Term Structure Field Models via Matrix Completion

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Abstract

For any type of term structure field model the essential input is the instantaneous correlation function. We suggest a construction of a strictly positive definite correlation function which is consistent with the observed sample correlation matrix, infinite-dimensional structure of the field, and monotonicity requirements. The construction is based on a solution of a constrained matrix completion problem.

Key words: Term structure of interest rates, random field models, strictly positive definite correlation functions, constrained matrix completion problems.

1 Introduction

In recent years several researchers have used the random field framework for modelling evolution of the term structure of interest rates. Random fields term structure models have several advantages over their multi-factor counterparts. The infinite number of risks associated with infinitely many instantaneous forward rates can be easily represented by the infinite-factor structure of random fields. In order to price or risk manage derivative securities in a field model,

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only an estimate of the covariance function of the rates is needed. Thus, random fields offer a much more parsimonious description of term structure dynamics than the multi-factor term structure models. Furthermore, the random field framework naturally accounts for the fact that the best hedging instrument for a bond is another bond of similar maturity. This is in stark contrast to finite factor models. The field models are also compatible with any sample of forward rates. That is, there is always a possible path for the random field over a finite interval, that can lead from the forward curve at the beginning of the interval to the forward curve at the end of the interval.

There are two ways to set up a term structure field[1] model. The first has been considered by Kennedy (1994). He suggested a model, in which the forward rates follow a continuous Gaussian random field so that they evolve as a continuous random surface. In particular, he modelled the instantaneous forward rates as

$$f_{s,t} = \mu_{s,t} + X_{s,t},$$

where the subscript s refers to the calender time, and t, depending on specification, either the time of maturity or the time to maturity. The drift function $\mu_{s,t}$ is deterministic and continuous, and $X_{s,t}$ is continuous Gaussian random field with covariance structure between X_{s_1,t_1} and X_{s_2,t_2} specified by $c(s_1 \wedge s_2, t_1, t_2)$. Furthermore, Kennedy (1994) derived a simple necessary and sufficient condition on the drift surface:

$$\mu_{s,t} = \mu_{0,t} + \int_0^t c(s \wedge \nu, \nu, t) d\nu,$$

which ensures that the discounted prices of zero-coupon bonds are martingales.

The second approach has been considered by Goldstein (2000), and Santa-Clara and Sornette (2001). They utilise the Stochastic Partial Differential Equations, whose solutions are random fields, to model the instantaneous forward rates as

$$df_{s,t} = \mu_{s,t} ds + \sigma_{s,t} dZ_{s,t}.$$

Here, the instantaneous forward rates evolve via shocks from a field $dZ_{s,t}$. For all dates s, the field describes a realisation of a random field $dZ_{s,t}$ for all t. The arbitrage-free condition can be again expressed in terms of the covariance function:

$$\mu_{s,t} = \sigma_{s,t} \int_s^t du \sigma_u(s) c(s, t, u),$$

where the function c is the covariance function between the the changes in the

[1] There has been interest in term structure field models in mathematical literature. For example, Hamza, Jacka and Klebaner (2003) used field models to generalize the fundamental theorem of asset pricing. Geometrical aspects have been considered by Filipovic and Teichmann (2004).

forward rates:

$$\mathrm{Corr}(df_{s,t_1}, df_{s,t_2}) = \mathrm{Corr}(dZ_{s,t_1}, dZ_{s,t_2}) = c(s, t_1, t_2)ds.$$

The application of random field models is not restricted to pricing. They should be more appreciated within risk management context, where impact of high number of factors is more crucial. Hodges and Weigel (2003) have considered these applications. Bouchaud *et al.* (1998) have also suggested SPDE type field models based on statistical properties of the forward rate curve.

To complete the description of the field models in any of the above frameworks the instantaneous covariance functions need to be specified. These functions should be strictly positive definite, thus providing a truly infinite-factor structure for the model. They also need to be flexible enough to fit the empirical sample covariance matrix observed in the market. The construction of such functions has not been addressed in the research literature and is the subject of this paper. We take the variance function as given [2], and concentrate only on the correlation function. Our idea comes from Matrix Completion Theory, Johnson (1990). We consider a matrix which includes the observed sample correlation matrix and the missing entries of correlations that are unknown. These missing entries can be thought of as free variables, or "unspecified entries" in the matrix completion literature.

This formulation allows us to cast the problem as a typical constrained matrix completion problem. In such a problem one looks for a choice of the unspecified entries so that the resulting ordinary matrix (called a completion of the given partial matrix) is of a desired type (e.g. positive definite). The unspecified entries are not independent free variables, but are subject to constraints (e.g. linear equation, or, as in our case, monotonicity).

Generally, the "shapes" of the sample correlation matrices in the interest rate market can be quite complex. However, they usually exhibit patterns such as monotonicity off the diagonal, i.e. $k \to \mathrm{Corr}(T_{i,j}, T_{i+k,j})$ is monotone decreasing. This is intuitive from economic point of view: we expect the correlation between, say 1 and 2 year rates to be higher than that between 1 and 10 years rates. A second pattern is the monotonicity along the diagonals, i.e. $j \to \mathrm{Corr}(T_{i,i+k}, T_{j,j+k})$ is monotone increasing. That is, rates for shorter maturities with the same distance apart usually have correlations lower than such rates with longer maturities. These patterns constraints the constraints in our completion problem.

In summary, our method produces an estimate matrix of the correlation func-

[2] The variance function can be easily constructed using curve approximation and interpolation techniques.

tion for any number of maturities. This matrix is positive definite, so it is compatible with an assumption that the correlation function be strictly positive definite (infinite-factor field model). It is also compatible with stylised monotonicity patterns usually observed in interest rate markets. Furthermore, the correlation estimates for the observed maturities are identical with the sample correlations.

The remainder of this paper is organised as follows. In the next section we set up and solve a constrained matrix completion problem. In Section 3 we discuss parameter estimation and implementation of the method. Furthermore, we present applications of the method based on both, stylised correlation matrices and sample correlation matrices estimated from empirical data. Section 4 concludes.

2 Constrained Positive Definite Completion

An $n \times n$ matrix is called *doubly nonnegative* (*positive*) if it is both positive semidefinite and entry-wise nonnegative (positive definite and entry-wise positive). We call a doubly positive correlation matrix *monotone* if its entries strictly decrease in each row leading away from the main diagonal. For example,

$$A = \begin{pmatrix} 1 & .3 & .2 & .1 \\ .3 & 1 & .4 & .2 \\ .2 & .4 & 1 & .5 \\ .1 & .2 & .5 & 1 \end{pmatrix}$$

is monotone. By symmetry, the entries also decrease in columns leading away from the diagonal. If, in addition, entries increase along diagonals parallel to the main diagonal, we call such a correlation matrix *doubly monotone.*

Question: May a row (column) be inserted between two consecutive rows (columns) of an $n \times n$ (doubly) monotone matrix to achieve an $(n+1) \times (n+1)$ monotone matrix? The answer is "yes", but we actually show more. First, some notation for this problem is useful. For an $n \times n$ matrix $A = (a_{ij})$, let $\hat{A}$ be the $(n+1) \times (n+1)$ partial matrix, for which $\hat{A}(\{k\})$, $\hat{A}$ with row and column k deleted, is A and all other entries of $\hat{A}$ are unspecified. Then, denote by A_α the particular completion of $\hat{A}$ for which row (column) k of A_α is α times row (column) $k-1$ of A_α plus $(1-\alpha)$ times row (column) $k+1$ of A_α. In other words a row and column is inserted into A by averaging two consecutive rows/columns with weight α.

We illustrate a simple way to calculate A_α when $n = 4$, A is a correlation matrix and $k = 3$. Let

$$A = \begin{pmatrix} 1 & a & d & f \\ a & 1 & b & e \\ d & b & 1 & c \\ f & e & c & 1 \end{pmatrix}$$

Then $\hat{A}$ is

$$\begin{pmatrix} 1 & a & ? & d & f \\ a & 1 & ? & b & e \\ ? & ? & ? & ? & ? \\ d & b & ? & 1 & c \\ f & e & ? & c & 1 \end{pmatrix},$$

and A_1 may be obtained, for example, by first replacing column 3 with column 2,

$$\begin{pmatrix} 1 & a & a & d & f \\ a & 1 & 1 & b & e \\ ? & ? & ? & ? & ? \\ d & b & b & 1 & c \\ f & e & e & c & 1 \end{pmatrix},$$

and then replacing row 3 with the (new) row 2:

$$A_1 = \begin{pmatrix} 1 & a & a & d & f \\ a & 1 & 1 & b & e \\ a & 1 & 1 & b & e \\ d & b & b & 1 & c \\ f & e & e & c & 1 \end{pmatrix}.$$

Note that, at first, the third diagonal entry is undetermined, but that it is determined and is 1 after the second stage. In a similar manner (replacing column 3 of $\hat{A}$ with column 4 and then row 3 with row 4), A_0 is determined as

$$A_0 = \begin{pmatrix} 1 & a & d & d & f \\ a & 1 & b & b & e \\ d & b & 1 & 1 & c \\ d & b & 1 & 1 & c \\ f & e & c & c & 1 \end{pmatrix}.$$

Then A_α is $\alpha A_1 + (1-\alpha)A_0$, which is

$$A_\alpha = \begin{pmatrix} 1 & a & \alpha a + (1-\alpha)d & d & f \\ a & 1 & \alpha + (1-\alpha)b & b & e \\ \alpha a + (1-\alpha)d & \alpha + (1-\alpha)b & 1 & \alpha b + (1-\alpha) & \alpha e + (1-\alpha)c \\ d & b & \alpha b + (1-\alpha) & 1 & c \\ f & e & \alpha e + (1-\alpha)c & c & 1 \end{pmatrix}.$$

Thus, A_α is a correlation matrix and meets the mentioned requirements.

Theorem 1 *If A is positive definite, and $0 < \alpha < 1$, then A_α is positive definite.*

Proof: Consider first A_1 and A_0. Each of these $(n+1) \times (n+1)$ matrices is positive semidefinite of rank n. Each is singular, because a row is repeated, and, therefore one eigenvalue is 0. On the other hand, each matrix is symmetric and, by the interlacing theorem for eigenvalues of symmetric matrices, Horn Johnson (1985), pp. 185-186, has n positive eigenvalues, as A, which is a principal submatrix, has n positive eigenvalues. Now, notice that the null space of A_1 is spanned by

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(entry 1 is at the k^{th} position) as row k is repeated, and the null space of A_0

is spanned by

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(entry 1 is at the $(k+1)^{\text{st}}$ position) as row $k+1$ is repeated. The symmetric matrix $A_\alpha = \alpha A_1 + (1-\alpha)A_0$ and, thus, is at least positive semidefinite when $0 \leq \alpha \leq 1$. However, the sum of two positive semidefinite matrices (in this case αA_1 and $(1-\alpha)A_0$, for $0 < \alpha < 1$) is positive definite, unless their null spaces intersect. In this case the null spaces do not intersect, so that $\alpha A_1 + (1-\alpha)A_0 = A_\alpha$ is positive definite for $0 < \alpha < 1$, as claimed. *q.e.d.*

Before continuing, we make two observations related to Theorem 1. First, if $\hat{A}$ is completed in some other way to produce symmetric $\tilde{A}$ so that the entries of row k lie between those of row $k-1$ and row $k+1$, $\tilde{A}$ may or may not be positive definite. It will be positive definite if and only if $\det(A) > 0$, as, again, the principal submatrix A is positive definite (so that if $\det(A) > 0$ there will be a full nested sequence of positive principal minors). Second, among the completions A_α, $0 \leq \alpha \leq 1$, there is a unique one with maximum determinant, namely $A_{\frac{1}{2}}$.

This may be proven as follows. Suppose, without loss of generality, that $A = (a_{ij})$ is a positive definite correlation matrix, and note that the column vector $b_k(\alpha)$ of off-diagonal entries that complete $\hat{A}$ is $b_k(\alpha) = \alpha a(k-1) + (1-\alpha)a(k)$, in which $a(j)$ denotes the $j-th$ column of A. Then, by Schur complements, Horn Johnson (1985), Ch. 0, we have $\det(A_\alpha) = (1 - b_k(\alpha)^T A^{-1} b_k(\alpha))\det(A)$. But, $det(A) > 0$ is constant w.r.t. α, and $A^{-1}b_k(\alpha) = A^{-1}(\alpha a(k-1) + (1-\alpha)a(k)) = \alpha e_{k-1} + (1-\alpha)e_k$, in which e_j denotes the j-th standard unit basis vector ($e_j^T = (0, \cdots, 0, 1, 0, \cdots, 0)$). Then, $b_k(\alpha)^T A^{-1} b_k(\alpha) = \alpha^2 a_{k-1,k-1} + (1-\alpha)^2 a_{kk} + \alpha(1-\alpha)a_{k-1,k} + \alpha(1-\alpha)a_{k,k-1} = \alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)a_{k-1,k}$. Thus, except for the constant factor $\det(A)$, $\det(A_\alpha) = 1 - \alpha^2 - (1-\alpha)^2 - 2\alpha(1-\alpha)a_{k-1,k}$, so that

$$\frac{d}{d\alpha}\det A_\alpha = -(2\alpha - 2(1-\alpha)) + 2(1-2\alpha)a_{k-1,k}) = (2-4\alpha)(1-a_{k-1,k}).$$

Since $1 - a_{k-1,k} > 0$, as A is a correlation matrix, the only critical point then occurs at $\alpha = \frac{1}{2}$. Since there is a maximum for $\det(A_\alpha)$ on $[0,1]$ by Weierstrass' theorem and it cannot occur at 0 or 1 as ($\det(A_0) = \det(A_1) = 0$), it must occur at $\alpha = \frac{1}{2}$, as claimed.

Now, we may apply theorem 1 to answer our original question. Notice that if A is a correlation matrix, both A_1 and A_0 are as well, so that A_α is also a correlation matrix. Further, if A is doubly positive, so is A_α, $0 < \alpha < 1$. We may conclude

Corollary 2 *If A is monotone, then each A_α, $0 < \alpha < 1$, is monotone as well.*

Proof: As noted above, A_α will be a doubly positive and a correlation matrix. It is easily checked that if the entries of A decrease, moving away from the diagonal, those of A_α, $0 < \alpha < 1$, will as well. *q.e.d.*

Finally, we may also conclude a parallel statement for doubly monotone matrices. The proof is again a matter of checking inequalities that govern the relative position of entries.

Corollary 3 *If A is doubly monotone, then each A_α, $0 < \alpha < 1$, is monotone as well.*

3 Implementation, Estimation, and Examples

We first discuss the implementation of our our completion method for a given value of the parameter α, and then the estimation of this parameter. The problem can be formulated as follows. For a given vector of the observed maturities $(T_1, \ldots, T_n)$, we need to find the correlations of these maturities with a new maturity $T \in [T_i, T_{i+1}]$, for $i \in \{1, \ldots n-1\}$. Unfortunately, the completion method we described in the previous section does not associate the completion with a particular maturity, it simply gives a completion between two neighbouring dates. We assign this completion the middle of the interval, i.e. $(T_i + T_{i+1})/2$. Having now associated completion with particular maturity we apply a bisection method to find a completion for any maturity $T \in [T_i, T_{i+1}]$, for $i \in \{1, \ldots n-1\}$. We simply keep bracketing maturity date T till we obtain the completion for either T itself or a date sufficiently close to it.

The only parameter we need to estimate is the convexity parameter α. We estimate α by minimising an objective function which we construct as follows. The idea is similar to the evaluations of the minors a matrix. We start with a $n \times n$ sample matrix A, with entries $a_{T_iT_j}$ signifying the correlations between maturities T_i and T_j. Consider matrices $A(T_i)$, $i = 2, \ldots, n-1$ which are obtained by deleting the row T_i and the column T_i in the matrix A. The

$(n-1)\times(n-1)$ matrices $A(T_i)$ are the same as A except that they don't include correlations with the maturity T_i. For each of the matrices $A(T_i)$ we construct a completion for maturity T_i and calculate how "far" this completed row and column is from the actual row and column in the sample correlation matrix A. That is, for each completion we sum up the absolute value of the differences between the completed rows and the actual rows in the sample correlation matrix A. The sum of these $(n-2)$ error terms constitutes the objective function with parameter α as the argument. Minimising this objective function we obtain the optimal parameter α.

To highlight the effect of the convexity parameter α in our completion method we consider a 2×2 stylised correlation matrix between maturities 1 and 9 years, with correlation between these maturities 0.5. We assume that we are interested in the maturity of 5 years. The completion for this maturity for parameter $\alpha = 0.5$ is the first matrix in (1).

$$\begin{pmatrix} 1.00 & 0.75 & 0.50 \\ 0.75 & 1.00 & 0.75 \\ 0.50 & 0.75 & 1.00 \end{pmatrix}, \quad \begin{pmatrix} 1.00 & 0.65 & 0.50 \\ 0.65 & 1.00 & 0.85 \\ 0.50 & 0.85 & 1.00 \end{pmatrix}. \tag{1}$$

For this choice of α we have achieved monotonic decorrelation off the diagonal. However, the correlations along diagonal are constant. Monotonic correlations along the diagonal, can be obtained with a different choice of α. For example, with $\alpha = 0.3$, we obtain the second completion matrix in (1). This completion is monotonic in both off the diagonal and along the diagonal.

To test our framework on real data we have used the sample correlation matrix arising from Japanese Yen for the period 2/12/96 to 25/10/01. The data is courtesy of LCH, London, and has been obtained by stripping the Japanese government issues.

In Figure 1 we plotted the sample covariance and correlation matrices. The plot of the sample correlation shows an expected decreasing behaviour: the further the distance between any two maturities, the lower the correlation. The actual values of sample correlations are in Table 4. Also, observe the high correlation between maturities. The figures fluctuate between 0.659 and 0.998. Observe the monotonic shape of the sample correlation matrix off the diagonal and along the diagonals.

We have estimated the parameter α by minimising the objective function described above, and obtained the value $\alpha = 0.408$. This value is close to 0.5 and is not unexpected: The sample correlation matrix is monotonically increasing along the diagonals and for most of maturities has a very small slope. We have completed this matrix for maturities that are middle points between the observed maturities. A part of this completed matrix is in Table 4. In Figure 2 we have plotted the completed matrix. Observe that the completed matrix maintains the same shape as uncompleted matrix in Figure 1.

So far, we have assumed the parameter α to be constant. This need not be the case. One straightforward generalisation is to vary α linearly between two neighbouring maturity. I.e., allow it to go from 0 to 1 as we move along from one observed maturity to the next. In this case we can avoid bisection altogether, as the parameter α for a given maturity will be a function of this maturity.

A more profound questions arising in this context relate to the properties of the limiting surface. We have suggested a method for construction of strictly positive surfaces however the properties of this surface are still unclear. E.g., if we believe in a smooth evolution of the fields, then, necessarily, the correlation surface need to be smooth as well. Second issue is about the choice of the convexity parameter α. Realistically, it needs to be a function of maturities and distances between neighbouring maturities. The precise functional form needs to be inferred empirically. We currently investigate these issues.

4 Conclusions

Our main objective was to construct strictly positive definite correlation functions for infinite-factor Gaussian field models. These functions should be able to fit sample correlation matrices observed in the practice. They also should exhibit monotonic patterns usually associated with such sample correlation matrices.

We have developed a method for construction of estimates of such functions for any set of maturities. This estimate is consistent with all the requirements we expect from a reasonable correlation function: it exactly fits the sample correlation matrix, is positive definite for any number of maturities and respects the usual patterns of sample correlation matrices. The method we develop allows calibration of field models to key market information, namely the covariation of the rates. Thus, we can capitalise on the main advantage of the field models, i.e. capturing the inter-dynamics of movements in the term structure. Our construction makes the random field methodology a much more practical tool.

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	2	3	4	5	6	7	8	9	10	12	15	20	25
3	0.964												
4	0.909	0.975											
5	0.856	0.940	0.985										
6	0.813	0.907	0.965	0.991									
7	0.773	0.872	0.942	0.976	0.993								
8	0.755	0.857	0.928	0.966	0.987	0.996							
9	0.739	0.841	0.915	0.955	0.980	0.992	0.996						
10	0.723	0.827	0.904	0.946	0.972	0.987	0.993	0.997					
12	0.707	0.811	0.888	0.932	0.961	0.977	0.985	0.991	0.996				
15	0.692	0.797	0.873	0.918	0.948	0.966	0.976	0.984	0.990	0.995			
20	0.673	0.778	0.855	0.902	0.933	0.953	0.964	0.974	0.981	0.988	0.995		
25	0.667	0.771	0.847	0.894	0.926	0.946	0.958	0.968	0.975	0.984	0.992	0.998	
30	0.659	0.762	0.837	0.884	0.916	0.936	0.948	0.959	0.967	0.976	0.985	0.993	0.998

Table 1
Sample correlation matrix of changes in interest rates.

	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8
2.5	0.979												
3	0.964	0.985											
3.5	0.931	0.963	0.985										
4	0.909	0.948	0.975	0.990									
4.5	0.877	0.923	0.954	0.976	0.991								
5	0.856	0.906	0.940	0.967	0.985	0.994							
5.5	0.831	0.884	0.920	0.952	0.973	0.986	0.995						
6	0.813	0.868	0.907	0.941	0.965	0.980	0.991	0.996					
6.5	0.789	0.847	0.886	0.925	0.951	0.969	0.982	0.990	0.996				
7	0.773	0.832	0.872	0.913	0.942	0.962	0.976	0.986	0.993	0.997			
7.5	0.762	0.822	0.863	0.905	0.934	0.955	0.970	0.981	0.989	0.994	0.998		
8	0.755	0.815	0.857	0.899	0.928	0.950	0.966	0.978	0.987	0.992	0.996	0.998	
8.5	0.745	0.806	0.848	0.891	0.921	0.944	0.960	0.973	0.983	0.989	0.994	0.996	0.998

Table 2
Sample correlation matrix of changes in interest rates together with completed correlations for selected maturities.

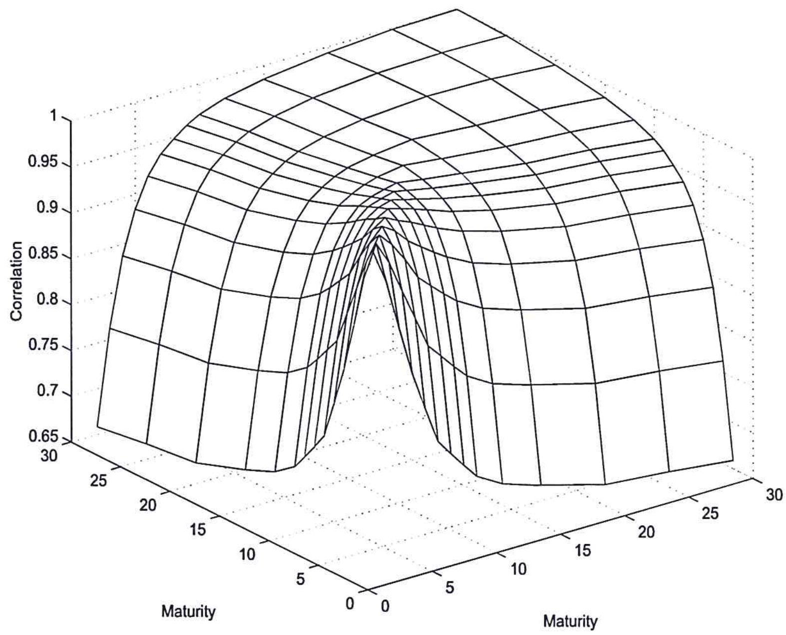

Fig. 1. Sample correlation matrix of changes in interest rates.

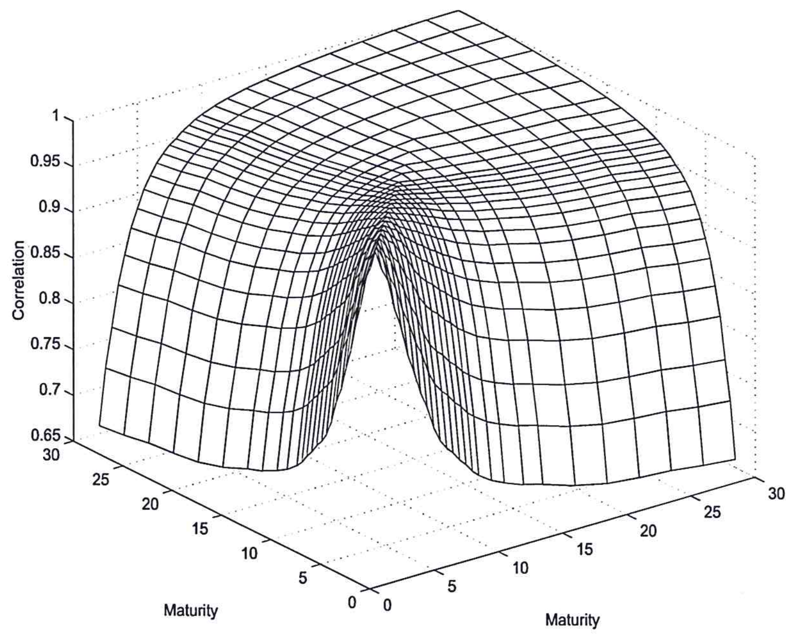

Fig. 2. Completed sample correlation matrix on with completed maturities being the middle points between the known maturities.