Equilibrium Model for Commodity Prices: Competitive and Monopolistic Markets*

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Abstract

In this article, we develop an equilibrium model for storable commodity prices. The model is formulated as a stochastic dynamic control problem and considers two state variables - the exogenous supply and the inventory. The inventory is a fully controllable endogenous variable. We assume that the uncertainty arises from the supply, which evolves as a Ornstein-Uhlenbeck stochastic process. This model is developed under a general framework which provides two distinct forms for the alternative economic scenarios of perfect competition and monopolistic storage. Since an analytical solution to the problem is not possible, we obtain a numerical solution that provides an optimal storage policy and generates the price dynamics. We also compute and analyse the equilibrium forward curves that result from the steady state optimal storage policy. The results are consistent with the theory of storage: the presence of storage in both economies stabilizes the natural prices. We also show that this effect is greater when the storage is competitive. The resulting forward curves take two fundamental shapes. If the initial spot price is greater than the long-run natural price we observe backwardation; otherwise the market is in contango. Furthermore, the degree of contango is greater in the competitive market.

`Keywords:` storage, structural model, price dynamics, equilibrium, continuous-time.

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1 Introduction

Over the past two decades energy markets such as electricity, natural gas, petroleum products and coal have undergone significant changes. The market has evolved from a monopolistic, stable pricing environment characterized by long term contracts with guaranteed margins to a competitive and volatile market environment. In this deregulated environment, market participants have found themselves increasingly exposed to price movements and to counterparty performance risk. This highlights the necessity to develop adequate models for energy prices that assist on designing market trading strategies and storage policies in the commodity industry.

Commodity prices in general are harder to model than other well developed conventional financial assets, such as equities. This is partly due to the fundamental price drivers in commodity markets which are more complex than in standard financial assets. In the case of energy commodities this difficulty is reinforced by the recent dramatic changes in the way energy is traded. In order to model the behaviour of commodity prices, it is necessary to understand the dynamic interplay between demand, supply and storage. In particular, storage plays a central role in shaping the behaviour of the prices of a storable commodity. On the supply side, storage plays a vital role in stabilizing spot prices by allowing an intertemporal shift of supply in response to shortage. As such, storage is one of the key elements that determines the degree of volatility in commodity prices, that is, the variance of spot price movements over time decreases with the amount in store and vice-versa. In addition, storage limitations may significantly increase the volatility of spot prices when there is a shortfall in commodity's availability. Similarly, storage also influences the extent to which the Samuelson (1965) effect is observed in commodity prices behaviour.

Commodity price dynamics and the economics of commodity storage has been the subject of numerous recent studies. Nevertheless, the research has been largely disjoint. On one side we assist to the development of discrete time structural models that focus on the behaviour of agricultural commodities and where storage takes a central role in the modelling process. On the other side, we assist to the recent development of reduced form models that emerged since the energy market became deregulated.

The equilibrium structural models are derived explicitly from economic principles and aim to replicate the equilibrium price for storable commodities. Most of the existing studies focus on establishing an equilibrium price model for agricultural commodities where the supply is determined by speculative storage and random behavior of harvests. The price is is obtained through numerical approximations that relate supply, demand and storage. This approach is standard and is described in Williams and Wright (1991) and is also adopted by Deaton and Laroque (1992, 1996), Chambers and Bailey (1996), Routledge et al. (2000).

The reduced form class of models dominates the current literature and practice on energy derivatives. Leading models include Gibson and Schwartz (1990), Schwartz (1997), Miltersen and Schwartz (1998) and Schwartz and Smith (2000). Generally, these models consider that the spot price and the convenience yield follow a joint stochastic process with constant correlation. The main focus of these models is to replicate the mean reversion in commodity spot prices and the dynamics of the convenience yield. Nevertheless, the use of current reduced form models in the literature to price energy contingent claims has not been effective. In particular, the convenience yield process seems to be misspecified since its specification ignores some crucial properties of commodity prices behaviour such as the dependency of prices variability on inventory levels (see Pirrong (1998) and Clewlow and Strickland (2000)). These misspecifications call for a better understanding of the supply, demand and storage roles on the dynamics of energy prices. Accordingly, the development of new structural models that take the key properties of the energy commodity markets into account is a fundamental tool that help financial managers to understand the dynamic interplay between the microeconomic factors that drive commodity markets. In this context, current research on structural models for commodity prices has been scarce and has not been following the rapid recent expansion of energy markets. As an example, none of the existing equilibrium models takes the mean reverting properties of commodity prices into account, which is a key characteristic in energy prices.

The model presented in this paper is a storage equilibrium model, which is formulated as a stochastic dynamic control problem in continuous time. The solution to this problem produces an optimal storage policy within each of the market contexts, which also generates the price dynamics. This model considers two state variables. One is the supply rate, which is a exogenous stochastic variable. The other is the inventory, which is an endogenous variable, whereby the storage policy is the decision variable in this model. This model expands the current literature on structural models for commodity prices by taking into account the mean reverting characteristics of commodity prices. One of the most innovative features of this model is that it establishes the link between the two major categories of the literature - the discrete time structural commodity price models and the continuous time reduced form models. More specifically, we develop a model based on the microeconomics of supply, demand and storage similarly to the structural models and add three important features. First, we consider a continuous time framework whereas the traditional models consider a discrete time framework. Second, we include the mean reverting characteristic of commodity spot prices in the dynamics of our model similar to those proposed in the reduced form models. Particularly, the mean reversion is introduced into the model by considering the exogenous supply as a mean-reverting O-U stochastic process. This mean-reverting process can be interpreted as the net supply, that is, the difference between the exogenous supply and the stochastic demand in the market. This interpretation is appropriate since uncertainty arises from the demand side in many commodity markets, such as energy. Although we do not consider seasonality in this model, we mention how to include this property in this model without adding complexity to the numerical computation of the solution. Third, we formulate and analyze separately the model for both competitive and monopolistic storage economies. This comparative analysis is appropriate since energy markets worldwide recently evolved from a regulated monopolistic environment to a competitive market. This comparison is not illustrated in the current literature, which only considers a competitive storage economy.

The analysis of this model is divided in two parts. The first focuses on the dynamic relationship between storage, supply and the price dynamics and draws the attention to the following issues: (i) the dependence of the storage on both the inventory level and the supply rate, (ii) how the storage policy affects changes the evolution of the commodity natural price[1], (iii) how different levels of inventory affect the commodity price variability and (iv) the differences between the competitive and the monopolistic storage policies in terms of (i), (ii) and (iii). The second part of the analysis focuses on the forward curve and convenience yield implied by this model. Specifically, we apply the steady state storage policy implied by this model to compute a trinomial tree for the commodity prices and the corresponding forward curves. With the exception of Routledge et al. (2000), the current structural models for commodity prices restrict the analysis to the properties of the spot prices as a function of the state variables and does not study the implied equilibrium forward curves. Although Routledge et al. (2000) present a structural model for commodity prices and present the analysis for the corresponding equilibrium forward curves, their study is limited. In particular, the authors limit their analysis to the case where the stochastic demand can only take

[1] By natural price we mean the commodity price evolution in the absence of storage.

two states - high and low - which is unrealistic. Moreover, the generalization of their results is difficult to obtain. In contrast, the numerical method presented here is relatively easy to implement and the corresponding analysis easy to generalize to any combinations of initial storage and supply levels.

The remaining of this paper is structured as follows. Section 2 formulates the model and describes the solution method. We present the model under a general framework and later unfold it into two distinct market scenarios: competitive and monopolistic market. Section 3 describes briefly the numerical implementation, provides numerical examples for both markets contexts and analyzes the results. Section 4 describes the numerical method used to compute the forward curves, provides numerical examples and examines the properties of the forward curves implied by this model. Section 5 concludes.

2 Storage Equilibrium

This model builds on and extends the discrete time framework formulated in Williams and Wright (1991). Williams and Wright develop a basic discrete time model for commodities in a pure competitive market using a discrete time dynamic programming approach. We use the basic storage formulation of Williams and Wright as a starting point and introduce three main features their framework and to the current structural models for the price of a storable commodity. First, we develop the model in continuous time instead of the discrete time setting used in the traditional literature. Second, we introduce mean-reverting properties in the price dynamics by modelling the exogenous supply rate as a mean reverting stochastic process of the O-U type. Finally, we extend the model to the separate case of a monopolistic storage economy and compare it with the competitive setting.

This model considers two state variables. One is the exogenous supply rate and the other is the inventory level. The stochastic supply rate can be interpreted as the difference between the exogenous supply and the stochastic part of the demand side. This interpretation is appropriate since demand is stochastic for many commodities. In the competitive storage economy, the storage decisions are made from a social planner perspective *as if* to maximize the expected present value of social welfare in the form of "consumer surplus". The planner's problem in the current period, t, is to select the current storage that will maximize the discounted stream of expected future surplus. The decision variable is the rate of storage, that is, the rate at which the commodity is bought or sold by the stockholder[2], which can be either positive or negative. For the monopolistic storage context, the competitive formulation is modified to consider that the decisions are made by the monopolistic stockholder, which maximizes the discounted stream of expected future cash-flows generated by his storage facility. For each market, the optimal storage policy is defined by specifying the rate of storage for each possible state of the world at each moment in the future.

2.1 Model Formulation

Both models are developed using the same basic framework. In one case we assume that the market (including storage) is perfectly competitive; in the other we assume that storage (only) is monopolistic. In the competitive equilibrium, we assume that the number of firms in the storage industry is sufficiently large for each to be a price taker. The storage decisions are made by a single identity, the "invisible hand". Under monopolistic storage, consumers can deal directly with producers through the market but neither group can store on its own. Only one firm has the right or the technology to store the commodity. A monopolistic firm is not the only source of the commodity for consumers since the producers also continuously supply the market. Hence, the monopolist does not extract its extra profits by holding

[2]Stockholder refers to the aggregate storage in the competitive market.

the commodity off the market to keep the price high. Likewise, the firm competes with consumers for any quantity it purchases on the market. The model is developed under the risk neutral measure whereby all the economic agents are risk neutral.

We introduce the model under a general formulation and later unfold it into the two distinct market scenarios. The general assumptions of the model are as follows:

- A single homogeneous commodity is produced and traded in continuous time, over a finite-time horizon T;
- Storage is purely speculative, whereby inventory decisions are driven by the single motive of trading profit;
- The supply has zero elasticity;
- The marginal storage cost, k, is constant; the storage cost is $k \times s$ per unit of time, where s is the current storage level;
- The one-period risk-free interest rate, $r \geq 0$ is constant.

We consider two state variables: the exogenous supply rate and the inventory level. The exogenous supply rate, z_t, is given by[3]:

$$dz_t = \alpha(\overline{z} - z_t)dt + \sigma dB_t, \quad (1)$$

where:

- α is the speed of mean reversion;
- $\overline{z}$ is the long-run mean, that is, the level to which z reverts as t goes to infinity;
- σ is the (constant) volatility;

[3] This model can be easily extended to incorporate seasonality by adding a seasonal component to the supply. In this case, the supply rate, x_t, would be given by: $x_t = z_t + c\sin(2\pi t)$. Therefore, the transition for x_t is $dx_t = (\alpha(\overline{z} - z_t) + 2\pi c\cos(2\pi t))dt + \sigma dB_t$. All the remaining theoretical results presented in this article hold if the exogenous supply rate is given by x_t instead of z_t.

- B_t is a standard Wiener process.

The aggregate storage level, s, is a fully controllable endogenous state variable and satisfies:

$$ds = u(s, z, t)dt, \qquad s \geq 0 \tag{2}$$

where u represents the rate of storage and is the decision variable in our problem. At each time t, the rate at which the commodity is stored depends on the amount already in storage, s, and on the exogenous supply, z.

Note that the decision $u(\cdot)$ is a function in $[0,\ T]$, which we call the *inventory management plan.* If the inventory capacity is $b > 0$, then the inventory level $s(t)$ must satisfy the constraint:

$$0 \leq s(t) \leq b \tag{3}$$

since negative storage is not allowed. On the other hand, if $z(t)$ is the supply rate at time t, then $u(\cdot)$ must not exceed this rate, that is:

$$u(t) \leq z(t). \tag{4}$$

Constraints (3) and (4) imply that the optimal storage rate, u^*, belongs to $[u_{\min}, u_{\max}]$ whereby the values $u_{\min}$ and $u_{\max}$ are such that these two constraints are satisfied. Any inventory management plan that satisfies these conditions is called an *admissible plan.* The total rate of consumption in the market, q, establishes the relationship between the state variables defined above and satisfies the equilibrium condition:

$$q = z - u \tag{5}$$

Moreover, the market price (or inverse demand function) is given by $p(q)$, where $\frac{\partial p}{\partial q} < 0$.

We consider a finite-time horizon T, at which there is no carryover and we work backwards in time. The following function is then to be maximized:

$$\begin{aligned} J(s_t, z_t, t; u(\cdot)) &= \{\underset{z}{E_t} \int_t^T e^{-r(l-t)} L\left(s_l, z_l, u_l, l\right) dl + \\ &\quad \Psi(s_T, z_T) | s = S,\ z = Z\}. \end{aligned} \tag{6}$$

The optimization is over all the admissible plans where $L(s_t, z_t, u_t, t)$ is the *instantaneous profit rate* and $\Psi(s_T, z_T)$ is the *salvage value* of having s_T and z_T as states at final time T. Without loss of generality, we consider $\Psi(s_T, z_T) = 0$. The crucial difference between the pure competitive and the monopolistic storage problem formulations consists in the definition of L, which we will describe later.

To find a solution to the problem we use the dynamic programming approach. Accordingly, we need to maximize a value function, $J(\cdot)$, in order to obtain the optimal set of carryover decisions through time. In other words, we apply the Bellman's principle of optimality (Bellman (1957)). This principle states that at any point of an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point. We then obtain the dynamic programming equation of the form (see derivation in appendix A):

$$-\frac{\partial V(s, z, t)}{\partial t} - H(s, z, V_s, V_z, V_{zz}) = 0 \tag{7}$$

where:

$$\begin{aligned} H(s, z, V_s, V_z, V_{zz}) &= \sup_{u \in [u_{min}, u_{max}]} \{L(s, z, u, t) + uV_s(s, z, t) + \\ &\quad \alpha(\overline{z} - z)V_z(s, z, t) + \frac{1}{2}\sigma^2 V_{zz}(s, z, t) - \\ &\quad rV(s, z, t)\} \end{aligned} \tag{8}$$

for the value function $V(s, z, t)$ with the boundary condition $V(s, z, T) = 0$. This yields the optimal u^*. Note that u^* needs to be formulated in such a way that the storage constraints are not violated, that is $u_{min} \leq u^* \leq u_{max}$. If u^{unc} represents unconstrained the maximum of the above dynamic programming equation, then:

$$u^* = u_{max}, \qquad \text{if} \qquad u_{max} \leq u^{unc}; \tag{9}$$

$$u^* = u^{unc}, \qquad \text{if} \qquad u_{min} \leq u^{unc} \leq u_{max}; \tag{10}$$

$$u^* = u_{min}, \qquad \text{if} \qquad u^{unc} \leq u_{max}; \tag{11}$$

Finally, the current price is given by:

$$p(q) = p(z - u^*) \tag{12}$$

In what follows, we separate the formulation into the competitive and the monopolistic scenarios. The distinction between these two formulations is imposed by the definition of the instantaneous profit rate, $L(s_t, z_t, u_t, t)$, which differs among these two contexts as mentioned above.

2.1.1 Competitive Market

As explained before, the competitive equilibrium evolves as if the maximization is made from a social planner perspective. This perspective is also adopted by Samuelson (1971), Robert E. Lucas and Prescott (1971) and Williams and Wright (1991). The social planner, in the current period t, aims to select the current rate of storage that will maximize the discounted stream of expected future consumer surplus. Let $p(q)$ represent the inverse demand function and also let:

$$f(x) = \int_0^x p(q)dq, \qquad \text{for} \qquad x \geq 0, \tag{13}$$

Accordingly:

$$L(s_t, z_t, u_t, t) = f(z_t - u_t) - ks_t \tag{14}$$

where k is the constant marginal storage cost per period. Accordingly, the functional to be maximized is obtained by substituting equation (14) in equation (8). By differentiating the right hand side of the resulting equation with respect to u, we obtain the first order condition that allow us to find the maximum:

$$-f'(z-u) + V_s(s,z,t) = 0 \tag{15}$$

A necessary, but not sufficient, condition in order to have a maximum is[4]:

$$f''(z-u) \leq 0 \tag{16}$$

where f' and f'' represent the first and the second order derivative of the function $f(\cdot)$ defined by equation (13).

Let $D(\cdot) = p^{-1}(x), \quad x \geq 0$ represent the demand function. If we initially ignore the fact that u^* needs to satisfy the storage constraint, the (unconstrained) maximum, u^{unc}, is given by:

$$u^{unc} = z - D(V_s^*) \tag{17}$$

Then, by taking into account the constraints given by (3) and (4) we obtain the optimal control value, u^*.

2.1.2 Monopolistic Market

We now specify the problem for the case of a monopolistic stockholder. In this case, the monopolistic storage manager in the current period, t, aims to select the current rate of storage that will maximize the discounted stream of expected future cash flows generated by the management of his storage facility. The control variable, u, represents the rate of storage, that is, the absolute change in inventory level over an infinitesimally small interval of

[4] A rigorous mathematical verification of existence and uniqueness of the solution requires additional technical work and is beyond the scope of this study.

time; hence $-u$ is the amount he sells over each period to generate profits. The instantaneous rate of profit is given by the proceeds from sales minus the cost incurred on the currently held stocks, that is:

$$L(s_t, z_t, u_t, t) = -u_t p(z_t - u_t) - ks_t \tag{18}$$

Accordingly, the functional to be maximized is obtained by substituting equation (18) in equation (8). We obtain the unconstrained control, u^{unc} by solving the first order condition of the right hand side of the above equation given by:

$$-p(z-u) + up'(z-u) + V_s = 0 \tag{19}$$

where p' denotes the first order derivative of the function $p(q)$. The necessary (but no sufficient) second condition to obtain maximum is[5]:

$$2p'(z-u) + up''(z-u) \leq 0 \tag{20}$$

where p'' denotes the second order derivative of the function $p(q)$.

Depending on the inverse demand function considered, there might not exist an explicit expression for u^{unc}, therefore equation (19) might need to be solved numerically. We then obtain the optimal control u^* taking into account the admissibility constraints.

2.1.3 Boundary Conditions

Since the Bellman equation is a backward equation, the temporal side condition is a final condition, rather than an initial condition. Supposing that no salvage value remains at the final time [6]:

$$V^*(s, z, T) = 0 \tag{21}$$

[5] As before, the proof of existence and uniqueness of the solution is beyond the scope of this study.
[6] Note that this assumption is for simplicity and not a restriction to the method.

In this problem there are no explicit boundary specifications, so the boundary values must be obtained by integrating the Bellman equations along the boundaries (see Hanson (1996)). The non-existence of explicit boundary conditions implies that there are no exterior circumstances in the nature of the problem that would force it to have specific solutions at the boundaries. Therefore, the boundary version of the Bellman equation will be the same as the interior version of the Bellman equation represented by equations (7) and (8) with the boundary values applied.

2.2 Linear Inverse Demand Function

Although the general formulation of our model allows for different definitions of the inverse demand function $p(q_t)$, we use a linear inverse demand function in the numerical implementation of the model for computational simplicity:

$$p(q_t) = a - bq_t, \qquad \alpha, \beta > 0. \tag{22}$$

Accordingly, for the competitive market, the unconstrained storage rate is given by:

$$u^{unc} = \frac{V_s^* + bz - a}{b} \tag{23}$$

which exists and is finite for any V_s^* and z.

For the monopolistic market, the optimal storage is given by:

$$u^{unc} = \frac{V_s^* + bz - a}{2b} \tag{24}$$

which exists and is finite for any V_s^* and z. The optimal storage rate, u^* is obtained by taking into account the state constraints defined by equations (3) and (4).

3 Numerical Implementation and Results

The solution to the general stochastic dynamic programming problem defined by equation (6) is obtained by solving the PDE given by equations (7) and (8) subject to the final condition $V(s, z, T) = 0$. This PDE does not have an analytical solution and therefore it is necessary to apply numerical methods to solve them. The optimal feedback control $u^*(s,\ z,\ t)$ is computed as the argument of the maximum in the functional control term $L(s, z, u, t)$.

Despite having a nonlinear partial differential equation (PDE) for both problems, the application of an explicit standard method (e.g. Morton and Mayers, 1994 Morton and Mayers (1994)) to obtain a numerical solution is as good as alternative methods which are more complex and imply a greater computational effort. For comparison we implemented both the explicit standard method and the hybrid extrapolated predictor-corrector Crank-Nicholson method, modified to account for the non-linearities and discontinuities in the PDEs (Hanson (1996) and Hanson and Ryan (1998)) were implemented. Both achieved very similar results. Therefore, we adopt the standard explicit numerical procedure to solve PDEs.

The results reported below consider the optimization problem formulated in equation (6). We compute and present separately the results for each alternative economic scenarios of perfect competition and monopolistic storage. We consider a very large final time, that is when $T \longrightarrow \infty$ in order to obtain the steady state equilibrium independent of time[7]. In other words, we consider T sufficiently large so that the influence of anticipation that storage will stop in period T becomes negligible and the decision rule becomes time independent. We consider $T = 50$ years, which gives a steady state equilibrium for the parameters considered below. As previously described, we perform the analysis for a linear price function: $p(q) = p\,(z-u) = a - b\,(z-u)\,, \quad a, b >.$

[7] We have also considered the possibility of developing this model for the steady state equilibrium by considering an infinite time horizon. However, the solution would be impossible to obtain without the knowledge of the boundary conditions for s and z.

Table 1 reports the parameters values used in the numerical implementation and Table 2 specifies the range for the annual supply rate[8] and the inventory capacity considered. When storage is not available, the price is a function of the exogenous supply only, that is, $p(q) = p(z) = a - bz = 100 - 10z$, $z \in [0.0, 9.0]$. Accordingly, in the long run, the price in the absence of storage follows a normal distribution with mean $\mu_p = 55$ and $\sigma_p = 8.2$. Numerous parameter combinations were implemented and analyzed beforehand to ensure that the results reported below are representative for the qualitative model properties.

a	b	α	σ	$\overline{z}$	k	r
100	10	12.0	4.0	4.5	5.0	0.05

Table 1: Value of the Parameters used to obtain the numerical solutions of both competitive and monopolistic markets.

z_{Min}	z_{Max}	s_{Max}
0.0	9.0	0.9

Table 2: z_{Min} and z_{Max} represent the lower and upper values of the grid for the exogenous supply rate, z. s_{Max} represents the storage capacity.

Next, we present the results for the optimal storage policy and the resulting price within each of the competitive and the monopolistic market contexts. We also represent graphically the results of the price variability as a function of the supply rate at different fixed levels of storage[9]. In the absence of storage, the price volatility is given by $b\sigma = 40$. In the presence of storage, the variability is calculated as σp_z as derived in appendix B, where p_z is the first order partial derivative of the price in order to z. This partial derivative is calculated numerically according to the approximation described before. This analysis illustrates the effect that the existence of storage in the economy

[8]In the long run, z has a gaussian stationary distribution with mean $\mu_z = \overline{z} = 4.5$ and standard deviation $\sigma_z = \frac{\sigma}{\sqrt{2\alpha}} = 0.82$. This range cover the steady state distribution of z.

[9]Because of the additive form of our model, we prefer to calculate the standard deviation of the commodity spot price process, dP, rather than the standard deviation of $\frac{dP}{P}$ as in a conventional volatility measure. Accordingly, we name this measure spot price variability instead of spot price volatility.

has on the variability of the commodity prices. We will also emphasize the differences between the competitive and the monopolistic markets. The results are analyzed separately for the competitive and the monopolistic markets and the results are compared thereafter.

Figure 1 shows the competitive optimal storage rate, u^*, as a function of both the storage level and the supply rate. Figure 2 illustrates the superposition of the corresponding price in the absence of storage and the price in the presence of storage in this economy as a function of both state variables. Figure 3 shows the variability of the price as a function of supply at different levels of storage.

The results confirm the intuition and the predictions in the theory of storage. Figure 1 shows that the storage rate increases with the value of the supply rate and decreases with the storage level. Figures 2 and 3 illustrate that the existence of storage stabilizes the prices. More specifically, if the commodity price is above the natural long-run mean (because supply is low), the existence of storage lowers the prices in relation to the natural price. On the other hand, if the price is below natural long-run mean (because supply is high), the existence of storage increases the prices in relation to the natural price. We conclude that storage affects the price dynamics by keeping the prices more stable and closer to the long-run mean, dampening down the slope of the original price function[10]. However, if the supply is high (the price is above the mean) and all the storage capacity has been used, the stockholders are being prevented from storing any further quantity of the commodity from the market and the price falls, behaving as in the case of non-storage. The spot price variability curves are represented in Figure 3, which corroborate the following results: the existence of storage significantly reduces the price variability as a function of supply. Moreover, this reduction is positively related to the inventory level. The exceptions occur when the aggregate storage

[10] The variability of the prices is directly proportional to the slope of the prices as a function of supply. Therefore damping down the slope means reducing the price variability.

facility is empty or when the full storage capacity is being used. In these two cases, the variability is equal to 40, which is the same value as it would be observed in a non-storage economy.

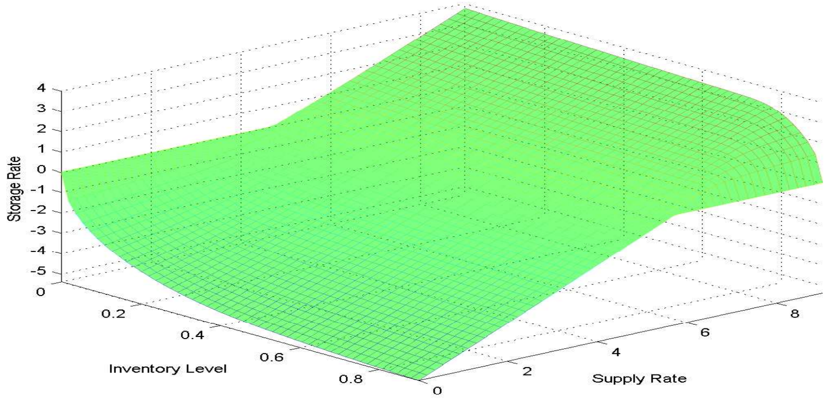

Figure 1: Competitive Case - Storage rate, u, as a function of the two state variables inventory level, s, and exogenous rate of supply, z.

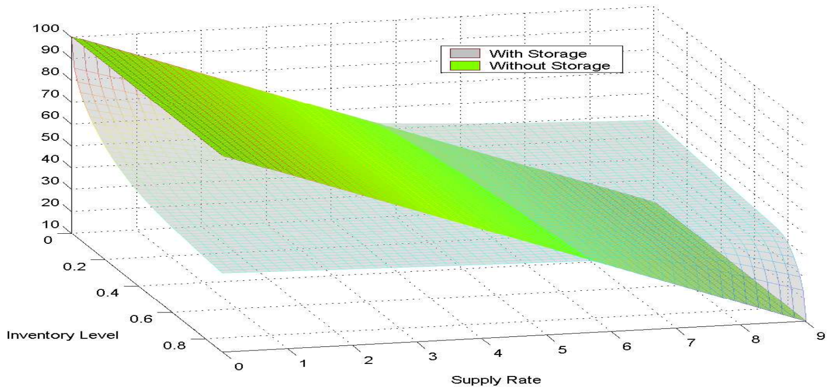

Figure 2: Competitive Case - Super-position of two graphs for the prices in the absence and in the presence of storage, respectively. The price is represented as a function of the two state variables inventory level, s, and exogenous rate of supply, z.

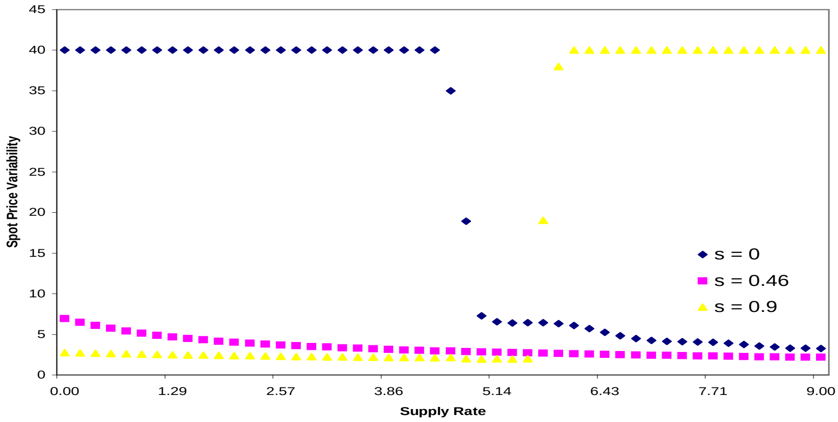

Figure 3: Competitive Case - Price variability as a function of supply rate, at different fixed levels of inventory.

Figures 4, 5 and 6 represent the equivalent results for the monopolistic case. Figure 4 shows that the storage rate increases with the rate of supply and decreases with the level of storage. When supply is high, the storage rate is relatively large and positive. However, when the supply rate is high *and* the inventory level is close to its capacity, the storage rate is forced to be reduced. Similarly to the competitive case, Figures 5 and 6 show that the existence of storage smoothes the price behaviour by comparison with the non-storage case. However, if the inventory level is close to capacity, the stockholder is prevented from buying additional stock, even if it would be optimal to do so. Similarly, when the inventory is empty, the stockholder cannot sell the commodity, even if it was profitable to do so, since commodity short sales are not allowed in a storage economy.

A comparison between Figures 1 and 4 show that the monopolist transacts less than the competitive stockholder at all supply levels. As a result the extent to which the monopolist actions smooth the price smaller than in the competitive case. This is observed by comparing Figures 2 and 3 with figures 5 and 6. Moreover, since the monopolist builds less inventory than the competitive stockholder, the capacity constrain on the storage policy for high supply levels is more prominent in the competitive market than in the monopolistic market.

These results show that the monopolistic stockholder benefits from performing less transactions than the competitive storer, thereby benefiting from a higher spread between the buying prices and the selling prices. The result of these policy differences is that the monopolist reduces less the variability of the natural commodity spot price behaviour than the competitive one.

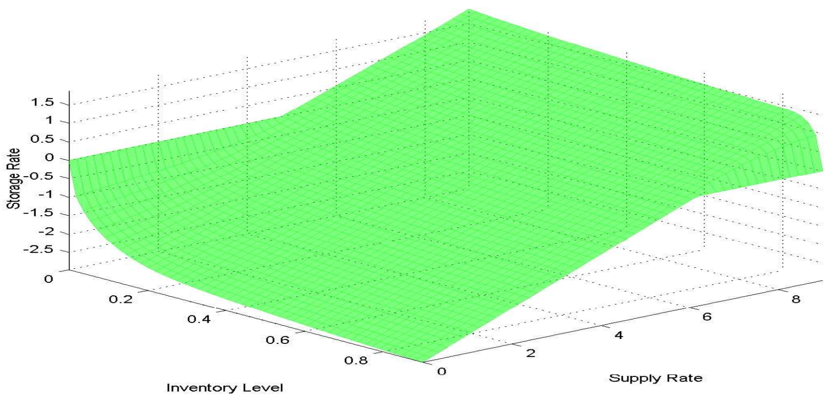

Figure 4: Monopolistic Case - Storage rate, u, as a function of the two state variables inventory level, s, and exogenous rate of supply, z.

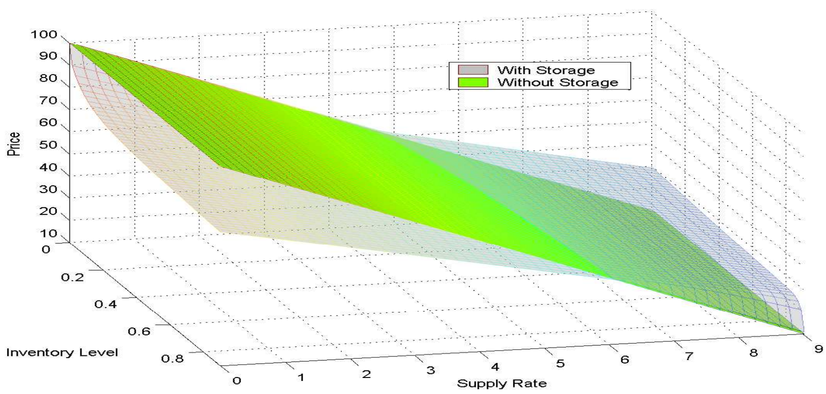

Figure 5: Monopolistic Case - Super-position of two graphs for the prices in the absence and in the presence of storage, respectively. The price is represented as a function of the two state variables inventory level, s, and exogenous rate of supply, z.

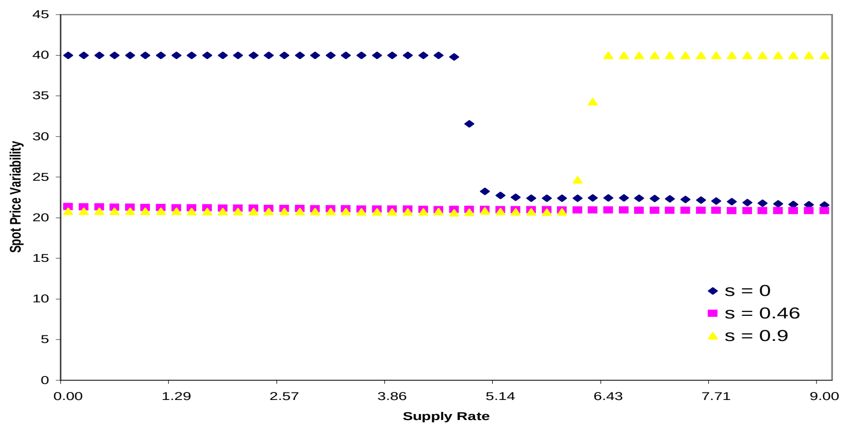

Figure 6: Monopolistic Case - Price variability as a function of supply rate, at different fixed levels of inventory.

4 Numerical Implementation and Analysis of the Forward Curve

In this section we implement and analyze the forward curve corresponding to the structural commodity price model presented above for both competitive and monopolistic markets. Using the steady-state optimal storage policy developed above, we construct a trinomial tree for the commodity prices and the corresponding forward curve that evolve by computing at each node the optimal combinations of inventory level and exogenous supply rate. We first build a trinomial tree for the Ornstein-Uhlenbeck process that describes the stochastic supply in the model presented in the previous chapter applying standard methods as described by Hull and White (1993, 1994). At time zero, we assume predetermined values for the supply rate and the level of storage. The storage levels for each node in the tree evolve from this starting point by computing the optimal rate of storage for each combination of supply and inventory. This optimal rate is calculated by interpolation using the steady optimal storage policy values obtained from the computational implementation of the structural model above. As the time evolves in the tree, the possible number of storage levels for each node representing the supply rate increases very rapidly. When the number of combinations is above a certain predetermined value, we merge the storage values that are combined with a particular supply value into a predetermined (smaller) number of values. The reduction in the number of nodes is subtle to avoid a large loss of information. The commodity price is calculated for each existing combination of supply and storage and the forward price at each instant of time is computed. The forward curve is calculated forward in time and not by moving backwards as in standard procedures.

We analyze and compare the different types of forward curves and the convenience yield obtained by varying the initial values of the inventory level in the model. Although we only illustrate the results for a unique initial value for the supply rate, the generalization for other values of initial supply can be easily made.

4.1 The Trinomial Tree

The tree that represents the evolution of commodity prices is the results of two main steps. The first is the construction of the tree representing the O-U process that describes the supply rate process as described by equation (1). The second is the calculation of the optimal storage levels which result from the application of the steady-state storage policy developed above. Each combination of storage and supply yields a unique commodity price with a certain probability.

At each time t, the rate at which the commodity is stored depends on the amount already in storage, s_t, and on the exogenous supply, z_t, as described before. Accordingly, at time t, for each combination of inventory level, s_t, and exogenous supply z_t, there exists an optimal storage rate, $u^*(s_t, z_t)$. This value is obtained through interpolation[11] using the long-run optimal storage

[11] We use the local Shepard interpolation method described in Chapter 9 of Engeln-Mullges and Uhlig (1996).

policy, $u^*(s,z)$ obtained numerically previously[12].

Taking into account the discrete time version of equation (2), given the storage level at time $t-1$ and the optimal storage rate u^*, the inventory level at time t is given by:

$$s_t = s_{t-1} + u^*(s_{t-1}, z_{t-1})\Delta t \tag{25}$$

Given the Markovian structure of both the exogenous supply rate and the storage process, it is always possible to compute $(s_{t+\Delta t}, z_{t+\Delta t})$ from (s_t, z_t).

We denote the k^{th} value of s at node (i,j) by $s_{i,j,k}$, for $k = 1, \cdots, k_{i,j}$ where $k_{i,j}$ is the number of possible (s,z) combinations at node (i,j). For $t \geq 2$, for each value of $z_{i,j}$ in the tree we have k values of inventory levels. Thus, the number of possible combinations of inventory level and supply rate, (s,z) at each node grows rapidly.

The spot price of the commodity is given by:

$$p(q) = p(z - u^*) \tag{26}$$

where z is the rate of exogenous supply evolving according to equation (1), and u^* is the optimal storage rate resulting from the optimal storage policy in the long run as described in the previous chapter. As mentioned above, for each specific combination (s,z) we calculate u^* by interpolating the values of the optimal storage policy designed in the previous chapter. Therefore, for each combination $(s_{i,j,k}, z_{i,j})$, $k = 1, ..., k_{i,j}$, where $k_{i,j}$ is the number of possible (s,z) combinations at node (i,j), there is an optimal storage rate associated with it, $u^*(i,j,k)$. Clearly, the probability associated with this optimal storage rate is the same as the probability associated with the combination of $(s_{i,j,k}, z_{i,j})$. This, in turn, also implies that this same probability is associated

[12] We are calculating u^* by interpolating the steady-state storage policy calculated previously, that is, when $T \to \infty$. Therefore u^* does not depend on time t.

with the resulting spot price, $p(s_{i,j,k}, z_{i,j})$. This information allows us to calculate the resulting price expectation at each time in the tree. Since we are working in the risk-neutral measure and the interest rate is non-stochastic we have that:

$$F_{t,T} = \underset{z}{E_t}[p_T], \tag{27}$$

where E_t denotes the conditional expectation under the risk neutral probabilities given the information at time t.

The probability attributed to a combination of (s, z) is calculated forward as the tree evolves. This enables us to compute the expected value of the spot price at each time in the tree without needing to calculate the expectation backwards. In this particular problem a forward calculation is simpler due to the merging process of nodes explained next.

4.2 The Node Merging Process

The numerical method described above implies that the number of possible combinations of the state variables, (s, z), grows very quickly with the size of the tree, becoming computationally inefficient. To avoid this problem we place a constraint on the number of the combinations (s, z) at each node of the basic tree that evolves exogenous supply process given by equation ((1). In other words, if the number of combinations (s, z) in a node exceeds say l then we merge these combinations into l_{New} combinations such that $l_{New} < l$. Before a merger takes place we first sort the storage levels to be merged by increasing order, starting with the smallest. This ensures that the mergers are effectuated between adjacent values of storage levels. This merging process is done using linear interpolation weighted by the corresponding probabilities, as described by equation (28) below. Note that we only merge the values of storage levels, s, while the corresponding value of z remains the same. Denote by $s_{i,j,k_{New}}$ the storage level that results from the merger of two nodes and by

s_{i,j,k_0} and s_{i,j,k_1} the two nodes to be merged. The resulting node is given by[13]:

$$s_{i,j,k_{New}} = \frac{pr_{i,j,k_0} * s_{i,j,k_0} + pr_{i,j,k_1} * s_{i,j,k_1}}{pr_{i,j,k_0} + pr_{i,j,k_1}} \tag{28}$$

the corresponding probability is the sum of the two probabilities of the corresponding nodes, that is:

$$pr_{i,j,k_{New}} = pr_{i,j,k_0} + pr_{i,j,k_1} \tag{29}$$

This process ensures consistency in the calculation of the expectation as described by equation (27) above. This process is repeated every time the number of combinations (s, z), m, is greater than a maximum of l. In this case, these combinations are merged in a predetermined number of nodes, $l_{New} < l$, according to the process described above. This ensures that the number of (s, z) does not grow beyond a certain limit. Note also that the reduction in the number of nodes involved in a merger should be subtle in order to keep accuracy in the resulting calculations.

4.3 Calculation of the Forward Curve and the Convenience Yield

Since we are working in the risk-neutral measure and the interest rate is non-stochastic, the forward curve is calculated using the expectation relationship between forward prices and spot prices as described by equation (27) above. Using this relationship we construct the forward curve starting at time $t = 0$ for the period of length T, conditional on a particular initial combination of exogenous supply rate and storage level (s_0, z_0).

The calculation of the convenience yield relies on the well known relationship between the futures and the spot price of a commodity when the interest

[13]Here we consider only two nodes to be merged for simplicity. However, this can be applied to an arbitrary number of nodes.

rate and the convenience yield are deterministic. If the amount of storage costs incurred between t and $t + dt$ is known and has a present value C at time t the convenience yield, δ, is defined as:

$$F_{t+\Delta t} = (p_t + C)e^{(r-\delta)\Delta t} \tag{30}$$

Based on this relationship, we calculate the annualized convenience yield for the time interval between t and $t + \Delta t$ by using pairs of adjacent maturities futures contracts according to the following formula:

$$\delta_{t,t+\Delta t} = r - \frac{1}{\Delta t} \ln \left(\frac{F_{t+\Delta t}}{F_t + C} \right) \tag{31}$$

A similar definition for the convenience yield is also used by Gibson and Schwartz (1990).

4.4 Results

This section presents and analyzes the commodity forward curves, which are generated by the application of the steady state storage policy. Both cases of competitive and monopolistic storage are considered.

Table 3 displays the time to maturity period, T, the time-step dt, the maximum number of combinations (s, z) allowed at each node of the tree, l, and the number of new combinations (s, z) after the merge takes place, l_{new}. We keep the reduction in the number of the nodes subtle in order to avoid a significant loss of information. Although we implement the tree using a time-step of 0.005, the results plotted in the figures correspond to sample time-intervals of 0.1. The annualized convenience yield is calculated according to equation (31) using two forward prices with consecutive maturities which differ by a time interval of 0.1. The parameter values used in the computation of the tree are displayed in Tables 4 and 5. Note that the values used here are the same as previously, with the exception of the supply rate limits, z_{Min} and z_{Max}, the marginal cost of storage, k, and the total storage capacity, s_{Max}. The supply rate limits are different because they are induced by the trinomial tree that represents the O-U stochastic process for the supply, starting at $z_0 = 4.5$[14] with time-step as above and space-step $dz = \sigma\sqrt{3dt}$. Finally and without loss of generality, we consider a marginal cost of storage equal to zero to avoid adding further numerical approximations in the calculation of the convenience yield. Specifically, the calculation of the storage costs incurred at each period of time involves the calculation of the total amount of storage at each period of time. This calculation, in turn, would be affected by numerical approximation resulting from the interpolations and the merging processes that occur. Moreover, keeping the storage costs equal to zero does not modify the results qualitatively.

T	dt	l	$lnew$
9	0.005	30	20

Table 3: T represents the maximum time to maturity considered, dt represents the time-step, l represents maximum number of combinations (s, z) allowed at each node of the tree, and l_{new} is the new number of (s, z) combinations after the merging takes place.

a	b	α	σ	$\overline{z}$	k	r
100	10	12.00	4.00	4.50	0	0.05

Table 4: Value of the parameters used to implement the tree for both competitive and monopolistic markets.

z_{Min}	z_{Max}	s_{Max}
0.09	8.9	0.9

Table 5: z_{Min} and z_{Max} represent the lower and upper values of the grid for the exogenous supply rate, z. s_{Max} represents the storage capacity.

[14] Although here we only present the case where the initial supply rate $z_0 = 4.5$, the results for different initial values of supply follow by analogy.

Each of the Figures 7 and 9 represent a series of three forward curves for the competitive and the monopolistic markets, respectively. Each of the forward curves displayed correspond to initial inventory levels $s_0 = 0, 0.225, 0.45$ and 0.9 respectively. The evolution of the corresponding convenience yield curves is displayed in Figures 8 and 10. When $s_0 = 0$, both forward curves are in backwardation since null inventory levels at time zero reflect the possibility of commodity shortages during the life of the forward contracts, inducing positive convenience yields. In the long-run, both curves move towards a state-independent (constant), long-term forward price, F_∞, which is equal to the long-run mean natural price[15] $\overline{P} = p(\bar{z}) = a - b\bar{z} = 55$ where $\bar{z}$ is the long-run mean of the Ornstein-Uhlenbeck stochastic process for the supply rate. This reflects the steady state equilibrium of both storage economies in which the expected total amount of commodity sold is equal to the expected total amount of the commodity bought by the (aggregate) storer. The corresponding convenience yield curves decrease with time, matching the shape of the forward curves. In particular, the convenience yield is at its maximum when the storage is empty and decreases convexly as the aggregate inventory increases with time, becoming equal to the riskless interest rate in the steady state long run. This is consistent with the predictions of the theory of storage which states that the convenience yield is a convex function of the aggregate inventory, that is, the convenience yield declines at a decreasing rate as the level of inventory increases.

For positive inventory at time zero, all the forward curves are in contango, as expected, since the commodity price at time zero is smaller than the long-run natural commodity price. We also observe that the smaller the initial inventory level is, the greater the initial commodity spot price is. This is due to the fact that the total availability of the commodity decreases. This is also reflected in the length of time at which each of the curves remain in

[15]The natural price is the price in the absence of storage.

contango. That is, the smaller the initial commodity price is the longer the forward curve will remain in contango until it reaches the unconditional forward price $F_\infty = \overline{P}$. This is observed because the slope of the forward curves is (approximately) the same[16] (within each of the competitive or monopolistic economies) independently of the initial level of storage.

Comparing the forward curves between the competitive and monopolistic markets we note that the slope of the competitive forward curves is greater than the slope of the monopolistic forward curves. This is also observed in the corresponding values of the convenience yield observed within each market. The convenience yield observed in the competitive market is (approximately) zero when the market is in contango. On the other hand, the convenience yield observed in the monopolistic market is positive (but small). This means that the annualized futures returns given by $\ln\left(\frac{F_{t+1}}{F_t}\right)$ is equal to the annualized risk free interest rate when storage is competitive. In contrast, the annualized futures returns are smaller than the interest rate when storage is monopolistic. This implies that the monopolistic storer has a positive benefit from holding inventory explained by the convenience yield. Moreover, this benefit is greater the greater the initial inventory is. This positive value is a result from the monopolistic storage policy. In particular, the monopolist restricts the quantity he buys since this strategy will guarantee him profitable spreads between the prices at which he buys and sells the commodity. Although the monopolist trades less than the competitive stockholder, these spreads guarantee him greater cash-flows than what we would get following the competitive trading strategy.

In summary, the commodity forward curves take two fundamental shapes depending on whether the initial commodity price is below or above the state-independent long-term forward price, $F_\infty = \overline{P}$. Specifically, if the commodity

[16]Note that the results presented are affected by numerical approximation and errors due to the successive interpolations to calculate the optimal storage policy for all the (s, z) combinations in the tree and to the merging process. Therefore the results are affected by some noise.

price is less than the long-run forward price, the curve will be in contango otherwise it will be in backwardation. In the example provided in this paper the initial inventory is equal to the long-run average supply, $\overline{z}$. In this case we observe the following two shapes: (i) when the initial inventory is zero, the forward curve is downward sloping (backwardation) for some time and declines towards the steady state long-term forward price, and (ii) when the initial inventory is positive the forward curve is upward sloping (contango) for some time and rises towards the steady state forward price. Moreover, the amount of time the curve remains in contango is positively related to the initial inventory level. In any case, the forward curve tends to the long-run forward price, $F_\infty = \overline{P}$. These results are consistent with the theory of storage and with the properties inherent to the structural models in the literature and in particular with the forward curve analysis in Routledge et al. (2000). These authors assume that the source of uncertainty comes from the demand, where the shocks are modelled by a 2-state Markov process, a *high* demand state and a *low* demand state. They assume an initial low (or zero) inventory level and observe the two following forward curve shape: (i) the curve is upward sloping when the demand state is *low*, which correspond to an low initial spot price and (ii) the curve is downward sloping when the demand is *high*, which corresponds to a high initial spot price. In both cases, the forward curve eventually becomes equal to the long-term forward price, F_∞.

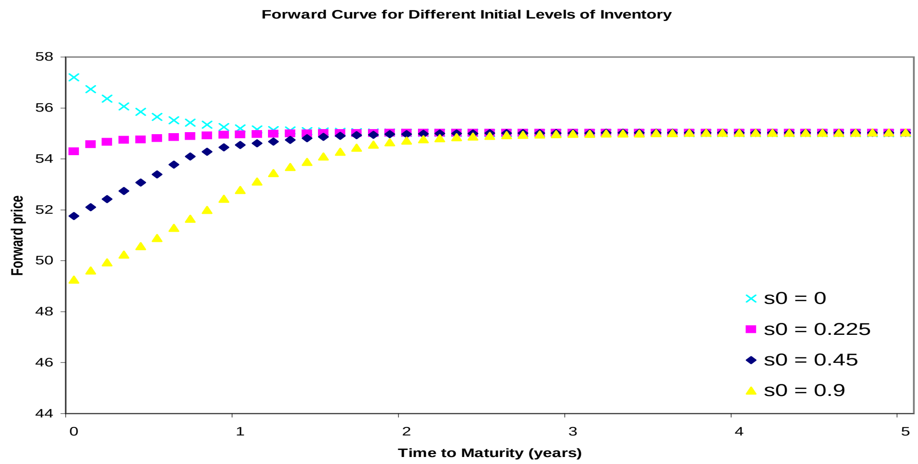

Figure 7: Competitive case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is equal to 0, 0.225, 0.45 and 0.9, respectively.

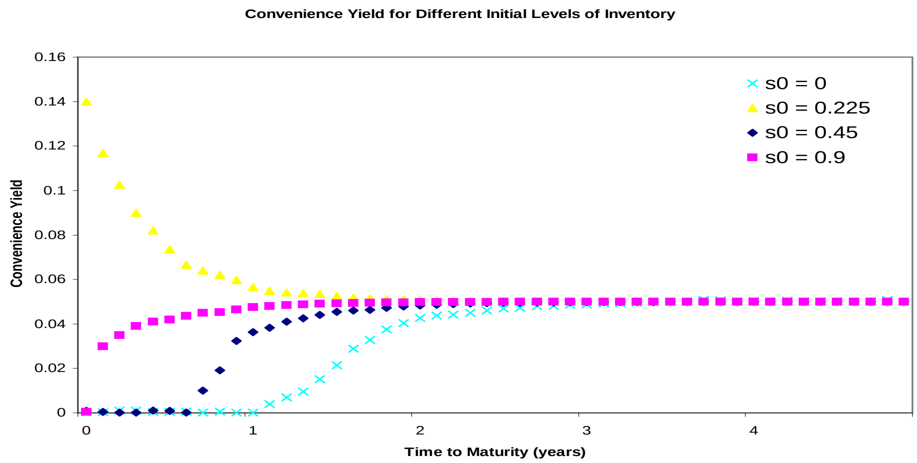

Figure 8: Competitive case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is equal to 0, 0.225, 0.45 and 0.9, respectively.

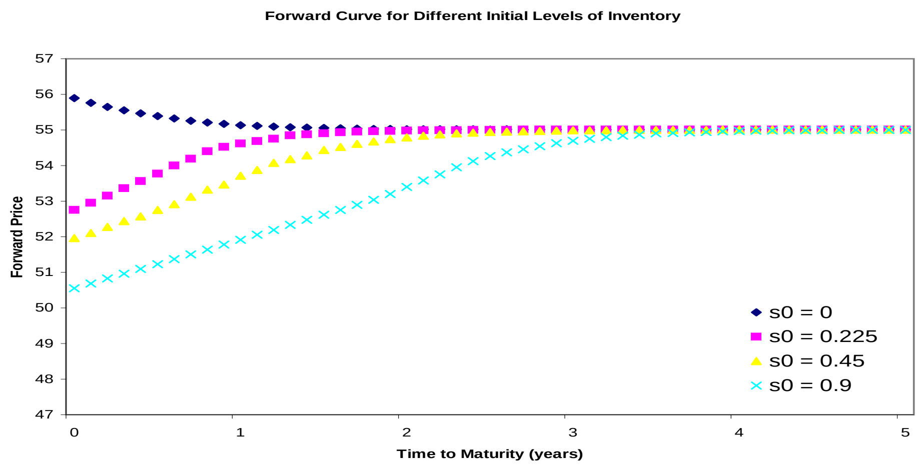

Figure 9: Monopolistic case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is equal to 0, 0.225, 0.45 and 0.9, respectively.

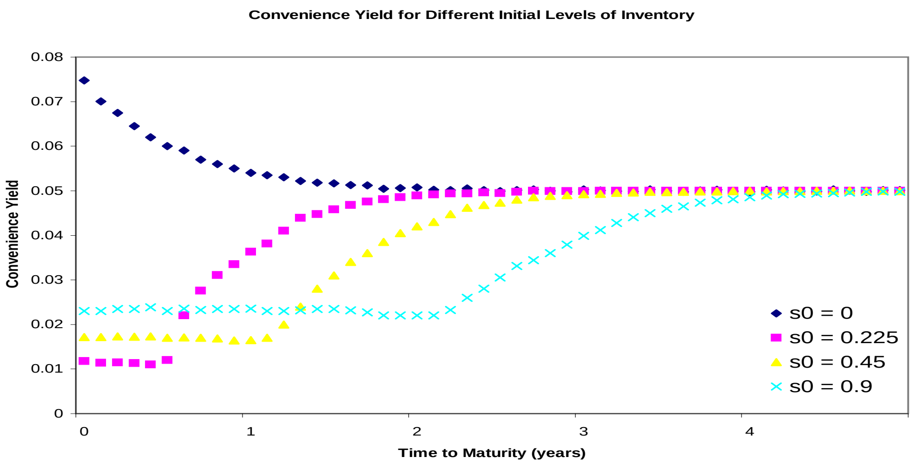

Figure 10: Monopolistic case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is equal 0, 0.225, 0.45 and 0.9, respectively.

5 Conclusion

In this paper we presented a continuous time stochastic equilibrium storage model suited for non-perishable storable commodity prices, where the source of uncertainty comes from the exogenous supply. This model builds upon the existing discrete time structural models. However, it also takes into account relevant features of reduced form models recently developed in the literature by accounting for the mean-reverting characteristics of spot commodity prices.

This model is formulated as a stochastic dynamic programming problem in continuous time and considers the existence of two state variables: (i) the exogenous stochastic supply, which evolves as a O-U stochastic process and (ii) the endogenous inventory level, which is a fully controllable variable. The decision variable is the rate of storage, which in turn determines the final commodity prices. In order to simplify the numerical computation, we considered a linear inverse demand function in the numerical examples provided. The model is initially formulated under a general framework and later unfolds into two distinct market scenarios - competitive and monopolistic markets.

The analysis provided in this paper is twofold. First, we analyzed the dynamic interplay between the optimal storage policy and the commodity price. Second we analyzed the forward curves and convenience yields implied by this model. The results provided are in accordance with the theory of storage. The presence of storage in the economy smoothes the spot price behavior by reducing the variability of the natural spot price from the no-storage case. Moreover, the degree of reduction in this variability is positively related to the level of inventory. This smoothing effect is more evident in the case of storage competition than it is in the case of monopolistic storage. This difference results from the observation that the monopolist performs less trading activity than the competitive storer since he benefits from having a greater spread between the buying prices and the sales prices. Another relevant result involves the effect of the storage capacity on the storage policy. In particular, if the storage capacity is fully used (or close to), the stockholders in both economies are not able to respond optimally to price variations. Consequently, the price dynamics will follow the natural price process. The resulting forward curves take two fundamental shapes: if the initial spot price is greater that the equilibrium long run commodity natural price we observe backwardation; otherwise the forward curve is in contango.

The model presented in this chapter makes several contributions to the current literature. First, it introduces a continuous time structural model that draws on specific microeconomics assumptions of the market environment and establishes a link with the existing reduced form models in the literature. That is, it builds on the structural models but it uses a continuous time framework and includes the mean reverting characteristics of commodity prices. This latter contribution is particularly relevant since none of the existing structural models has included the mean reversion property of commodity prices. Second, this model is developed under a very flexible framework which allows for different extensions of the model to be adapted to different commodities. For example, this model can be extended to accommodate other type of supply/demand functions. In particular, we mentioned how seasonality could be included in the model without adding extra complexity to the solution method. Third this paper provides a renewing and comprehensive analysis of the forward curves implied by this model. With the exception of Routledge et al. (2000), the existing literature in structural model for commodity prices does not provide the analysis of the forward curve. Although Routledge et al. (2000) provide a study of equilibrium forward prices their analysis is limited since the stochastic state variable is restricted to two states and a generalization to a more realistic number of possible state values is not obvious.

Finally, this model is formulated and analyzed for both competitive and monopolistic storage environments. This comparison renewing since it is not illustrated in the current literature, which only considers a competitive storage economy. Since the energy markets have evolved from a monopolistic to a competitive environment in recent years, we stress the importance to analyze both storage economies in order to understand the implications of the market evolution in the price dynamics.

One of the directions for further work should include the extension of the analysis to encompass non-linear demand functions, although this would imply a more arduous numerical implementation to solve the Bellman equation. Another direction of future work is to study the case where the supply includes jumps since one of the energy price characteristic is the occurrence of occasional spikes. This could easily be included in the model by adding a Poisson process to the supply stochastic process in the spirit of the jump diffusion process presented by Merton (1976). Allowing capacity investment is also worth exploring. The integration of a real options model like that of Dixit and Pindyck (1994) with the richer environment of this model is an interesting, and certainly challenging, possibility. Another direction is the development of a steady state version of the model presented in this chapter. Although this seems to be extremely difficult to obtain under realistic assumptions, it would be interesting to find a method to study the steady state case directly.

A Derivation of the Stochastic Dynamic Programming Equation

We derive the stochastic dynamic programming equation resulting from maximizing the following functional:

$$J(s_t, z_t, t; u(\cdot)) = \underset{z}{E_t}\{\int_t^T e^{-r(l-t)} L\left(s_l, z_l, u_l, l\right) dl| + \Psi(s_T, z_T)|s = S, z = Z\} \tag{32}$$

over all the admissible plans where the state variables s and z satisfy the following transition equations:

$$dz_t = \alpha(\overline{z} - z_t)dt + \sigma dB_t, \qquad t \geq 0; \tag{33}$$

where B_t is a standard Wiener process defined on the underlying filtered probability space $\left(\Omega, F, \{F_t\}_{t\geq 0}, P\right)$.

$$ds = u(s, z, t)dt, \qquad 0 \leq s \leq b; \tag{34}$$

and $L(s_t, z_t, u_t, t)$ is the instantaneous profit rate and $\Psi(s_T, z_T)$ is the salvage value of having s_T and z_T as states at final time T. Without loss of generality, we consider $\Psi(s_T, z_T) = 0$.

To solve the problem defined by equation (32), let $V(s, z, t)$, known as the

Value Function be the expected value of the objective function in (32) form t to T when an optimal policy is followed from t to T, given $s_t = S$ and $z_t = Z$. Then, by the *principle of optimality*,

$$\begin{aligned} V(s,z,t) &= \sup_{u\in[u_{\min},u_{\max}]} \underset{z}{E}\{L(s_t,z_t,u_t,t)dt + \\ &\quad e^{-rdt}V(s+ds,z+dz,t+dt|s=S,z=Z)\} \end{aligned} \tag{35}$$

where $[u_{\min}, u_{\max}]$ is defined in Section 3.1.

Multiplying both sides of the equation by e^{rdt} and noting that $e^{rdt} \simeq 1 + rh$ we obtain:

$$\begin{aligned} (1+rdt)\,V(s,z,t) &= \sup_{u\in[u_{\min},u_{\max}]} \underset{z}{E}\{L(s_t,z_t,u_t,t)dt + \\ &\quad V(s+ds,z+dz,t+dt|s=S,z=Z))\} \end{aligned} \tag{36}$$

That is:

$$\begin{aligned} rdtV(s,z,t) &= \sup_{u\in[u_{\min},u_{\max}]} \{L(s_t,z_t,u_t,t)dt + \underset{z}{E}(V(s+ds,z+dz,t+dt) - \\ &\quad V(s,z,t)|s=S,z=Z)\} \qquad (37) \\ &= \sup_{u\in[u_{\min},u_{\max}]} \left\{L(s_t,z_t,u_t,t)dt + \underset{z}{E}\left(dV(s,z,t)|s=S,z=Z\right)\right\} \end{aligned}$$

Applying Ito's calculus, we have:

$$dV(s,z,t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial s}ds + \frac{\partial V}{\partial z}dz + \frac{1}{2}\frac{\partial^2 V}{\partial z^2}(dz)^2 \tag{38}$$

where ds and dz are as above and $dz^2 = \sigma^2 dt$, which gives:

$$dV(s,z,t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial s}udt + (\alpha(\overline{z} - z_t)dt + \sigma dB_t)\frac{\partial V}{\partial z} + \frac{1}{2}\sigma^2 dt \frac{\partial V^2}{\partial z^2} \quad (39)$$

which implies that

$$E\left(dV(s,z,t)|s = S, z = Z\right) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}u + \alpha(\overline{z} - z)\frac{\partial V}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial z^2}\right) dt \quad (40)$$

Replacing (40) into equation (37) and dividing by dt gives:

$$\begin{aligned} rV(s,z,t) &= \sup_{u \in [u_{\min}, u_{\max}]} \{L(s_t, z_t, u_t, t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}u + \alpha(\bar{z} - z)\frac{\partial V}{\partial z} + \\ & \frac{1}{2}\sigma^2 \frac{\partial V^2}{\partial z^2}\}, \end{aligned} \quad (41)$$

which is the stochastic dynamic programming equation we need to solve. This equation can also be written in the following form:

$$-\frac{\partial V(s,z,t)}{\partial t} - H(s,z,V_s,V_z,V_z z) = 0 \quad (42)$$

where:

$$\begin{aligned} H(s,z,V_s,V_z,V_z z) &= \sup_{u \in [u_{\min}, u_{\max}]} \{L(s_t, z_t, u_t, t) + uV_s(s,z,t) + \\ & \alpha(\bar{z} - z)V_z(s,z,t) + \frac{1}{2}\sigma^2 V_{zz}(s,z,t) - rV(s,z,t)\} \end{aligned} \quad (43)$$

B Derivation of the Spot Prices Variability

First we point out that, because of the additive form of our model, we prefer to calculate the standard deviation of the commodity prices process dP, rather than the standard deviation of dP/P as in a conventional volatility measure.

The price function is $p(q) = a - bq$. If we write the price as a function of the two state variables in the models s and z, which are the inventory level and the supply rate and apply Ito's lemma we get:

$$dP(s,z) = P_s ds + P_z dz + \frac{1}{2} P_{zz} (dz)^2 \tag{44}$$

where ds and dz are the transition equation of s and z respectively and are given by:

$$ds_t = u(s,z,t)dt, \qquad 0 \leq s \leq b \tag{45}$$

$$dz_t = \alpha(\overline{z} - z_t)dt + \sigma dW_t \tag{46}$$

where W_t is a standard Wiener process. Substituting equations (45) and (46) in equation (44) we get:

$$dP(s,z) = (u_t P_s + \alpha(\overline{z} - z_t) + \frac{1}{2}\sigma^2 P_{zz})dt + \sigma P_z dW_t \tag{47}$$

Ignoring the deterministic terms of the above equation, the variability of the resulting price, σ_p is given by

$$\sigma_p = \sigma P_z \tag{48}$$

where P_z is calculated numerically as explained in Section 3.3.1. In the absence of storage, the volatility of the spot price would be a function of the exogenous supply only, that is, $p(z)$. In the particular case of the linear function considered here we would have $p(z) = a - b(z), \quad a, b \geq 0$. Applying

Ito's

$$dP = P_z dz_t = -b\alpha(\overline{z} - z_t)dt - b\sigma dW_t \tag{49}$$

The variability of the price in the absence of storage is then $b\sigma$.

References

Bellman, R. (1957). *Dynamic Programming.* Princeton University Press, Princeton, N.J.

Chambers, M. J. and Bailey, R. E. (1996). A theory of commodity price fluctuations. *Journal of Political Economy*, 104(5):924–957.

Clewlow, L. and Strickland, C. (2000). *Energy Derivatives: Pricing and Risk Management.* Lacima Publications, London, England.

Deaton, A. and Laroque, G. (1992). On the behaviour of commodity prices. *Review of Economic Studies*, 59(198):1–23.

Deaton, A. and Laroque, G. (1996). Competitive storage and commodity price dynamics. *Journal of Political Economy*, 104(5):896–923.

Dixit, A. K. and Pindyck, R. S. (1994). *Investment Under Uncertainty.* Princeton University Press, Princeton, New Jersey.

Engeln-Mullges, G. and Uhlig, F. (1996). *Numerical Algorithms with C.* Springer-Verlag.

Gibson, R. and Schwartz, E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims. *Journal of Finance*, 45(3):959–976.

Hanson, F. B. (1996). *Stochastic Digital Control System*, volume 76 of *Control and Dynamic Systems: Advances in Theory and Applications*, chapter Techniques in Computational Stochastic Dynamic Programming, pages 103–162. Academic Press.

Hanson, F. B. and Ryan, D. (1998). Optimal harvesting with both population and price dynamics. *Mathematical BioSciences*, 148:129–146.

Hull, J. and White, A. (1993). One-factor interest rate models and the valuation of interest-rate derivative securities. *Journal of Financial and Quantitative Analysis*, 28(2):235–254.

Hull, J. and White, A. (1994). Numerical procedures for implementing term structure models i: Single-factor models. *Journal of Derivatives*, Fall:7–16.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144.

Miltersen, K. R. and Schwartz, E. S. (1998). Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates. *Journal of Financial and Quantitative Analysis*, 33(1):33–59.

Morton, K. W. and Mayers, D. F. (1994). *Numerical Solution for Partial Differential Equations: an introduction*. Cambridge: Cambridge University Press.

Pirrong, S. C. (1998). Price dynamics and derivatives prices for continuously produced, storable commodities. Working Paper, Washington University.

Robert E. Lucas, J. and Prescott, E. C. (1971). Investment under uncertainty. *Econometrica*, 39(5):659–681.

Routledge, B., Seppi, D. J., and Spatt, C. (2000). Equilibrium forward curves for commodities. *The Journal of Finance*, 55(3):1297–1338.

Samuelson, P. A. (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6(2):41–49.

Samuelson, P. A. (1971). Stochastic speculative price. *Proceedings of the National Academy of Sciences of the United States of America*, 68(2):335–337.

Schwartz, E. S. (1997). The stochastic behaviour of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52(3):923–973.

Schwartz, E. S. and Smith, J. E. (2000). Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46(7):893–911.

Williams, J. C. and Wright, B. D. (1991). *Storage and Commodity Markets*. Cambrige University Press, Cambrige, England.