ can be exploited in order to solve integrals via Monte Carlo integration without the need to simulate from the original distribution; all that is needed is to simulate iid uniform $[0,1]$ pseudo random variables. In particular the technique arises from the copula representation of multivariate distributions and the use of the marginal quantile function of the data. The procedure is further simplified if the quantile function has closed form. Several financial applications are considered in order to highlight the scope of this numerical techniques for financial problems.

JEL: C15, G11, G12

Keywords: Copula, Martingale, Monte Carlo Integral, Quantile Transform, Utility Function.

1 Introduction

Mathematical and computer modelling in financial problems almost always requires the calculation of some sort of expectation in the form of integral and its optimization; i.e. expected utility, expected payoffs, expected losses. The purpose of this

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paper is to provide a general technique for the numerical solution of multidimensional integrals via Monte Carlo (MC) integration. The standard MC integral only requires us to evaluate the function at uniform random points over the support of the integrand. However, this can be inefficient. A partial solution to the problem is to use a kernel density function from which to simulate data (e.g. Geweke, 1996). The function is chosen to minimize the error of the integrand intuitively giving more weight to regions that give a larger contribution to the integral. This kind of MC integral is called the importance sampling MC (ISMC) integral. The method proposed in this paper provides a new practical approach to the implementation of the ISMC integral.

We consider expectations of payoff functions with respect to arbitrary multidimensional distributions. Unlike conventional ISMC integration, we do not need to simulate from the multidimensional distribution. We only need to be able to simulate iid uniform $[0, 1]$ random variables, as we would usually do with the standard MC integral, or to construct quasi-random numbers in order to reduce the number of simulations (e.g. Spanier and Maize, 1994).

Consider the expectation of some function with respect to the multivariate Gaussian density. We cannot directly simulate from a multivariate normal distribution with given covariance matrix. Therefore, ISMC integration could be problematic. The simplest alternative is to use standard MC integration. However, to do so, we need to consider a subset, say $A^K \subset \mathbb{R}^K$, such that its Lebesgue measure $\left|A^K\right| < \infty$. It is well known that as $K \to \infty$ the error in this approximation goes to infinity: this is the well known curse of dimensionality (e.g. Scott, 1992). Fortunately, the fast decay of the tails of the Gaussian distribution justifies the approximation for any finite K. However, problems arise with other joint pdfs which are more suitable in finance. Our method avoids approximation of the support whenever we can numerically compute the quantile function of the marginal distributions at a finite number of points. Clearly, the focus is not on the Gaussian case. (But, we consider the case of continuous martingales. Since any continuous martingale is a time change of Brownian motion, we will consider the Gaussian transition distribution.) Rather, we look at computations that are more suitable to financial problems that try to capture the salient features of financial returns, e.g. fat tails and nonlinear dependence. However the use of these distributions is common in fields other than economics, so the approach used in this paper can be extended to other cases of interest.

The Gaussian distribution allows us to derive closed form solutions for several financial problems, e.g. portfolio optimization with suitable choice of utility. Using more general distributions, analytical closed form solutions are very rare and confined to specific cases. For this reason Gaussianity has received great attention. Unfortunately, computational problems that arise from a more general specification of financial problems often prevent us from using interesting techniques which in principle could provide more satisfactory answers. Our approach finds further justification in this context. For this reason we consider several financial applications to illustrate the scope of the numerical technique proposed.

The reluctance to use more sophisticated tools is such that normality and linear measures of dependence are used even when empirical evidence seems to disprove them. As is well known, financial returns are found to be leptokurtic (e.g. Mandelbrot, 1963, 1997, and Fama, 1965), they exhibit complex nonlinear time dependence (e.g. Ding et al., 1993) and their cross dependence among countries (e.g. US and European Markets) and shares within the same market is highly dependent during downside movements and not as much during bull markets (e.g. Longin and Solnik, 2001).

Several solutions have been proposed to some of these problems. Our method takes full advantage of these solutions. In fact, using the copula function representation of multivariate distributions, we show that the integral of any nonlinear payoff function with respect to the distribution of the payoff can be transformed from an integral in, say $\mathbb{R}^K$, to an integral in $[0,1]^K$. Then, we can easily compute the expected value over a compact interval. When the copula is absolutely continuous, this only requires us to compute expectation with respect to the Lebesgue measure over $[0,1]^K$. Simulating $[0,1]^K$ observations, the expected value is just given by the mean of the integrand times the copula density evaluated at these simulated points. Once a simulated sample is available (notice that the sample can be the same from application to application) we only need to compute a mean like statistic.

The procedure requires us to compute the univariate quantile function of the variables concerned. This is a feasible numerical problem. However, in order to keep our method simpler, we may like to use marginal distributions which have inverses with a closed form expression. It is well known that rational powers of the argument of the exponential distribution lead to distributions with closed form quantile function. Incidentally, for the intermediate time scale, e.g. daily returns, the use of densities proportional to an exponential of a rational power have been advocated as the most reasonable choice of distributions (e.g. Frisch and Sornette, 1997, and Laherrère and Sornette, 1998).

Therefore, we can couple ideal numerical properties with a judicious modelling approach. For this reason we also propose a family of distributions that both possess these properties and satisfies many others, which are relevant for unconditional returns distributions. A special case of this family of distributions can be found in Sancetta and Satchell (2001). All we do is to generalize it using an approach in the spirit of Knight et al. (1995).

Therefore, our method has the following advantages. 1. It makes direct use of the copula function. This is a benefit, as all distributions with continuous marginals have a unique copula representation. Only a few multivariate densities (e.g. multivariate normal or t) can be written analytically without need to use the copula representation. Unfortunately, this densities exhibit symmetric dependence and are not particularly adequate for financial applications. Therefore, the copula function is a quite important modelling tool. 2. The procedure is directly applicable to integrals over $\mathbb{R}^K$. 3. It does not require to simulate from multivariate distributions. This is often a difficult task. 4. It is a universal transformation which does not change from situation to situation. 5. Our theoretical and numerical studies show that the error can be diminished with respect to other MC integration procedures when our methodology is employed. 6. Unlike competitive approaches based on importance sampling, our approach directly extend to the use of quasi-random numbers (e.g. Spanier and Maize, 1994, for a definition of quasi-random numbers)

The plan for the paper is as follows. In Section 2 we introduce the copula based MC integration. In Section 3 we study the method both theoretically and by three numerical examples. Section 4 further discusses the copula based MC integration: several financial applications are provided to highlight the scope of the technique. Some appendices provide background material.

In the paper we will use De Finetti's linear functional notation. Suppose $\mathbf{X}$ is a random variable with values in $\left(S^K, \mathcal{S}^K\right)$, $K \geq 1$, some subspace of $\left(\mathbb{R}^K, \mathcal{R}^K\right)$ or $\left(\mathbb{R}^K, \mathcal{R}^K\right)$ itself. Let $\mathbb{Q}$ be a measure. Then we write $\mathbb{Q}\mathbf{X}$ for the integral of $\mathbf{X}$ with respect to $\mathbb{Q}$ over S^K, i.e. the expectation of $\mathbf{X}$ with respect to $\mathbb{Q}$. Define λ^K to be the Lebesgue measure on S^K and $\mathbb{P}_N^{\mathbf{x}_i} := N^{-1} \sum_{i=1}^N \delta_{\mathbf{x}_i}$ where $\delta_{\mathbf{x}_i}$ is the Dirac measure at $\mathbf{x}_i$ (i.e. $\delta_{\mathbf{x}_i}\mathbf{X} = 1$ if $\mathbf{X} = \mathbf{x}_i$, 0 otherwise) to be the empirical measure. Moreover, I_A stands for the indicator function of a set A and $|A|$ for its Lebesgue measure. Moreover, we will use $\mathfrak{G}$ to denote a class of functions. For example, $\mathfrak{G}$ could be a family of functions which depends on some parameters in a compact space, e.g. a portfolio variance with portfolio weights in the unit simplex.

2 Copula-Based Monte Carlo Integration

Our technique relies on a non-linear transform for arbitrary integrals. This transform is obtained using the copula representation of multivariate distribution functions. This representation is central to the development of the whole paper and exploits the fact that simulation from univariate distributions is an easy task. In multidimensional MC integration over an infinite support, one uses a multivariate kernel density function from which the data are simulated (e.g. Geweke, 1996, p. 757, for more details on kernel density functions for Monte Carlo integration). Univariate distribution functions are one to one mappings, for this reason numerical simulation can be easily implemented. On the other hand, simulation from multivariate distributions can be very challenging.

Suppose $g : S^K \to \mathbb{R}$, and we want to compute its integral with respect to the probability measure $\mathbb{P}$. Suppose P is the K dimensional joint distribution function related to $\mathbb{P}$ with density with respect to the Lebesgue measure equal to p. Then,

$$\begin{aligned} \mathbb{P}g(\mathbf{X}) &= \int_{S^K} g(\mathbf{x})\, p(\mathbf{x})\, d\mathbf{x} \\ &= \lambda^K [g(\mathbf{x})\, p(\mathbf{x})], \end{aligned}$$

Suppose $\left|S^K\right| < \infty$, and $\mathbf{x}_1, ..., \mathbf{x}_N$ is a sample of pseudo-uniform random variables in S^K; then $\mathbb{P}_N^{\mathbf{x}_i}\left[G(\mathbf{X})\left|A^K\right| p(\mathbf{X})\right]$ is an estimator for $\lambda^K [G(\mathbf{X})\, p(\mathbf{X})]$ in the statistical sense. This is the basic result which justify Monte Carlo integration. More generally, we may be interested in integral optimization problems, i.e.

$$\sup_{g \in \mathfrak{G}} \mathbb{P}g(\mathbf{X}),$$

where $g : S^K \to \mathbb{R}$ is a function in some class of functions $\mathfrak{G}$. We have in mind the maximization of expected utility and many other simulation problems in financial economics. This procedure requires uniform convergence over $\mathfrak{G}$.

In most financial applications, $S^K = \mathbb{R}^K$. In this case, the usual MC integral for functions with infinite support requires us either to simulate data from a multivariate kernel density function, or to truncate the support to a level, say A^K, where $\left|A^K\right| < \infty$ such that $\mathbb{P}A^K = 1 - \epsilon$ for ϵ very small. A measure $\mathbb{P}$ which satisfies this requirement for A^K compact and arbitrary $\epsilon > 0$ is called tight (e.g. van der Vaart and Wellner, 2000, for details). Tightness of measures allows us to apply MC integration to random variables with infinite support by truncation to a suitable level.

Consistency of MC integration follows from results in the statistical literature and in particular from the empirical process theory. In particular, the MC integral can be shown to be Glivenko-Cantelli (i.e. uniformly almost surely convergent) over a wide class of functions and Donsker (i.e. converges weakly to a Gaussian process). The reader is referred to van der Vaart and Wellner (2000) for general details on uniform convergence. Weak convergence (i.e. convergence in distribution) is a desirable property used to bound the error in the integration by probabilistic arguments when pseudo-random numbers are used.

However, it is well known that the convergence rate attained using uniform random numbers is of order $N^{-\frac{1}{2}}$ as opposed to the rate attained by the use of quasi-random numbers. In this case, the error is $O\left(N^{-1}(\ln N)^K\right)$ whenever the integrand with respect to the Lebesgue measure is of bounded variation (e.g. Spanier and Maize, 1994). The method used in this paper is clearly not restricted to the use of pseudo-random numbers. However, for simplicity of exposition, we will usually only refer to the use of uniform pseudo-random numbers.

2.1 Linear Transformations for MC Integration

Let $\mathbf{X} := (X_1, ..., X_K)$ be a random vector with values in $\left(S^K, \mathcal{S}^K\right)$ and $g : S^K \to S$. Let $\mathbb{P}$ be a probability measure (clearly, the discussion is valid for any positive finite measure). Suppose that the Radon-Nikodyn derivative of $\mathbb{P}$ with respect to λ^K exists. Then, consider the following

$$\mathbb{P}g(\mathbf{X}) = \int_{S^K} g(\mathbf{x})\, p(\mathbf{x})\, d\mathbf{x},$$

where $p(\mathbf{x})$ is the density corresponding to $\mathbb{P}$. If $\left|S^K\right| < \infty$, then there exist a linear transformation $\mathcal{T} : [0,1]^K \to S^K$ such that $\mathbf{x} := \mathcal{T}\mathbf{u}$ $\left(\mathbf{u} \in [0,1]^K\right)$ and

$$\mathbb{P}g(\mathbf{X}) = \lambda^K \left[p(\mathcal{T}\mathbf{u})\left|S^K\right| G(\mathcal{T}\mathbf{u})\right];$$

e.g. for $S^K := [0, b]^K$

$$(x_1, ..., x_K) = (bu_1, ..., bu_K).$$

Then we can generate $\mathbf{u}_1, ..., \mathbf{u}_N$, which are $[0,1]^K$ pseudo-uniform random numbers. The Monte Carlo integral is then defined as

$$\mathbb{P}_N^{\mathbf{u}_i}\left[p(\mathcal{T}\mathbf{u})\left|S^K\right| g(\mathcal{T}\mathbf{u})\right].$$

Whenever possible, one may generate a sample $\mathbf{x}_1, ..., \mathbf{x}_N$ from $p(\mathbf{x})$ and define the MC integral as $\mathbb{P}_N^{\mathbf{x}_i} g(\mathbf{X})$ directly. This is an importance sampling MC integral.

2.2 Importance Sampling via Non-Linear Change of Variables

Under the same conditions of the previous subsection, there exist a non-linear transformation $\mathcal{Q} : [0,1]^K \to S^K$ such that for $\mathbf{x} :=\mathcal{Q}\mathbf{u}$ and the Jacobian $J(\mathbf{u}) := \det |d\mathbf{x}/d\mathbf{u}|$

$$\mathbb{P}g(\mathbf{X}) = \lambda^K [p(\mathcal{Q}\mathbf{u}) J(\mathbf{u}) g(\mathcal{Q}\mathbf{u})].$$

This transformation does not require $|S^K| < \infty$. Then, by usual MC integration, we generate $\mathbf{u}_1, ..., \mathbf{u}_N$, which are $[0,1]^K$ uniform pseudo random numbers. The MC integral is then defined as

$$\mathbb{P}_N [p(\mathcal{Q}\mathbf{u}) J(\mathbf{u}) g(\mathcal{Q}\mathbf{u})].$$

A natural non-linear transformation is based on the quantile transform, which is related to the copula representation of joint distributions. We recall the definition of copula function. Let P be a K dimensional distribution function with one dimensional marginals $F_1, ..., F_K$. There then exists a function C from the unit K cube to the unit interval such that

$$P(x_1, ..., x_K) = C(F_1(x_1), ..., F_K(x_K));$$

C is referred to as the K-Copula. If each F_j is continuous, the copula is unique (e.g. Sklar, 1973).

Since the whole theory and construction of copula functions is independent of the marginals we may define joint distributions by looking at the copula and the marginals separately. There are many different copulae. For financial applications, one would look at those that allow us to model increasing dependence as we move to the tails together with asymmetry between lower and upper tail (see Joe, 1997, for a list of copulae). More details on the copula function can be found in Appendix A.

Let $P(\mathbf{x})$ be the joint distribution of the vector random variable $\mathbf{X} = (X_1, ..., X_K)$ with respective marginal distribution functions $F_1(x_1), ..., F_K(x_K)$. Further, let $C(u_1, ..., u_K)$ be their copula with copula density $c(u_1, ..., u_K)$. If the marginals are continuous, we know that the copula is unique. Let $g : S^K \to S$ be a uniformly integrable function, and suppose we want to compute $\mathbb{P}g(\mathbf{X})$. If $\mathbf{X}$ had compact support, it would be quite simple to compute the corresponding integral. However, if $S^K = \mathbb{R}^K$, using the copula function, $\mathbb{P}g(\mathbf{X})$ can be written as

$$\begin{aligned}\mathbb{P}g(\mathbf{X}) &= \int_{\mathbb{R}^K} g(\mathbf{x})\, p(\mathbf{x})\, d\mathbf{x} \\ &= \int_{[0,1]^K} g\left(F_1^{-1}(u_1), ..., F_K^{-1}(u_K)\right) c(u_1, ..., u_K)\, du_1, ..., u_K,\end{aligned}$$

where F_j^{-1} is the quantile function, i.e. the inverse of F_j. This shows that the Riemann Stieltjes integral over $\mathbb{R}^K$ can be transformed into an equivalent integral over $[0,1]^K$. Therefore, the above integral is just the expectation of

$$g\left(F_1^{-1}(u_1), ..., F_K^{-1}(u_K)\right) c(u_1, ..., u_K) \tag{1}$$

with respect to the Lebesgue measure λ^K in $[0,1]^K$. We can directly solve the integral by Monte Carlo simulation. This is done by simulating N $(K \times 1)$-vectors of iid uniform $[0,1]$ random variables and evaluating (1) at these points under the empirical measure, i.e. $\mathbb{P}_N^{\mathbf{u}_i}$. This makes the solution of such a problem trivial. (This procedure implicitly requires that F^{-1} can be computed either in closed form or numerically at each simulated point. Choosing a marginal distribution with closed form quantile function may considerably simplify the procedure.) It is apparent the similarity of our approach to importance sampling MC integration. The non-linear change of variables is such that the uniform marginal density over $[0,1]^K$ becomes the natural choice of kernel density from which to simulate.

If the copula is not differentiable, we can simulate from $C(u_1, ..., u_K)$ and then use those N simulated $(K \times 1)$-vectors of $[0,1]$ uniforms to compute the expectation of $g(\mathcal{Q}\mathbf{u})$ under the empirical measure. Simulation from a copula is a standard procedure. (For convenience we collect some of the results in Appendix C.) However, simulation from a copula may lead to several problems. In fact, a closed form solution for the inverse of a conditional copula may not exist, and this would also require further numerical computations. However, (1) allows us to circumvent these problems, only requiring a large enough sample of iid $[0,1]^K$ uniform random variables. (Notice that simulating from a copula requires it to be differentiable. Therefore, there is no advantage in computing the expectation of g with respect to dC instead of using (1) as integrand and computing its expectation with respect to the Lebesgue measure. However, when using data simulated from a copula, we have to evaluate a simpler function, i.e. g, instead of a more complicated one, i.e. (1).) Clearly, the simulated sample may be the same from application to application, keeping the implementation simple and similar to the calculation of mean statistics. Moreover, our method is perfectly viable even when quasi-random numbers are used in order to further reduce the error.

3 Properties of the Method

In this section we look more in depth at the properties of the method by both theoretical and numerical evidence. In the case of pseudo random numbers, the error in the MC integral can be estimated by

$$\begin{aligned} s_N^2 &= \frac{1}{N-1}\sum_{i=1}^{N} g\left(\mathbf{x}_i\right)^2 - \frac{N}{N-1}\left[\sum_{i=1}^{N} g\left(\mathbf{x}_i\right)\right]^2 \\ &= \frac{2}{N\left(N-1\right)}\sum_{1\leq i<j\leq N}\left[g\left(\mathbf{x}_i\right)-g\left(\mathbf{x}_j\right)\right]^2. \end{aligned}$$

To provide more intuition on our results, suppose g is Lipschitz of order one in each argument (e.g. DeVore and Lorentz, 1993, for definitions). Let $\langle , \rangle$ be the inner product and $\mathbf{C}_K$ a K-dimensional vector such that $\langle \mathbf{C}, \mathbf{C}\rangle < \infty$. Then, for $\mathbf{C}_K$ being the Lipschitz constant of g,

$$\left|g\left(\mathbf{x}_i\right)-g\left(\mathbf{x}_j\right)\right| \leq \left\langle \mathbf{C}_K, \left|\mathbf{x}_i-\mathbf{x}_j\right|\right\rangle \tag{2}$$

so that s_N^2 depends on both the smoothness of g and on the points $\mathbf{x}_1, ..., \mathbf{x}_N$. On the other hand, should we use quasi-random numbers, then the Koksma-Hlawka inequality gives an upperbound in terms of the variation of g and the discrepancy of $\mathbf{x}_1, ..., \mathbf{x}_N$ (e.g. Spanier and Maize, 1994, for an excellent review and definitions).

From (2), we may deduce that for a given N, it is reasonable to use a high proportion of data for those areas where the integrand is less smooth, and a relatively smaller proportion for the rest, in order to improve on the MC methodology. After discussing some of the properties of the method in details, we will outline a procedure that takes advantage of previous knowledge concerning the smoothness of the integrand in order to reduce the error.

3.1 Comparing the Error of the MC Integral under Different Transformations

As mentioned above, the error involved in MC integration can be bounded using probabilistic arguments if iid random numbers are used, or the Koksma-Hlawka inequality in the case of quasi-random numbers. Here we will compare the usual MC integral with our new approach. Consider the following integral

$$\int_{A^K} f(\mathbf{x})\, d\mathbf{x}$$

and a discrete approximation of A^K ($\left|A^K\right| < \infty$) by N hyperrectangles $\mathbf{h}_i$, $i = 1, ..., N$ such that $|\mathbf{h}_i| \leq h^K$. Therefore, $f(\mathbf{x})\, d\mathbf{x} \simeq f(\mathbf{x})\, |\mathbf{h_i}|$. Consider the integral transformation $\mathcal{Q}\mathbf{u} := \left(F_1^{-1}(u_1), ..., F_K^{-1}(u_K)\right)$ for $F_k : \mathbb{R} \to [0,1]$ and write f_k for the derivative of F_k with respect to the Lebesgue measure $(k = 1, ..., K)$,

$$\int_{[0,1]^K} f(\mathbf{x}) \det \left|\frac{d\mathbf{x}}{d\mathbf{u}}\right| d\mathbf{u}.$$

Then, we consider a discrete approximation of $[0,1]^K$ by N hyperrectangles $\mathbf{h}_i^*$, $i = 1, ..., N$ such that $|\mathbf{h}_i^*| \leq h^K / \left|A^K\right|$. Therefore, $f(\mathbf{x}) \det \left|\frac{d\mathbf{x}}{d\mathbf{u}}\right| d\mathbf{u} \simeq f(\mathbf{x}) \det \left|\frac{d\mathbf{x}}{d\mathbf{u}}\right| \mathbf{h}_i^*$, and $f(\mathbf{x})\, |\mathbf{h}_i| - f(\mathbf{x}) \det \left|\frac{d\mathbf{x}}{d\mathbf{u}}\right| |\mathbf{h}_i^*| = f(\mathbf{x}) h^K \left(1 - \det \left|\frac{d\mathbf{x}}{d\mathbf{u}}\right| / \left|A^K\right|\right)$. We need to consider $\left(1 - \det \left|\frac{d\mathbf{x}}{d\mathbf{u}}\right| / \left|A^K\right|\right)$ or equivalently to $\det \left|\frac{d\mathbf{u}}{d\mathbf{x}}\right| - 1/\left|A^K\right|$. If F_k is a distribution function, then $\det \left|\frac{d\mathbf{u}}{d\mathbf{x}}\right| = \prod_{k=1}^K f_k$ is a joint pdf of random variables which are independent, while $1/\left|A^K\right|$ corresponds to the joint pdf of random variables that are independent and uniformly distributed over A^K. Therefore, the sign of $\left(1 - \det \left|\frac{d\mathbf{x}}{d\mathbf{u}}\right| / \left|A^K\right|\right)$ will depend on the shape of the marginals f_k $k = 1, ..., K$, and the size of the set A^K. For example, suppose the f_k's are monotonically decreasing in x. Then, $\det \left|\frac{d\mathbf{u}}{d\mathbf{x}}\right| - 1/\left|A^K\right| > 0$ for $\|\mathbf{x}\| \to 0$ and $\det \left|\frac{d\mathbf{u}}{d\mathbf{x}}\right| - 1/\left|A^K\right| < 0$ for $\|\mathbf{x}\| \to \infty$. Clearly, as either $K \to \infty$ or $|A| \to \infty$, $\det \left|\frac{d\mathbf{u}}{d\mathbf{x}}\right| - 1/\left|A^K\right| > 0$. These remarks show that we may expect the copula based MC integral to be prone to error in the tails of the distribution.

3.2 A Modified Integration Procedure

The copula based MC integral leads to a change of variables that produces the Jacobian $J(\mathcal{Q}\mathbf{x}) = \left[\prod_{k=1}^K pdf_k(x_k)\right]^{-1}$. Therefore, $J(\mathbf{u}) \to \infty$ on any edge of $[0,1]^K$ for any *pdf* with support on the real line (this is just a consequence of the integrability condition for *pdf*'s). This makes the integration possibly unstable in these areas. For this reason, we may advocate the following modified procedure, which is further justified in light of the previous remarks. Consider the following integral transformation

$$\begin{aligned} \mathbb{P}G(\mathbf{X}) &= \int\limits_{[\epsilon,1-\epsilon]^K} G\left(F_1^{-1}(u_1), ..., F_K^{-1}(u_K)\right) c(u_1, ..., u_K)\, du \\ &+ \int\limits_{[0,1]^K \backslash [\epsilon,1-\epsilon]^K} G\left(F_1^{-1}(u_1), ..., F_K^{-1}(u_k)\right) c(u_1, ..., u_k)\, du \end{aligned}$$

so that each integral can be approximated by a copula based MC integral. For $s \geq 1$, generate sN uniform random variables in $[0,1]^K$, say $(\mathbf{u}_1, ..., \mathbf{u}_{sN})$. Choose $\epsilon \geq 0$ such that $A^K = [\epsilon, 1-\epsilon]^K \subseteq [0,1]^K$. For $j = 1, ..., sN$, if $\mathbf{u}_j \in [\epsilon, 1-\epsilon]^K$ delete $\mathbf{u}_j$. Then, we would end up with a new series, say $(\mathbf{u}'_1, ..., \mathbf{u}'_{N'})$ where $\mathbf{u}'_j \notin [\epsilon, 1-\epsilon]^K$, $j = 1, ..., N'$, and $\mathbb{P}N' = sN \left|[0,1]^K - A^K\right|$. Then, define $N'' = N - sN \left|[0,1]^K \backslash A^K\right|$, and generate N'' uniform random variables in $[\epsilon, 1-\epsilon]^K$, say $(\mathbf{u}''_1, ..., \mathbf{u}''_{N''})$. Therefore, $N = \mathbb{P}N' + N''$. Then, we perform the MC integral over the two regions A^K and $[0,1]^K \backslash A^K$:

$$\mathbb{P}G(\mathbf{X}) \simeq \mathbb{P}_{N'}^{\mathbf{u}'_j}\left[p(\mathcal{Q}\mathbf{u})\, J(\mathbf{u}) \left|[0,1]^K \backslash A^K\right| G(\mathcal{Q}\mathbf{u})\right] + \mathbb{P}_{N''}^{\mathbf{u}''_j}\left[p(\mathcal{Q}\mathbf{u})\, J(\mathbf{u}) \left|A^K\right| G(\mathcal{Q}\mathbf{u})\right]. \tag{3}$$

For $s = 1$, the procedure collapses to the direct copula based MC integration, i.e. $A^K = [0,1]^K$. Therefore, s is used to increase the proportion of uniform pseudo-random in $[0,1]^K \backslash A^K$, where the integrand has larger variation.

3.3 Some Numerical Evidence

Let X and Y be two random variables with values in $\mathbb{R}$. Suppose we are facing the following problem $\mathbb{P}XY$. We consider three different cases for the joint density of X and Y:

1. the bivariate standard normal density with correlation ρ,

$$\phi(x, y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{(x^2 + y^2 - 2\rho xy)}{2(1-\rho^2)}\right\};$$

2. the Kimeldorf-Sampson (KS) copula density with standard normal marginals

$$(1+\delta)(uv)^{-\delta-1}\left(u^{-\delta} + v^{-\delta} - 1\right)^{-\frac{1}{\delta}-2} \phi(x)\, \phi(x), \tag{4}$$

where ϕ is the univariate standard normal density, $u = \Phi(x)$, $v = \Phi(y)$, and Φ is the Gaussian distribution;

3. the translated Kimeldorf-Sampson (KS) copula density with exponential marginals

$$(1+\delta)(\bar{u}\bar{v})^{-\delta-1}\left(\bar{u}^{-\delta}+\bar{v}^{-\delta}-1\right)^{-\frac{1}{\delta}-2}\exp\{-x\}\exp\{-y\}, \tag{5}$$

where $\bar{u}=(1-u)$, $\bar{v}=(1-v)$ with $u=1-\exp\{-x\}$, and $v=1-\exp\{-v\}$. Notice that this is just the usual KS copula density (e.g. Joe, 1997), but rotated by an angle equal to π, so that the lower tail (at $[0,0]$) now corresponds to the upper one (at $[1,1]$).

The normal density is the most common choice of density and results are often derived under normality assumptions. For this reason we think it is necessary to evaluate the properties of our copula based approach in this case despite the fact that this distribution is found to be not particularly appropriate to model financial returns.

The second choice of joint density retains the normal distribution for the marginals, but allows for a more complex dependence in the variables. In fact, the KS copula captures non-linear dependence in terms of lower tail dependence. This means specifically that random variables become more dependent as they tend to minus infinity (i.e. large losses). This appears to be an important property when modelling joint financial returns: asset returns are more dependent when they are negative (Silvapulle and Granger, 2000, Fortin and Kuzmics, 2002 for the case of daily financial returns). To give a feeling of this parametric specification, Figure I shows the counterplot of the KS copula.

[Figure I]

For our third choice of joint density we had financial applications in mind. We were thinking about financial losses, so that we model positive random variables. For example, if we were to model daily financial returns, the modified Weibull distribution or the double gamma could be good choices (e.g. Knight et al., 1995, and Laherrere and Sornette, 1998). The exponential distribution is nested into these distributions, so it might provide a suitable choice to model losses in real problems. Again, the choice of KS copula is of interest. We notice that, once it is translated, this copula density exhibits upper tail dependence. This means specifically that random variables become more dependent as they tend to infinity (i.e. as losses increase).

In our numerical examples we choose $N=10,000$, $100,000$. Then, for each N, we compute the expectation of the MC integral and its standard deviation by use of 500 simulations of size N. Further we just set $\epsilon=.05$, and $s=1.5$; while it could be interesting to try different values of s and ϵ, for simplicity we do not pursue this route.

Then, results are reported for the mean and the variance of the following 5 approaches:

1. The approximate MC integral (AMC_A), truncated over a set A^2,

$$\left|A^2\right| \mathbb{P}_N^{\mathbf{u}_i}\left[xyp\left(x,y\right)\right],$$

where $x := |A|\left(u_1 - \min\left(A\right)\right)$ $y := |A|\left(u_2 - \min\left(A\right)\right)$. In this case the standard errors are not reliable as they are computed over a finite subset of the actual support of the functions considered.

2. The importance sampling MC integral (ISMC), where the marginals are used as kernel densities to simulate the random points, i.e.

$$\mathbb{P}_N^{x_i y_i}\left[xyc\left(F_X\left(x\right), F_Y\left(y\right)\right)\right],$$

where $p\left(x,y\right) = c\left(F_X\left(x\right), F_Y\left(y\right)\right) f_X\left(x\right) f_Y\left(y\right)$ and $F_X\left(x\right), f_X\left(x\right)$ are the marginal distribution and density of X (similarly for Y).

3. The copula based MC integral (CMC),

$$\mathbb{P}_N^{\mathbf{u}_i}\left[F_X^{-1}\left(u_1\right) F_Y^{-1}\left(u_2\right) c\left(u_1,u_2\right)\right].$$

4. The modified CMC (MCMC) defined as MCMC:=MCMC_1+MCMC_2 where MCMC_1 is the modified CMC over A^2, and MCMC_2 is the modified CMC over $[0,1]^2 \backslash A^2$ (see (3) above).

3.3.1 The Bivariate Normal

In this numerical investigation, we consider the following sets $A^2 = [-2.5, 2.5]^2$, $[-5,5]^2$, $[-10,10]^2$, and the following values for $\rho = .2$, .5, .8.

[Table I]

3.3.2 The KS Copula with Normal Marginals

Again, we consider the following sets $A^2 = [-2.5, 2.5]^2$, $[-5,5]^2$, $[-10,10]^2$ and the following values for the dependence parameter in the copula, $\delta = .31$, .76, 1.55, 3.19, which correspond to the following values of Spearman's rho $\rho_S = .2$, .4, .6, .8. Spearman's rho is a measure of dependence that is invariant of the marginals, as it corresponds to the covariance between distribution functions (Joe, 1997, for details). Results are reported in Table II.

[Table II]

3.3.3 The KS Copula with Exponential Marginals

In this other numerical investigation, we consider the following sets $A^2 = [0, 10]^2$, $[0, 20]^2$, $[0, 30]^2$ and the following values for the dependence parameter in the copula, $\delta = .31$, .76, 1.55, 3.19, which are the same values as in the previous example. Results are reported in Table III.

[Table III]

3.3.4 Comments on Results

The results from Tables I-III show that the AMC has lower variance, but this is just due to truncation of the support. Therefore, the standard error is not a reliable indicator of the error. Further, as we increase the size of the approximating support (i.e. A^2), the variance appears to grow linearly in $|A^2|$. In a sense, AMC corresponds to an extreme case of importance sampling, i.e. we just do not sample from the region outside the truncation. As mentioned in the introduction, this approach can be "dangerously" inaccurate when the dimension increases due to the curse of dimensionality.

Using the copula function, a more natural approach is the ISMC integral, where we use the marginal pdfs as kernel density functions from which to sample the data. The results show that this is a clear competitor to our method. However, the following is worth noticing. Our integral transformation leads to surfaces which are usually simpler to characterize. For example consider (4). Then, our integrand, is given by

$$xy\,(1+\delta)\,(\Phi(x)\,\Phi(y))^{-\delta-1}\left(\Phi(x)^{-\delta}+\Phi(y)^{-\delta}-1\right)^{-\frac{1}{\delta}-2}\phi(x)\,\phi(x)\,,$$

and Figure II shows its plot before and after the transformation. The surface becomes much more regular with a clear monotonic behaviour on the tails. Though we do not claim that this may always be the case, the discussion in Section 3.1 shows that this pattern can be expected when our approach is used. This makes automatic use of importance sampling in our approach. Our integrand will be expected to exhibit higher variation on the edges of the hypercube under most standard integral problems. Therefore, one can directly use this previous knowledge to achieve fast and simpler MC integral estimates. While we did not make any attempt to "calibrate" the parameters of the MCMC integral, results appear to be quite promising as the sample increases, providing similar answers but with a (usually) lower error. We might further reduce the error in the MCMC integral from $O\left(N^{-\frac{1}{2}}\right)$ to $O\left((\ln N)^K N^{-1}\right)$ by use of quasi-random numbers.

[Figure II]

4 Further Discussion and Financial Applications

The scope and application to finance of the method presented in this paper is quite large. In fact, the copula function allows us to model financial problems in a fashion that is more realistic and flexible. It is convenient to discuss some further details about the copula function in a time series context and the use of distributions with close form quantile function. This discussion is required in order to enhance the possibility of solving many financial problems. The Section concludes with examples of applications.

4.1 Conditional Integration for Discrete Time Stationary Markov Processes

The copula based MC integration can be applied to financial problems of different nature, e.g. discrete time stationary Markov processes. Let $p(X_0, X_1, ..., X_T)$ be the joint density of the Markov process $(X_t)_{t\leq T}$ with invariant distribution function $F(X_t)$ and density $f(X_t)$. Further let $C(u_t, u_{t+1})$ and $C_{,ij}(u_t, u_{t+1})$ be, respectively, the copula function and the partial derivative of the $i = 0, 1$ $j = 0, 2$ argument of the copula of (X_t, X_{t+1}), $t-1 \leq T$. Let $g : \mathbb{R}^k \to \mathbb{R}$ be some uniformly integrable function. Again we may want to compute its expectation, but this time conditional on the information at time 0, i.e. $\mathbb{P}[g(X_1, ..., X_T)|\mathcal{F}_0]$ with respect to the joint density $p(X_0, X_1, ..., X_T)$, where $\mathcal{F}_0$ is the filtration at 0. Using the fact that the process is Markovian,

$$\begin{aligned}
\mathbb{P}\left[g(X_1, ..., X_T)\middle| X_0\right] &= \int\limits_{\mathbb{R}^T} g(x_1, ..., x_T)\frac{p(x_0, x_1, ..., x_T)}{f(x_0)}dx_1\cdots dx_T \\
&= \int\limits_{\mathbb{R}^T} g(x_1, ..., x_T)\prod_{t=0}^{T-1} C_{,12}(F(x_t), F(x_{t+1}))\, f(x_{t+1})\, dx_{t+1} \\
&= \int\limits_{[0,1]^T} g\left(F^{-1}(u_1), ..., F^{-1}(u_T)\right)\prod_{t=0}^{T-1} C_{,12}(u_t, u_{t+1})\, du_{t+1}.
\end{aligned}$$

By the same argument as above, this integral is just the expectation of

$$g\left(F^{-1}\left(u_{1}\right),...,F^{-1}\left(u_{T}\right)\right)\prod_{t=0}^{T-1}C_{,12}\left(u_{t},u_{t+1}\right)$$

with respect to the Lebesgue measure over $[0,1]^{T}$ and it can be dealt exactly as in (1). If one had to solve the integral, simulating from the copula, the procedure would be simplified by the Markov condition, but difficulties would be likely to arise in an arbitrary context.

4.2 Expectations of Functions of Continuous Martingales

We would not provide a complete setup for our integral transform if we did not consider the continuous time framework which is central to the theoretical development of mathematical finance.

Let $(X_{t})_{t\geq0}$ be a continuous process which admits the following decomposition

$$X_{t}=A_{t}+M_{t},$$

where A_{t} is increasing and deterministic, while M_{t} is a martingale. This setup is equivalent to

$$M_{t}=X_{t}-A_{t},$$

i.e., it only requires us to consider expectation with respect to M_{t}. From (16) in Appendix A.2, M_{t} has time copula

$$C^{|t-s|}\left(u,v\right)=\int\limits_{[0,u]}\Phi\left(\frac{\left(\sqrt{\langle M\rangle_{t}}\Phi^{-1}\left(v\right)-\sqrt{\langle M\rangle_{s}}\Phi^{-1}\left(h\right)\right)}{\sqrt{\left|\langle M\rangle_{t}-\langle M\rangle_{s}\right|}}\right)dh,$$

with copula density (see (18) in Appendix A.2 and (13) for the definition of $\langle M\rangle_{t}$)

$$C_{,12}^{|t-s|}\left(u,v\right)=\phi\left(\frac{\left(\sqrt{\langle M\rangle_{t}}\Phi^{-1}\left(v\right)-\sqrt{\langle M\rangle_{s}}\Phi^{-1}\left(u\right)\right)}{\sqrt{\left|\langle M\rangle_{t}-\langle M\rangle_{s}\right|}}\right)\frac{\sqrt{\langle M\rangle_{t}}}{\phi\left(\Phi^{-1}\left(v\right)\right)\sqrt{\left|\langle M\rangle_{t}-\langle M\rangle_{s}\right|}},$$

where $\phi\left(...\right)$ defines the standard normal density.

Let $g:\mathbb{R}\rightarrow\mathbb{R}$ be a uniformly integrable function, then $\mathbb{P}\left[g\left(X_{t}\right)|\mathcal{F}_{s}\right]$, $s<t$, can be written as

$$\begin{aligned}\mathbb{P}\left[g\left(A_{t}+M_{t}\right)|\mathcal{F}_{s}\right] &= \int\limits_{\mathbb{R}}g\left(A_{t}+y\right)\frac{1}{\sqrt{\langle M\rangle_{t}-\langle M\rangle_{s}}}\phi\left(\frac{y-x}{\sqrt{\langle M\rangle_{t}-\langle M\rangle_{s}}}\right)dy\\ &= \int\limits_{[0,1]}g\left(A_{t}+\Phi^{-1}\left(v\right)\right)C_{,12}^{|t-s|}\left(u,v\right)dv.\end{aligned}$$

Therefore, we can calculate the Monte Carlo integral of

$$g\left(A_t + \Phi^{-1}(v)\right) C_{,12}^{|t-s|}(u,v)$$

using a series of iid uniform $[0,1]$ random variables where $u = \Phi\left(\frac{x}{\sqrt{\langle M\rangle_s}}\right)$.

4.3 Closed Form Quantile Functions

Monte Carlo integration based on our proposed transformation can be further simplified if the quantile function of the random variables has a closed form. Here we provide a family of distributions that satisfies these requirements and is particularly well suited as a distribution for financial returns. We only need to consider unconditional marginal distributions. The following modified Weibull distribution can be used for the assets' geometric returns x,

$$\begin{aligned} pdf(x) &= qI_{\{x\geq\mu\}}ab(x-\mu)^{b-1}\exp\left\{-a(x-\mu)^b\right\} \\ &\quad + (1-q)I_{\{x<\mu\}}a'b'(\mu-x)^{b'-1}\exp\left\{-a'(\mu-x)^{b'}\right\} \end{aligned}$$

where $x \in \mathbb{R}$, $q = \Pr(x \geq \mu)$. (Although in some cases it is appropriate to use arithmetic returns, we use geometric returns. This is the common approach in econometrics. Appendix C comments on this and simple extensions of the modified Weibull that would allow us to describe arithmetic or relative returns.) The returns are seen to be possibly asymmetric. In the symmetric case, the following alternative can be considered,

$$\frac{\lambda}{2}ab(|x-\mu|+\delta)^{b-1}\exp\left\{-a(|x-\mu|+\delta)^b\right\},$$

where $\delta = \left(\frac{b-1}{ab}\right)^{\frac{1}{b}}$ and $\lambda = \exp\left\{\frac{b-1}{b}\right\}$ if $b > 1$, $\delta = 0$ and $\lambda = 1$ otherwise. Therefore, the general model can be written as

$$\begin{aligned} pdf(x) &= qI_{\{x\geq\mu\}}\lambda ab(x-\mu+\delta)^{b-1}\exp\left\{-a(x-\mu+\delta)^b\right\} \\ &\quad + (1-q)I_{\{x<\mu\}}\lambda a'b'(\mu-x+\delta)^{b'-1}\exp\left\{-a'(\mu-x+\delta)^{b'}\right\} \end{aligned} \tag{6}$$

where the two possible models are embedded. A particular case of (6) has already appeared in Sancetta and Satchell (2001). Further, as mentioned in the Introduction, the use of rational powers of the argument of the exponential function has been advocated for modelling unconditional financial returns by Frisch and Sornette (1997) and Laherrère and Sornette (1998), particularly for the intermediate time scale.

The following features emerge from (6). For $b > 1$, the distribution is not unimodal, unless we use δ, in which case symmetry must be assumed. Therefore, we can allow for unimodality only at the cost of giving up asymmetry. Fortunately, data at higher frequencies tend to have $b < 1$, in which case the density is unimodal and asymmetry is more common at high frequencies. On the other hand, data at lower frequencies have $b > 1$ in which case it is known that they converge to normal, and asymmetry is less of a problem. Notice that the degree of bimodality is very low for b greater than, but close to, one. (There is no general evidence about the extent of unimodality in financial returns; in fact some series may exhibit some bimodality. Therefore, being able to capture bimodality, if required, can turn out to be an advantage.)

It is important to notice that the quantile function of (6) has a closed form. This property has important consequences for the computational issues discussed here. To our knowledge, other flexible parametric specifications for the unconditional distribution of asset returns do not have a closed form for the distribution function. Among these, we recall the double gamma of Knight et al. (1995) and the modified version of the Weibull distribution as given in Sornette et al. (1999) and Andersen and Sornette (1999).

4.4 Financial Applications

We conclude the section with three illustrative examples, which highlight the range of financial applications: 1. the optimal portfolio for an agent with linear plus exponential Bernoulli utility function, 2. the optimal VaR portfolio, 3. the value of an Asian option.

4.4.1 Example 1: Optimal portfolio with the linear plus exponential Bernoulli utility

Assume we have a portfolio $Z_k := \sum_{j=1}^{k} w_j X_j$ and we want to find the optimal weights $(w_1, ..., w_k)$ restricted to be in the simplex for an agent with the linear plus utility

$$u(Z_k) = \alpha Z_k - \beta \exp\{-\gamma Z_k\}, \tag{7}$$

where $\alpha, \beta, \gamma \in \mathbb{R}_{++}$. This Bernoulli utility function has been considered by Bell (1988). Let u' stands for first derivative of $u(Z_k)$ and similarly for the higher derivatives u'' and u'''. Then, (7) is one of the only two possible Bernoulli utility functions that satisfy $u' > 0$, $u'' < 0$, $u''u'' < u'u'''$ (i.e. increasing, concave, and decreasingly risk averse) and allow us to rank gambles in terms of riskiness in a way that is independent of wealth (Bell, 1988). Further, (7) is the only Bernoulli utility function that satisfies what is called the strong one switch condition. The strong one switch condition says that the independence axiom can be violated because of a change in perception of risk after an equal certain amount of money has been added to each lottery. However, such a change can occur only once (see Bell and Fishburn, 2001). Similar conditions lead to the identification of specific Bernoulli utility functions in the case of multiplicative gambles, i.e. relative prices (see Pedersen and Satchell, 2001, for an exhaustive treatment). Because of these optimal properties, (7) has been proposed by Bell (1995) as the reference Bernoulli utility for financial economics when gambles are additive. (Clearly, our framework works equally well for relative prices.)

Using the notation above, the optimal portfolio problem for an agent with Bernoulli utility (7) is given by

$$\sup_{\substack{w_k \geq 0 \\ 1 \leq k \leq K}} \left\{ \int_{\mathbb{R}_+^K} \left[\alpha \left(\sum_{k=1}^{K} w_k x_k \right) - \beta \exp\left\{ -\gamma \left(\sum_{k=1}^{K} w_k x_k \right) \right\} \right] dP\left(x_1, ..., x_K\right) \right\}$$

with

$$\sum_{k=1}^{K} w_k = 1.$$

Using the simple quantile change of variables, we just need to compute

$$\int_{[0,1]^K} \left[\alpha \left(\sum_{k=1}^{K} w_k F_k^{-1}\left(u_k\right) \right) - \beta \exp\left\{ -\gamma \left(\sum_{k=1}^{K} w_k F_k^{-1}\left(u_k\right) \right) \right\} \right] c\left(u_1, ..., u_K\right) du_1 \cdots du_K. \tag{8}$$

This can be done very easily by Monte Carlo integration if we have an iid sample of uniform $[0,1]^K$ random variables. For arbitrary distributions $P\left(x_1, ..., x_K\right)$, the problem would require us to compute the Laplace transform of a portfolio, and closed form solutions might not exist.

4.4.2 Example 2: Optimal value at risk portfolio

Tail probabilities and quantiles are of great interest to risk managers. Here we consider how our simulation procedure can be applied in this context.

Let $X_k^- = -X_k$, where $X_k, 1 \leq k \leq K$, is a random variable with values in $\mathbb{R}$. Consider the cumulant generating function of

$$Z_K^- := \sum_{k=1}^{K} w_k X_k^-,$$

i.e.

$$\ln\left(\mathbb{P}\exp\left\{\alpha\sum_{k=1}^{K} w_k X_k^-\right\}\right),$$

which is a convex function in α. We define

$$\hat{\alpha} := \arg\max\left\{\alpha z - \ln\left(\mathbb{P}\exp\left\{\alpha\sum_{k=1}^{K} w_k X_k^-\right\}\right)\right\} \tag{9}$$

for some real $z > 0$. By the usual saddlepoint approximation for tail probabilities (see Daniels, 1954, or Barndorff-Nielsen and Cox, 1989, p. 182) we have

$$\Pr\left(Z_k^- \geq z\right) \simeq \frac{\exp\left\{-f\left(\hat{\alpha}, z^-, w\right)\right\}}{\hat{\alpha}\left(2\pi g\left(\hat{\alpha}, w\right)\right)^{\frac{1}{2}}}, \tag{10}$$

where

$$f\left(\hat{\alpha}, z, w\right) := \left\{\hat{\alpha} z - \ln\left(\mathbb{P}\exp\left\{\hat{\alpha}\sum_{k=1}^{K} w_k X_k^-\right\}\right)\right\},$$

and

$$g\left(\hat{\alpha}, w\right) := \left[\left(\frac{\partial^2}{\partial\alpha^2}\right)\ln\left(\mathbb{P}\exp\left\{\hat{\alpha}\sum_{k=1}^{K} w_k X_k^-\right\}\right)\right]_{\lambda=\hat{\lambda}}.$$

(The formal validity of (10) requires a four times differentiable cumulant generating function. In which case, if $w_j \to 0$, $\forall j \in J$, as $card\left(J\right) \to \infty$ (i.e. as the number of assets in the portfolio tends to infinity), the approximation can be shown to be asymptotically valid. In particular, we require $\frac{\kappa_4}{\sigma^2} \to 0$, where κ_4 is the kurtosis of the portfolio and σ^2 is its variance. The quality of the approximation is determined by the speed at which the fourth cumulant converges to zero. It is interesting to compare the condition $\frac{\kappa_4}{\sigma^2} \to 0$ with the remarks in Sornette et al. (1999) and the problems related to straight minimization of σ^2 due to its negative influence on $\frac{\kappa_4}{\sigma^2}$, i.e. large portfolio risk.) Notice that we are using a leading term approximation for the distribution and not the density; for this reason we standardize by $\hat{\alpha}$ (see Daniels, 1987, equation 3.3). This result is very much related to the large deviation principle. In fact, by Markov's inequality,

$$\Pr\left(Z_K^- \geq z\right) \leq \exp\left(-\alpha z\right)\mathbb{P}\exp\left(\alpha Z_K^-\right) \tag{11}$$

and we need to find the optimal α, i.e. minimize with respect to α. The only difference lies in the fact that we are not standardizing by a factor proportional to $g(\hat{\alpha}, w)$, which is a function of the portfolio weights. Results related to (11) have been considered by Stutzer (2002) in a utility context.

If we want to minimize the probability of our portfolio being below some level z, we need to compute (9) where $\mathbb{P}\exp\left\{\alpha \sum_{k=1}^{K} w_k X_k^-\right\}$ can be calculated exactly in the same way as in (8). Then, the optimization part is standard and consistency is assured by uniform convergence.

4.4.3 Example 3: The Value of an Asian Option

Here we provide a specific example on how the quantile transform can be used to value an Asian option under the Esscher transform in a discrete time setting. Although many pricing kernels could be used, the one obtained via the Esscher transform is a popular one that can be justified on an expected utility ground. Indeed this approach can be seen as a special case of a more general approach based on utility optimization (see Jackwerth, 2000, and the applications in Knight et al., 2000).

Assume that we want to price, at time 0, a contingent claim written on some arbitrary payoff function $g(S_T)$, where

$$S_T = S_0 + \sum_{t=1}^{T} X_t$$

and $(X_t)_{t\leq T}$ is a discrete time homogeneous Markov process which has a financial interpretation as a price increment process. This means that the price process is not Markovian, i.e. the increments are not independent. However, these price increments are assumed to be stationary. By appropriate choice of copula function, this set up may cover cases where relatively high and low returns are likely to be followed by large absolute returns allowing also for asymmetry. (For example, this would capture the salient features of models like the asymmetric GARCH.) Let $Z_T := \sum_{t=1}^{T} X_t$, $\mathcal{F}_0$ be the natural filtration of S_T at time 0, and write $pdf\left(Z_T|\mathcal{F}_0\right)$ for the pdf under the objective probability. Then, using the Esscher transform (e.g. Bühlmann et al., 1998, for this specific application of the conjugate density) a risk neutral density is given by

$$pdf^*\left(S_T|\mathcal{F}_0\right) = \frac{\exp\left\{\hat{\alpha} S_T\right\}}{\mathbb{P}\left[\exp\left\{\hat{\alpha} S_T\right\}\right]} pdf\left(S_T|\mathcal{F}_0\right) \tag{12}$$

where

$$\hat{\alpha} := \arg\max\left(\alpha \bar{Z}_T - \ln \mathbb{P}\left[\exp\left\{\alpha Z_T\right\}\right]\right),$$

and $\bar{Z}_T := \mathbb{P}Z_T$. The expectation of the payoff function $g(S_T)$ will have to be calculated with respect to $pdf^*(z_T|\mathcal{F}_0)$. Now, define

$$\begin{aligned} g(S_T) &= \max\left[\sum_{t=1}^{T}\frac{S_t}{T} - K, 0\right] \\ &= \left[\sum_{t=1}^{T}\frac{S_t}{T} - K\right]^+, \end{aligned}$$

which is the payoff function for a discrete time Asian option. (Notice that in reality, Asian options can only be in discrete time, and not in continuous time as usually postulated in theory.) Since we condition at time 0, we can write

$$g(S_T) = \left[\sum_{t=1}^{T} t\frac{X_t}{T} - (K - S_0)\right]^+,$$

so that we have

$$\begin{aligned} \mathbb{P}\left[g(P_T)|\mathcal{F}_0\right] &= \int_{\mathbb{R}}\left[\sum_{t=1}^{T} t\frac{x_t}{T} - (K - S_0)\right]^+ \frac{\exp\{\hat{\alpha} z_T\}}{\mathbb{P}\left[\exp\{\hat{\alpha} Z_T\}\right]} pdf(z_T|\mathcal{F}_0)\, dz_T \\ &= \int_{\mathbb{R}^T}\left[\sum_{t=1}^{T} t\frac{x_t}{T} - (K - S_0)\right]^+ \\ &\quad \times \frac{\exp\left\{\hat{\alpha}\sum_{t=1}^{T} x_t\right\}}{\mathbb{P}\left[\exp\{\hat{\alpha} Z_T\}\right]} pdf(x_1, ..., x_T|\mathcal{F}_0)\, dx_1 \cdots dx_T; \end{aligned}$$

but using the Markov property, we have

$$pdf(x_1, ..., x_T|\mathcal{F}_0) = \prod_{j=1}^{T}\frac{pdf(x_{j-1}, x_j)}{pdf(x_{j-1})}$$

where $pdf(x)$ is the time invariant density for any X. If the process $(X_t)_{t\leq T}$ has copula density $c(u_t, u_{t+1})$, and writing $F(x)$ for the invariant distribution function, we have

$$\begin{aligned} \mathbb{P}\left[g(S_T)\right] &= \int_{[0,1]^T}\left[\sum_{t=1}^{T}\frac{t}{T}F^{-1}(u_t) - (K - S_0)\right]^+ \\ &\quad \frac{\exp\left\{\hat{\alpha}\sum_{t=0}^{T} F^{-1}(u_t)\right\}}{\mathbb{P}\left[\exp\{\hat{\alpha} Z_T\}\right]}\prod_{t=1}^{T} c(u_{t-1}, u_t)\, du_t, \end{aligned}$$

where we assume $\hat{\alpha}$ had been previously calculated in a way very similar to the previous example. Clearly, the integral may be very complex, but amenable to Monte Carlo integration. With a simulated sample of iid uniform $[0,1]^T$ random variables, the integral with respect to $\prod_{t=1}^{T} du_t$ can be calculated straightforwardly as a mean-like statistic, and the error bounded by means of probabilistic arguments.(Unless α is given, using the α obtained by a previous maximization of a Monte Carlo integral leads to an incidental parameter problem. Since we may control the standard deviation of α by choosing the size of the simulation, this is not a problem in practice.) Again, no simulation from the joint distribution is required.

5 Conclusion

We have shown that the quantile transform arises as a natural application of the copula function. The copula representation allows us to transform arbitrary integrals into integrals on the hypercube. The copula based MC integration is conceptually related to importance sampling MC integration, but simpler to use in general situations. Moreover, we showed that the transformation leads to a somehow predictable shape of the integrand, i.e. more mass and variability can be expected on the tails, while little variability on the centre of the hypercube. In a sense this is the opposite of what happens when we deal with integrands like the one depicted on the top panel of Figure II. This predictability can be further exploited by a modified copula based Monte Carlo approach as described in the paper. Numerical results support the use of this approach. While we did not focus on using quasi-random numbers, it is clear that the theory about quasi-random numbers directly applies within the context of our approach.

Since integrals on the hypercube are amenable to numerical computation via Monte Carlo integration, a great number of problems in finance can be solved by simply using this technique. The technique can be employed in cross sectional kind of problems, discrete time Markov processes and for continuous martingales. In particular, to show the scope of the applications in finance, we provided several illustrative examples in the context of portfolio optimization and contingent claims valuation. The procedure is further simplified if marginal distributions with closed form quantile function are considered. We have shown that a family of distributions very naturally arises in this context, i.e. modified versions of the Weibull distribution. Since modified versions of the Weibull distribution have been advocated by some authors to model daily asset returns, this choice provides both computational simplicity and a realistic parametric specification.

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A Copula Representation of Multivariate Distributions

The purpose of this appendix is to present some results about the copula function and Markov processes. At first, we recall a few properties of the copula function itself (e.g. Scarsini, 1989):

(1) *C is increasing in all its arguments. Notice that we use increasing to mean nondecreasing;*
(2) *C satisfies the Fréchet bounds, i.e.*

$$\max\left(0, u_1 + ... + u_K - (K-1)\right) \leq C\left(u_1, ..., u_K\right) \leq \min\left(u_1, ..., u_K\right),$$

which implies C_B is grounded: i.e. $C(u_1, ..., u_K) = 0$ if $u_j = 0$ for at least one j, and $C(1, ..., 1, u_j, 1, ..., 1) = u_j$, $\forall j$;
(3) $\prod_{j=1}^{K} u_j$ *is a copula for independent random variables, i.e. the product copula;*
(4) *C is Lipschitz with constant one, i.e.*

$$\left|C\left(x_1, ..., x_K\right) - C\left(y_1, ..., y_K\right)\right| \leq \sum_{j=1}^{K} \left|x_j - y_j\right|;$$

(5) *For $j = 1, ..., K$, let $u_j = F_j\left(x_j\right)$ be the distribution function of x_j and $v_j = G_j\left(y_j\right)$ be the distribution function of $y_j = g_j\left(x_j\right)$, where $g_j\left(...\right)$ is an increasing function , then $u_1, ..., u_K$ and $v_1, ..., v_K$ have same copula function.*

A.1 Discrete Time Markov Processes and the *-Product

Financial time series are obtained in discrete time. For this reason, it is common in econometrics to use discrete time models. Given the copula $C\left(u, v\right)$, the conditional copula is given by

$$C\left(v|u\right) = \frac{\partial C\left(u, v\right)}{\partial u}.$$

To see this, in the case of random variables with differentiable and strictly increasing distribution functions, use the chain rule,

$$\frac{\partial C\left(u, v\right)}{\partial x} = \frac{\partial C\left(u, v\right)}{\partial u} \frac{\partial u\left(x\right)}{\partial x},$$

and notice that $\frac{\partial u(x)}{\partial x}$ is the density of x (see Theorem 3.1 in Darsow et al., 1992, for the proof in the general case). In the sequel, we will write $C_{,j}$ to denote the partial derivative of C with respect to its j^{th} argument. In order to define Markov processes in terms of the copula function we need to define the *-product and use results from Darsow et al. (1992).

Definition 1. *(Darsow et al., 1992). Let A and B be 2-copulae, then*

$$\left(A * B\right)\left(u_1, u_2\right) := \int_0^1 A_{,2}\left(u_1, v\right) B_{,1}\left(v, u_2\right) dv.$$

Theorem 1. *(Theorem 3.2 in Darsow et al., 1992) Let* $(X_t)_{t\in T}$, *be a real stochastic process where for each* $s<u<t$, X_s *and* X_t *have copula* C_{st}. *Then, the following are equivalent:*
(i.) the transition probabilities $P(s,x,t,A)=\Pr(X_t\in A|X_s=x)$ *satisfy the Chapman-Kolmogorov equations*

$$P(s,x,t,A)=\int_{-\infty}^{\infty} P(u,\xi,t,A)\,P(s,x,u,d\xi)$$

for all Borel sets A, $s<u<t$, $s,u,t\in T$, *and* $x\in\mathbb{R}$;
(ii.) for $s<u<t$, $s,u,t\in T$,

$$C_{st}=C_{su}*C_{ut}.$$

We note the following generalization of the *-product (see Darsow et al., 1992, for details).

Definition 2. *(Darsow et al., 1992). Let* $A(u_1,...,u_{n-1},v)$ *and* $B(v,u_{n+1},...,u_{n+m})$ *be, respectively, a n-copula and a m-copula, then*

$$(A\star B)(u_1,...,u_{n+m}):=\int_0^{u_n} A_{,n}(u_1,...,u_{n-1},v)\,B_{,1}(v,u_{n+1},...,u_{n+m})\,dv,$$

where $(A\star B)(u_1,...,u_{n+m})$ *is a n+m-copula.*

Remark. *Clearly* $(A\star B)(u,1,v)=(A*B)(u,v)$.

Theorem 1 is a necessary condition for the existence of a Markov process. For this reason we need the following.

Theorem 2. *(Theorem 3.3 in Darsow et al., 1992).* $(X_t)_{t\in T}$ *is a Markov process if and only if*

$$C_{t_1\ldots t_n}=C_{t_1t_2}\star C_{t_2t_3}\star\cdots\star C_{t_{n-1}t_n}.$$

The set of all 2-copulae defines a Markov algebra with respect to the *-product (Darsow et al., 1992). This result is very useful for applications where one defines a dynamical system that can be characterized in terms of Markov processes.

Let $\mathcal{C}$ be the set of all copulae. Then, the following establishes that any copula generated through means of the *-product is a copula.

Theorem 3. *(Darsow et al., 1992). If A,B*$\in\mathcal{C}$, *then A*B*$\in\mathcal{C}$.

Theorem 1 and 3 allows us to generalize multiplication of finite dimensional discrete time Markov chains to infinite dimensional ones. This is all we need in order to work with the transition densities of discrete time Markov processes. (Issues in terms of continuity may arise when working with the *-product, see Darsow et al. (1992).)

A.2 The Time Copula for the Transition Distribution of Continuous Martingales

While the discrete time setting is quite natural and common in econometrics, a great part of financial mathematics uses continuous time Markov processes for the price dynamics. This implies that returns are independently distributed. The copula for continuous time Markov processes is also referred to as the time copula. Darsow et al. (1992) appear to be the first to have given an example of the time copula, i.e. the copula for Brownian motion. Following Këllezi and Webber (2001), we can identify the conditional copula function for continuous martingales. These last authors derived the time copula for some semimartingales. Here we only consider the case of continuous martingales. Let $(M_t)_{t\geq 0}$, be an $\mathcal{F}_t$ adapted continuous martingale such that $\lim_{t\to\infty} \langle M \rangle_t = \infty$ almost surely, where

$$\langle M \rangle_t := \int_0^t (M_s)^2 \, ds \tag{13}$$

is the quadratic variation. Then, we have the following well known representation for $(M_t)_{t\geq 0}$

$$M_t = B_{\langle M \rangle_t}, \tag{14}$$

almost surely for $0 \leq t < \infty$, where B_t is Brownian motion at time t (e.g. Karatzas and Shreve, 1991, p. 174). Now, following Darsow et al. (1992) the time copula for the $|t-s|$ increments of a Brownian motion is given by

$$C^{|t-s|}(u,v) = \int_{[0,u]} \Phi\left(\frac{\left(\sqrt{t}\Phi^{-1}(v) - \sqrt{s}\Phi^{-1}(h)\right)}{\sqrt{|t-s|}}\right) dh, \tag{15}$$

where $\Phi(...)$ is standard normal cumulative distribution and $\Phi^{-1}(...)$ its quantile function. Using (14) and (15), we have that the $|t-s|$ increments of $(M_t)_{t\geq 0}$ have the following time copula

$$C^{|t-s|}(u,v) = \int_{[0,u]} \Phi\left(\frac{\left(\sqrt{\langle M \rangle_t}\Phi^{-1}(v) - \sqrt{\langle M \rangle_s}\Phi^{-1}(h)\right)}{\sqrt{|\langle M \rangle_t - \langle M \rangle_s|}}\right) dh, \tag{16}$$

which is a special case of quantile transform for Gaussian random variables.

The copula density is directly obtained from (16) by differentiating with respect to each argument and making the change of variables $u = \Phi\left(\frac{x}{\sqrt{\langle M \rangle_s}}\right)$ and $v =$

$\Phi\left(\frac{y}{\sqrt{\langle M\rangle_t}}\right)$ with $\frac{dx}{du}=\sqrt{\langle M\rangle_s}\left[\phi\left(\frac{x}{\sqrt{\langle M\rangle_s}}\right)\right]^{-1}$ and $\frac{dy}{dv}=\sqrt{\langle M\rangle_t}\left[\phi\left(\frac{y}{\sqrt{\langle M\rangle_t}}\right)\right]^{-1}$, where $\phi(...)$ is the standard normal density, i.e.

$$
\begin{aligned}
C_{,12}^{|t-s|}(u,v) &= \frac{\exp\left\{\frac{(y-x)^2}{2\left|\langle M\rangle_t-\langle M\rangle_s\right|}\right\}}{\sqrt{2\pi\left|\langle M\rangle_t-\langle M\rangle_s\right|}}\frac{\exp\left\{\frac{x^2}{2\langle M\rangle_s}\right\}}{\sqrt{2\pi\langle M\rangle_s}} \\
&\quad\times\sqrt{\langle M\rangle_t}\left[\phi\left(\frac{y}{\sqrt{\langle M\rangle_t}}\right)\right]^{-1}\sqrt{\langle M\rangle_s}\left[\phi\left(\frac{x}{\sqrt{\langle M\rangle_s}}\right)\right]^{-1} \qquad (17)\\
&= \frac{\exp\left\{\frac{(y-x)^2}{2\left|\langle M\rangle_t-\langle M\rangle_s\right|}\right\}}{\sqrt{2\pi\left|\langle M\rangle_t-\langle M\rangle_s\right|}}\sqrt{\langle M\rangle_t}\left[\phi\left(\frac{y}{\sqrt{\langle M\rangle_t}}\right)\right]^{-1} \\
&= \frac{\sqrt{\langle M\rangle_t}}{\sqrt{\left|\langle M\rangle_t-\langle M\rangle_s\right|}}\phi\left(\frac{y-x}{\sqrt{\left|\langle M\rangle_t-\langle M\rangle_s\right|}}\right)\left[\phi\left(\frac{y}{\sqrt{\langle M\rangle_t}}\right)\right]^{-1} \\
&= \frac{\sqrt{\langle M\rangle_t}}{\phi\left(\Phi^{-1}(v)\right)\sqrt{\left|\langle M\rangle_t-\langle M\rangle_s\right|}}\phi\left(\frac{\sqrt{t}\Phi^{-1}(v)-\Phi^{-1}(u)}{\sqrt{\left|\langle M\rangle_t-\langle M\rangle_s\right|}}\right). \qquad (18)
\end{aligned}
$$

B Simulating from Multivariate Distributions

We give some information on how to simulate observations from an arbitrary K dimensional multivariate joint distribution. A recent reference is Li et al. (1996). At first, K independent vectors of independent uniform pseudo random numbers are generated. The j^{th} vector of random variables is then generated by inverting a copula function conditional on an increasing number of variables. If P is a K dimensional distribution function with one dimensional marginals $F_1,...,F_K$, the copula function C is such that

$$P(x_1,...,x_K)=C(F_1(x_1),...,F_K(x_K)).$$

Let $(X_1,...,X_K)$ be a K vector of random variables with joint distribution P and marginals F_j, $1\le j\le K$, and $(U_1,...,U_K)$ be a vector of independent uniform $[0,1]$ random variables. Then,

$$
\begin{aligned}
(U_1,...,U_K) &= \Psi_P(X_1,...,X_K) \\
&= \left(F_1,F_{2|1},...,F_{K|1,...,K-1}\right),
\end{aligned}
$$

where $F_{j|1,...,j-1}$ stands for the conditional distribution of X_j given $\sigma(X_1,...,X_{j-1})$, and $\sigma(X_1,...,X_{j-1})$ is the sigma algebra generated by $(X_1,...,X_{j-1})$. Denoting

$F^{-1}_{j|1,...,j-1}$ to be the inverse of $F_{j|1,...,j-1}$, we can recursively define our simulated sample from the distribution P to be

$$\begin{aligned} X_1 &= F_1^{-1}(u_1) \\ &\vdots \\ X_K &= F^{-1}_{K|1,...,K-1}(u_K|X_1,...,X_{K-1}). \end{aligned}$$

Let $C_K(...)$ be the copula of $F_1(X_1)$, ..., $F_K(X_K)$, and $C_{K-1}(...)$ be the copula of $F_1(X_1)$, ..., $F_{K-1}(X_{K-1})$, i.e.

$$C_{K-1}(F_1(X_1),...,F_{K-1}(X_{K-1})) = C_K(F_1(X_1),...,F_{K-1}(X_{K-1}),F_K(\infty))$$

then

$$F_{K|1,...,K-1}(X_1,...,X_K) = \frac{\partial^{K-1} C_K(u_1,...,u_K)}{\partial u_1 \cdots \partial u_{K-1}} \frac{\partial u_1 \cdots \partial u_{K-1}}{\partial^{K-1} C_{K-1}(u_1,...,u_{K-1})},$$

where $u_1,...,u_K$ are the univariate marginals. If $K = 2$, then

$$F_{2|1}(X_1,X_2) = \frac{\partial C(u_1,u_2)}{\partial u_1}.$$

See Darsow et al. (1992) for the proof when $K = 2$.

While it is conceptually simple, sometimes it may not be possible to invert the conditional copula. In these cases, the values have to be obtained by numerical solution of the nonlinear equation; e.g. Newton-Raphson iteration.

C Geometric Versus Arithmetic Returns and Relative Prices

Geometric returns may lead to some problems, and in some cases it is preferable to use arithmetic returns. Actually, if x is a log-return, we could just substitute $\exp\{x\} - 1$ for the arithmetic returns or just $\exp\{x\}$ for relative prices. Since this changes only the integrand, there is no difficulty involved. However, it is well known that evaluation of exponential functions is slower. Therefore, from a computational point of view, it would not be advisable to use this method when time is an issue.

Because of this and possibly other theoretical problems, it may be convenient to model arithmetic returns directly. Fortunately, arithmetic returns can still be represented using (6) by simple truncation below -1. The model can be written in more general terms as

$$\omega ab\left(|x-\mu|+\delta\right)^{b-1}\exp\left\{-a\left(|x-\mu|+\delta\right)^{b}\right\},\ \text{if}\ \tau \leq x < \infty \tag{19}$$

where τ is the truncation level, i.e. $\tau = -1$; however, we may also find it convenient to use $\tau = 0$ when dealing with relative prices. Further, ω serves the same purpose as λ in (6) and is given by $\exp\{a(\mu - \tau + \delta)\}$, where all the variables have already been defined in (6). Notice that this specification is symmetric. A more general asymmetric model can be defined by modification of ω.

Figure I. KS Copula $(\delta = 0.76)$.

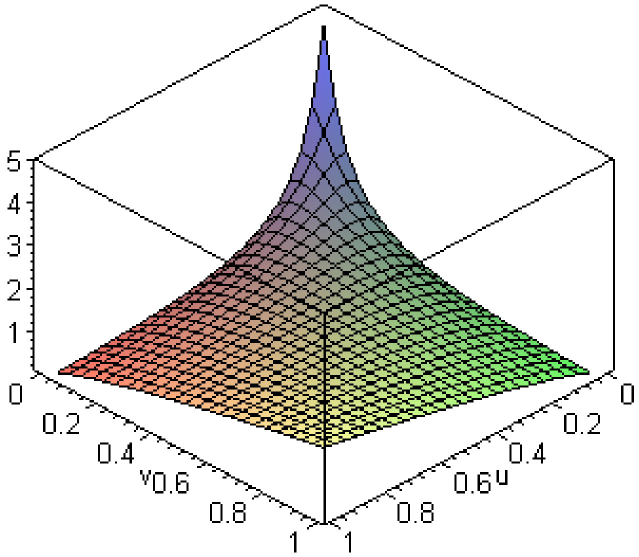
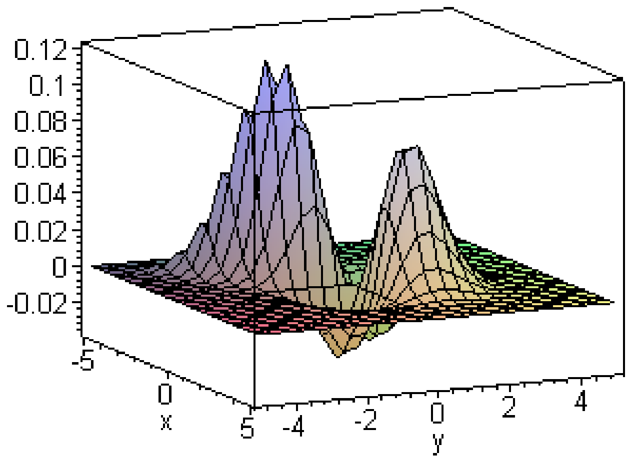


Figure II. Integrand Transform

Original Integrand.



Transformed Integrand.

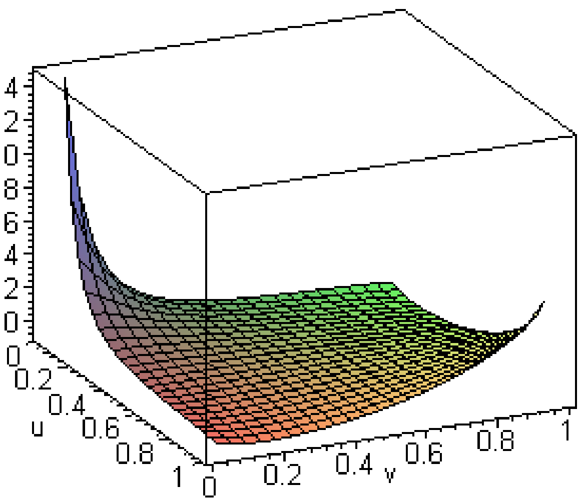


Table I. Bivariate Normal.

rho	$AMC_{[-2.5,2.5]}$	$AMC_{[-5,5]}$	$AMC_{[-10,10]}$	ISMC	CMC	$MCMC_1$	$MCMC_2$	MCMC
				N=10000				
0.2	0.1624	0.1998	0.2010	0.2005	0.2002	0.0639	0.1357	0.1996
sd	0.0075	0.0155	0.0290	0.0127	0.0123	0.0064	0.0090	0.0113
0.4	0.3272	0.3999	0.4011	0.4009	0.4002	0.1349	0.2645	0.3994
sd	0.0081	0.0173	0.0329	0.0203	0.0195	0.0071	0.0159	0.0178
0.6	0.4974	0.6001	0.6012	0.6012	0.6000	0.2216	0.3777	0.5993
sd	0.0093	0.0206	0.0402	0.0356	0.0346	0.0085	0.0308	0.0322
0.8	0.6778	0.8001	0.8014	0.8009	0.7991	0.3371	0.4630	0.8001
sd	0.0122	0.0273	0.0535	0.0666	0.0655	0.0113	0.0642	0.0651
rho				N=100000				
0.2	0.1624	0.2000	0.1998	0.1999	0.1999	0.0641	0.1360	0.2001
sd	0.0020	0.0046	0.0093	0.0038	0.0039	0.0020	0.0027	0.0034
0.4	0.3272	0.3999	0.3996	0.3997	0.3997	0.1352	0.2649	0.4001
sd	0.0021	0.0052	0.0107	0.0060	0.0065	0.0022	0.0046	0.0052
0.6	0.4974	0.5999	0.5994	0.5990	0.5992	0.2221	0.3780	0.6002
sd	0.0025	0.0063	0.0129	0.0108	0.0123	0.0025	0.0090	0.0096
0.8	0.6776	0.7998	0.7990	0.7975	0.7979	0.3378	0.4623	0.8002
sd	0.0035	0.0083	0.0170	0.0219	0.0246	0.0034	0.0185	0.0192

Table II. KS Copula with Normal Marginals.

delta	$AMC_{[-2.5,2.5]}$	$AMC_{[-5,5]}$	$AMC_{[-10,10]}$	ISMC	CMC	$MCMC_1$	$MCMC_2$	MCMC
				N=10000				
0.31	0.1709	0.2165	0.2176	0.2166	0.2159	0.0663	0.1499	0.2162
sd	0.0077	0.0161	0.0296	0.0248	0.0196	0.0065	0.0223	0.0233
0.76	0.3499	0.4227	0.4242	0.4231	0.4208	0.1512	0.2706	0.4217
sd	0.0090	0.0195	0.0361	0.0613	0.0463	0.0078	0.0508	0.0514
1.55	0.5339	0.6184	0.6210	0.6198	0.6182	0.2612	0.3555	0.6167
sd	0.0119	0.0260	0.0494	0.1087	0.0813	0.0106	0.0786	0.0792
3.19	0.6952	0.7876	0.7915	0.7889	0.7916	0.3755	0.4094	0.7849
sd	0.0172	0.0363	0.0723	0.1759	0.1158	0.0155	0.1158	0.1171
delta				N=100000				
0.31	0.1709	0.2166	0.2163	0.2159	0.2160	0.0666	0.1501	0.2167
sd	0.0020	0.0047	0.0096	0.0079	0.0087	0.0020	0.0068	0.0072
0.76	0.3499	0.4228	0.4223	0.4205	0.4207	0.1516	0.2714	0.4230
sd	0.0025	0.0059	0.0120	0.0216	0.0197	0.0023	0.0173	0.0177
1.55	0.5339	0.6185	0.6175	0.6157	0.6152	0.2621	0.3569	0.6190
sd	0.0035	0.0081	0.0165	0.0384	0.0318	0.0032	0.0301	0.0307
3.19	0.6948	0.7867	0.7850	0.7842	0.7830	0.3769	0.4108	0.7877
sd	0.0053	0.0118	0.0234	0.0548	0.0503	0.0049	0.0466	0.0472

Table III. KS Copula with Exponential Marginals.

	$AMC_{[0,10]}$	$AMC_{[0,20]}$	$AMC_{[0,30]}$	ISMC	CMC	$MCMC_1$	$MCMC_2$	MCMC
delta				N=10000				
0.31	1.2805	1.2858	1.2863	1.2892	1.2867	0.6841	0.6016	1.2857
sd	0.0258	0.0550	0.0817	0.2058	0.1156	0.0129	0.1117	0.1125
0.76	1.5484	1.5556	1.5570	1.5682	1.5607	0.7614	0.8023	1.5636
sd	0.0308	0.0670	0.1001	0.6286	0.3573	0.0185	0.3320	0.3322
1.55	1.7646	1.7728	1.7773	1.7733	1.7890	0.8777	0.9067	1.7844
sd	0.0425	0.0892	0.1346	0.6919	0.6889	0.0288	0.5035	0.5031
3.19	1.9044	1.9182	1.9320	1.8835	1.9407	1.0063	0.9035	1.9098
sd	0.0623	0.1276	0.1919	0.6562	1.1167	0.0452	0.6764	0.6762
delta				N=100000				
0.31	1.2806	1.2859	1.2860	1.2892	1.2842	0.6844	0.5997	1.2841
sd	0.0079	0.0169	0.0253	0.0648	0.0454	0.0041	0.0305	0.0308
0.76	1.5485	1.5554	1.5553	1.5691	1.5527	0.7618	0.7895	1.5513
sd	0.0094	0.0208	0.0310	0.2251	0.1399	0.0058	0.0870	0.0872
1.55	1.7648	1.7710	1.7705	1.7953	1.7722	0.8779	0.8902	1.7681
sd	0.0132	0.0280	0.0413	0.3674	0.2544	0.0090	0.1518	0.1521
3.19	1.9040	1.9088	1.9072	1.9354	1.9155	1.0055	0.9028	1.9083
sd	0.0201	0.0392	0.0582	0.3838	0.3885	0.0144	0.2438	0.2441

List of other working papers:

2004

1. Xiaohong Chen, Yanqin Fan and Andrew Patton, Simple Tests for Models of Dependence Between Multiple Financial Time Series, with Applications to U.S. Equity Returns and Exchange Rates, WP04-19
2. Valentina Corradi and Walter Distaso, Testing for One-Factor Models versus Stochastic Volatility Models, WP04-18
3. Valentina Corradi and Walter Distaso, Estimating and Testing Sochastic Volatility Models using Realized Measures, WP04-17
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