Implied Volatility Processes: Evidence from the Volatility Derivatives Markets*

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Abstract

We explore the ability of alternative popular continuous-time diffusion and jump diffusion processes to capture the dynamics of implied volatility over time. The performance of the volatility processes is assessed under both econometric and financial metrics. To this end, data are employed from major European and American implied volatility indices and the rapidly growing CBOE volatility futures market. We find that the simplest diffusion/jump diffusion models perform best under both metrics. Mean reversion is of second order importance. The results are consistent across the various markets.

JEL Classification: G11, G12, G13.

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1. Introduction

The dynamics of the instantaneous and implied volatility are of crucial importance for option pricing and risk management purposes. While the dynamics of the former notion of volatility have been considered extensively, this is not the case for those of implied volatility[1]. This study fills this void by exploring the ability of alternative popular diffusion and jump diffusion processes to capture the dynamics of implied volatility indices over time.

There is a voluminous literature on the specification of the stochastic process that governs the dynamics of instantaneous volatility in continuous time. This literature first emerged in a stochastic volatility option-pricing context. In this setup, the underlying asset price and the instantaneous volatility of the underlying asset returns are modeled jointly. In the late eighties, a plethora of stochastic volatility option pricing models were developed by assuming a volatility process with continuous paths (see e.g., Hull and White, 1987, Johnson and Shanno, 1987, Scott, 1987, Wiggins, 1987, Stein and Stein, 1991, Heston, 1993, and Jones, 2003 for a more flexible specification, among others); the underlying asset price was also assumed to follow a diffusion process. In the late nineties, new type stochastic volatility option pricing models were introduced based on a jump diffusion process for the underlying asset price and a diffusion volatility process (see e.g., Bakshi et al, 1997, Bates, 1996, 2000, Andersen et al. 2002, and Pan, 2002). Recently, there have appeared option-pricing models where both the underlying asset price and the instantaneous volatility follow jump-diffusion processes (see e.g., Duffie et al., 2000, Bakshi and Cao, 2004, Broadie et al., 2004, Eraker, 2004). The validity of the specification of the process of the instantaneous volatility has also been examined jointly with that of the underlying asset price by using data only from the underlying assets' market (see e.g., Eraker et al., 2003), and in a Value-at-Risk framework (Lehar et al., 2002).

On the other hand, not much attention has been devoted to the *complete* specification of the autonomous process that implied volatility follows in continuous

[1] The implied volatility is usually used as a proxy for the instantaneous volatility. Usually, it is interpreted as the average instantaneous volatility to be realized over the life of the option. However, this is strictly true in a Hull and White (1987) world, where the instantaneous volatility is uncorrelated with the asset price, the market price of volatility risk is zero, and the option is linear with respect to volatility (i.e. at-the-money). In the case where these conditions do not hold, the implied differs from the instantaneous volatility.

time[2]. In a discrete-time context, Poterba and Summers (1986) were among the first to document that implied volatility mean-reverts. The paper by Franks and Schwartz (1991) suggests that the changes in implied volatility can be regarded as being stochastic, since they are attributed to shocks to various economic variables. Within a continuous time setup, Merville and Pieptea (1989), Moraux et al. (1999), and Daouk and Guo (2004) estimated implied volatility mean-reverting processes. All these papers have considered at-the-money short maturity implied volatilities. However, no comparison with alternative specification of the implied volatility processes was performed, and no jumps in volatility were considered. Wagner and Szimayer (2004) were the first to investigate the presence of jumps in implied volatility by estimating an autonomous mean reverting jump diffusion process using data on the implied volatility indices VIX and VDAX. They found evidence of significant positive jumps in implied volatilities. However, they adopted the rather restrictive assumption that the volatility jump size is constant rather than being random. Again, their specification was not compared with alternative ones. Finally, in a very recent paper, Bakshi et al. (2005) estimated various general specifications of the autonomous instantaneous variance diffusion process. To this end, they used the squared implied volatility index VIX as a proxy to the unobserved instantaneous variance. However, their study did not address the empirically documented presence of jumps in implied volatility (see e.g., Malz, 2000, Wagner and Szimayer, 2004).

The complete specification of the autonomous implied volatility process as well as deciding on whether the process is diffusion and/or presents jumps is important for a number of reasons. First, understanding the dynamics of implied volatility is a step towards understanding the dynamics of the equity risk premium; Merton (1980) showed that there is a linear relationship between the equity risk premium and the variance of equity returns (see also Poterba and Summers, 1986, for an empirical analysis). Second, knowledge of the process that governs the evolution of implied volatility over time is particularly useful for (volatility) trading and hedging purposes. This is because the implied volatility is a reparameterisation of the market option price, and is used as an input to calculate the sensitivities of the option price with respect to various risk factors

[2] Kamal and Derman (1997), Skiadopoulos et al. (1999), Alexander (2001), Ané and Labidi (2001), Cont and da Fonseca (2002), and Fengler et al. (2003) have studied only the volatility structure of diffusion implied volatility processes, i.e. the number and form of shocks that drive implied volatilities over time. Their analysis has been placed in multivariate context where they investigate the evolution of the implied volatility surface. These studies have left unanswered the specification and estimation of the drift, though.

(hedge ratios). Third, specification of the implied volatility process is necessary so as to price and hedge derivatives written on implied volatility; these derivatives fall in the class of volatility derivatives. Volatility derivatives depend on some measure of volatility. They are traded over the counter for a long time and very recently, in March 2004, CBOE introduced volatility futures on the implied volatility measured by VIX. CBOE has also announced the imminent introduction of volatility futures on the implied volatility index VXD and volatility options. Various volatility option-pricing models have already been developed (see e.g., Whaley, 1993; Grünbichler and Longstaff, 1996, Detemple and Osakwe, 2000)[3]. The models rely on different specifications of the process that implied volatility follows in continuous time and the volatility risk premium. However, these alternative specifications and the corresponding option pricing models have not been assessed empirically; the comparative empirical examination of various implied volatility process will shed light on which model to use[4]. Finally, the implied volatility process is an indispensable tool to measure the market risk of positions in volatility derivatives, e.g., calculate Value-at-Risk by means of Monte Carlo simulation.

This paper explores for the first time the ability of alternative univariate diffusion and jump diffusion processes to capture the dynamics of implied volatility indices over time[5]. The choice of the specifications for the implied volatility processes is motivated by the extensive use of the corresponding instantaneous volatility processes in the option pricing literature. Three affine volatility diffusion processes are examined: the standard Geometric Brownian motion, the mean reverting, and the square root mean reverting process. In the jump-diffusion setting, the three volatility diffusion processes are augmented by adding a jump component.

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[3] The growing interest in volatility derivatives has emerged after the 1987 crash. Brenner and Galai (1989, 1993) first suggested options written on a measure of volatility that would serve as the underlying asset. Other types of volatility derivatives include variance/volatility swaps and variance options that are traded over-the-counter (see Demeterfi et al., 1999, Chriss and Morokoff, 1999, and Carr and Lee, 2005, for details on the pricing and hedging aspects of variance/volatility swaps, and Carr et al., 2005, for the pricing of variance options). Brenner et al. (2006) proposed and priced an option written on a straddle.

[4] Daouk and Guo (2004) have investigated the impact of model error to the pricing performance of only one (Grünbichler and Longstaff, 1996) of the developed volatility option pricing models. Psychoyios and Skiadopoulos (2006) have looked at the hedging and pricing performance of various volatility option pricing models. However, their application is placed within a simulation setup where no market data are employed.

[5] In a very recent paper, Wu (2005) has also investigated the presence of jumps in the specification of the process that dictates the dynamics of the variance over time. To this end, he has also considered the process of the variance independently of the process that governs the dynamics of the asset price. However, he has examined specifications of the process of the instantaneous variance rather than that of the implied volatility.

The validity of the six alternative implied volatility processes is assessed under both econometric and financial metrics. The use of various metrics is necessary so as to get a full understanding of the properties of the volatility processes (see for instance, Daouk and Guo, 2004, for a similar approach). It is often the case that the performance of a model is not consistent under a statistical and a financial criterion. The performance of the model may also depend on a particular data set. To check whether there is such a dependence, data on a plethora of European and American implied volatility indices (VIX, VXN, VXO, VXD, VDAX, VX1 & VX6, and VSTOXX) over various time periods, and the rapidly evolving CBOE volatility futures market are employed.

On any given point in time, an implied volatility index represents the implied volatility of a synthetic option that has constant time to maturity. The data on the implied volatility indices are the natural choice to estimate the unobservable parameters of the implied volatility process. This is because the various methods to construct the index are informative and precise. They take as input the implied volatilities of options with various strikes and expiries, and they "average" them so as to minimize the notorious measurement errors in implied volatilities (see Hentschel, 2003, for a study on the construction method of VXO). Moreover, the fact that the data sets on most implied volatility indices are extensive allows incorporating periods of market stress and hence learning about rare events such as jumps, as Broadie et al. (2004) argue. Furthermore, the study of the properties of an implied volatility index in continuous time deserves attention since the index is of great importance to both academics and practitioners. This is because it can be used in a number of applications. It serves as the underlying asset to volatility options and futures. In addition, it affects the pricing and hedging of variance/volatility swaps; an implied volatility index can be interpreted as the variance/volatility swap rate (see Carr and Wu, 2004b, 2004c, and the references therein) that affects the market value of these volatility derivatives (Chriss and Morokoff, 1999). The implied volatility index can also be used for Value-at-Risk purposes (Giot, 2005), to identify buying/selling opportunities in the stock market (Whaley, 2000), and to forecast the future market volatility (see e.g., Fleming et al., 1995, Moraux et al., 1999, Simon, 2003, Giot, 2005).

Within the econometric framework, (conditional) Maximum Likelihood Estimation (MLE) is used to estimate the parameters of the various volatility processes. In the case where the conditional density function does not have a closed-form, the characteristic function is derived. Then, Fourier inversion of the characteristic function is employed. Standard statistical tests are used to compare the alternative processes. From the econometric perspective, our paper is analogous to studies that have been conducted in the interest rate literature where the validity of alternative processes for the short-term interest rate has been investigated (see e.g., Chan et al., 1992).

Within the financial metric, the CBOE volatility futures market is used to rank the alternative processes within a futures pricing context. Under the risk-adjusted probability measure, the volatility futures price equals the expected value of volatility. Hence, the valuation of volatility futures is not model-free; for any given process, the pricing performance of the corresponding volatility futures pricing model is examined. To the best of our knowledge, Zhang and Zhu (2006) is the only study that has investigated the pricing of volatility futures empirically. However, they do this for a specific model without conducting a horse race among different models.

The econometric analysis finds discontinuities in implied volatility while mean reversion is of second order importance. The simplest Merton type (1976) jump diffusion model performs best. The results obtained from the financial metric confirm this finding.

The remainder of the paper is structured as follows. In the next Section the specifications of the implied volatility processes are presented. Section 3 describes the data set. Next, the econometric methodology is outlined. Section 5 discusses the results from the econometric estimation, and it checks their robustness. In Section 6, the properties of the various volatility futures pricing models are discussed and the alternative processes are ranked based on the evidence from the volatility futures markets. The last Section concludes, it presents the implications of the study and it suggests directions for future research.

2. The Processes

We examine diffusion and jump diffusion implied volatility processes. The diffusion processes are nested in the general stochastic volatility process described by the following equation:

$$dV_t = \mu\left(V_t, t\right)dt + \sigma(V_t, t)dW_t \tag{1}$$

where V_t is the value of the implied volatility index at time t, dW_t is a standard Wiener process, $\mu(V_t,t)$ is the drift, and $\sigma(V_t,t)$ is the diffusion coefficient (i.e. the volatility of volatility). The jump diffusion processes are nested in the general jump diffusion stochastic volatility process, described by the following equation:

$$dV_t = \mu(V_t,t)dt + \sigma(V_t,t)dW_t + y(V_t,t)dq_t \tag{2}$$

where dq_t is a Poisson process with constant arrival parameter λ (intensity), that is $\Pr\{dq_t=1\}= \lambda dt$, and $\Pr\{dq_t=0\}= 1-\lambda dt$, and y is the jump amplitude. *dW, dq and y* are assumed to be mutually independent processes. Equations (1) and (2) are defined under the actual probability measure *P*. The drift, diffusion and jump size coefficients are assumed to be general functions of time and volatility. The following specifications are examined:

Geometric Brownian Motion Process (GBMP)	$dV_t = \mu V_t dt + \sigma V_t dW_t$
Mean-Reverting Gaussian Process (MRGP)	$dV_t = k(\theta - V_t)dt + \sigma dW_t$
Mean Reverting Square-Root Process (MRSRP)	$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t$
Geometric Brownian Motion Process augmented by Jumps (GBMPJ)	$dV_t = V_t\mu dt + \sigma V_t dW_t + (y-1)V_t dq_t$
Mean-Reverting Gaussian Process augmented by Jumps (MRGPJ)	$dV = k(\theta - V_t)dt + \sigma dW_t + y dq_t$
Mean Reverting Square-Root Process augmented by Jumps (MRSRPJ)	$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t + y dq_t$

The analogous processes that researchers have used to model the evolution of instantaneous volatility/variance in a stochastic volatility option pricing setting motivate the specifications of the processes that are considered in this paper. For instance, Hull and White (1987) and Johnson and Shanno (1987) have assumed a GBMP. Similarly, the MRGP has been used by Hull and White (1987), Scott (1987), Stein and Stein (1991), and Brenner et al. (2006), among others. The MRSRP has been proposed as an alternative to the MRGP, so as to constrain volatility from taking negative values (see e.g., Hull and White, 1988, Heston, 1993, Grünbichler and Longstaff, 1996, Bates, 1996, 2000, Andersen et al., 2002, and Pan, 2002, among others). The presence of mean reversion has been documented empirically.

To examine the possible presence of jumps in implied volatilities, the GBMPJ, MRGPJ, and the MRSRPJ are the natural extensions to their diffusion analogues. The GBMPJ is a Merton (1976) type of model, while the MRGPJ has been used by Das (2002) to model the evolution interest rates over time. The MRSRPJ has been used by Duffie et al. (2000), Eraker et al. (2003), Bakshi and Cao (2004), Broadie et al. (2004), and Eraker (2004) to model the dynamics of the instantaneous volatility.

3. Data Description

3.1 Implied Volatility Indices

We use daily data (closing prices) on four major American and four European implied volatility indices: VIX, VXO, VXN, VXD, VDAX, VX1, VX6, and VSTOXX. For any given index, the data cover a period from the first date that there is an available quote until 24/03/2004.

The first four indices are traded in CBOE. VIX and VXO are constructed from the implied volatilities of options on the S&P 500 and S&P 100, respectively. VXN and VXD are based on the implied volatilities of options on the Nasdaq 100 and on Dow Jones 100, respectively. VDAX is constructed from the implied volatilities of options on DAX (Germany), while VX1 and VX6 are constructed from the implied volatilities of options on CAC 40 (France). VSTOXX is constructed from the implied volatilities of options on the DJ EURO STOXX 50 index. The data for VDAX are obtained from Bloomberg while for the other indices are obtained from the websites of the corresponding exchanges. All indices represent the implied volatility of a synthetic option that has fixed strike (or incorporates information from all strikes) and constant time-to-maturity at any point in time. The constant time to maturity is the same (thirty calendar days) for almost all indices under scrutiny. The only exceptions are VDAX (45 days) and VX6 (185 days). Table 1 provides a synopsis of the methods that are used to construct the various implied volatility indices (see the web sites of the various exchanges for further details on the construction of the indices). The use of implied volatility indices that are based on different construction methods, have different horizons, and cover different time periods allows us to check whether their properties are affected by these factors.

Figure 1 shows the evolution of the eight indices over time (up to 24/03/2004). We can see that *prima facie*, there is evidence of mean reversion and (up and down) spikes. Table 2 shows the summary statistics for the eight implied volatility indices (in levels and daily differences, Panels A and B, respectively), as well as the time periods over which the data are collected and the number of observations. Application of the Jarque-Bera test rejects the null hypothesis of normality. The evidence of non-normality may be attributed to the presence of jumps in implied volatility. The first order autocorrelation coefficients of the daily differences are also reported in Panel B (one asterisk denotes significance at a 5% confidence level). They are all negative indicating the presence of mean reversion; the only exception is that of VXN[6]. The dependence of the observations should be taken into account by the method that will be used to estimate the parameters of the various processes. The sample size ranges from 787-4,608 observations across indices. This ensures that we obtain reliable ML estimates since the results on the statistical inference of the ML estimators hold asymptotically.

3.2 Volatility Futures

Daily data (settlement prices) on the CBOE volatility index futures (ticker name VX) are used over the period from 26/03/2004 to 17/06/2005. CBOE volatility futures were introduced in March 26, 2004. The underlying asset is an “Increased-Value index” (VBI) that is 10 times the value of VIX at any point in time. The contract size of the volatility futures is $100 times the value of VBI. On any day, four futures contracts are traded: the two near-term contract months plus two contract months on the February quarterly cycle (February, May, August, and November). The contracts are cash settled on the Wednesday prior to the third Friday of the expiring month.

Three time series of futures prices were constructed by ranking the data according to their expiry date: the shortest, second shortest and third shortest maturity series. To minimise the impact of noisy data, we roll to the second shortest series in the case where the shortest contract has less than five days to maturity. Also quotes that correspond to a volume of less than five contracts were omitted. Table 2 (Panel C) shows the summary statistics of the first three shortest futures series (in levels and daily differences). We can

[6] Alternatively, the negative sign may be attributed to the presence of measurement errors in implied volatilities (see Harvey and Whaley, 1992).

see that the daily changes of the volatility futures prices are not normally distributed; the departure from normality is greatest for the shortest futures series.

4. The Econometric Methodology

The parameters of the various processes are estimated by the conditional Maximum Likelihood (ML) method (see Hamilton, 1994, for a description of the method). We rely on the conditional MLE because it takes into account the empirically documented dependence of the observations. MLE has been commonly used to estimate continuous time models in finance (see Sundaresan, 2000, for a review, and the references therein). This is because it has desirable statistical properties. The set $\hat{\Theta}$ of the ML estimators are consistent, asymptotically efficient achieving the Cramer-Rao lower bound for consistent estimators, and they are normally distributed $\hat{\Theta} \sim N[\Theta, \{I(\Theta)^{-1}\}]$, where $[I(\Theta)]^{-1} = (-E[\frac{\partial^2 \log L(\Theta)}{\partial\Theta\partial\Theta'}])^{-1}$. Moreover, Ait-Sahalia (2004) has shown that alternative estimation methods such as the generalized method of moments cannot attain the efficiency of the ML estimators. Moreover, he has found that MLE can disentangle the diffusion from the jump component.

The conditional MLE requires the conditional (transition) density function $f[V(t+\tau)|V(t),\Theta]$ (τ>0) of the process V_t, where τ denotes the sampling frequency of observations and Θ is the set of parameters to be estimated. For a sample $\{V(t)\}_{t=1}^{T}$, the log-likelihood function that is maximized is given by

$$\Im = \max_{\Theta} \sum_{t=1}^{T-\tau} \log\left(f\left(V(t+\tau)\right) \middle| V(t), \Theta \right) \tag{3}$$

The standard errors of the ML estimators are retrieved from the inverse Hessian, evaluated at the obtained estimates.

In the cases where the conditional density function does not exist in a closed form (MRGPJ and MRSRPJ), then the corresponding conditional characteristic function is derived. The required conditional density function is obtained by Fourier inversion of the characteristic function. Maximizing the likelihood function via Fourier inversion, though computationally intensive, provides asymptotically efficient estimates of the unknown parameters (see Singleton, 2001, for a discussion and applications). Moreover, even though the likelihood function of jump diffusion processes may be unbounded, as in the case of the GBMPJ (Honoré, 1998), its Fourier transform is always bounded (Yu, 2004).

To make the exposition concrete, let $\{V_t\}_{t=1}^{T}$ be a discretely-sampled time series of implied volatilities drawn from an affine diffusion/jump diffusion process. Let $F(V_t,\tau;s) \equiv E(e^{isV_{t+\tau}} \vert V_t;\Theta)$ be the conditional characteristic function of V_t, where $i=\sqrt{-1}$. Duffie et al. (2000, page 1351) prove that under technical regularity conditions, the characteristic function for affine diffusion/jump diffusion processes has an exponential affine form (see Singleton, 2001, for an extensive discussion)

$$F(V_t,\tau;s,\Theta) = \exp\left(A(\tau;s) + B(\tau;s)V_t\right) \tag{4}$$

where the functions $A(\bullet), B(\bullet)$ satisfy complex-valued ordinary differential equations that may or may not have explicit solutions.

Assume we stand at time t. The Fourier inversion of the characteristic function provides the required conditional density function $f[V(t+\tau)\vert V(t)]$, i.e.

$$f[V(t+\tau)\vert V(t),\Theta] = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}[e^{-isV(t+\tau)}F(V(t),\tau;s,\Theta)]ds\,, \tag{5}$$

where Re denotes the real part of complex numbers. Then, the conditional log-likelihood function is maximised (equation (3)).

4.1 Estimation of Diffusion Processes

Geometric Brownian Motion Process

Under the GBMP, V_t is log-normally distributed. The conditional density of the GBMP is given by:

$$f(\log\left(V(t+\tau)\right)\vert\log\left(V(t)\right),\Theta) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(-\frac{\left(x_t-\alpha\tau\right)^2}{(2\sigma^2\tau)}\right) \tag{6}$$

where $a=\mu-\frac{1}{2}\sigma^2$ and $x_t=\log\left(\frac{V(t+\tau)}{V(t)}\right)\sim \mathrm{N}\left(\alpha\tau,\sigma^2\tau\right)$. The set of parameters Θ to be estimated is $\Theta=\{\mu,\sigma\}$.

The Mean-Reverting Gaussian Process (MRGP)

Under the MRGP, V_t is normally distributed. The conditional density of the MRGP processes is given by:

$$f(V(t+\tau)|V(t),\Theta)=\frac{1}{\sqrt{2\pi s^2}}\exp\left(-\frac{\left(V(t+\tau)-m_t\right)^2}{2s^2}\right), \tag{7}$$

where $m_t \equiv \theta + \left(V_t - \theta\right)e^{-k\tau}$, and $s^2 \equiv \frac{\sigma^2}{2k}\left(1-e^{-2k\tau}\right)$. The set of parameters Θ to be estimated is $\Theta=\{\kappa, \theta, \sigma\}$.

The Mean-Reverting Square Root Process (MRSRP)

Under the MRSRP process, the implied volatility is distributed according to a non-central chi-squared distribution (Cox et al., 1985). The transition density is given by:

$$f(V(t+\tau)|V(t),\Theta)=ce^{-u-v}(v/u)^{q/2}I_q\left(2(uv)^{1/2}\right) \tag{8}$$

where $c \equiv 2k/(\sigma^2(1-e^{-k\tau}))$, $u \equiv cV(t)e^{-k\tau}$, $v \equiv cV(t+\tau)$, $q = 2k\theta/\sigma^2 - 1$ and $I_q(\bullet)$ is the modified Bessel function of the first kind of order q. The set of parameters Θ to be estimated is $\Theta=\{\kappa, \theta, \sigma\}$.

4.2 Estimation of Jump-Diffusion Processes

A critical issue in the complete specification of jump diffusion processes is the assumption about the distribution of the jump size. This assumption needs to be both numerically tractable for the estimation purposes and realistic, i.e. to account for the possible presence of both up and down jumps in implied volatility.

In the GBMPJ model, we follow Merton (1976) by assuming that the logarithm of the jump is distributed normally. This assumption delivers the transition density in closed-form. It also allows for both up and down jumps in implied volatility, and it can be used as a benchmark against alternative assumptions for the distribution of the jump size that have been proposed recently. In the MRGPJ and the MRSRPJ models, we follow Kou (2002) by assuming that the jump size is drawn from an asymmetric double exponential distribution:

$$f(y) = p\eta_1 e^{-\eta_1 y}1_{\{y\geq 0\}} + q\eta_2 e^{\eta_2 y}1_{\{y<0\}} \tag{9}$$

where p, $q \geq 0$ and $p+q=1$, represent the probabilities of the upward and downward jump, respectively, and $1/\eta_1$, $1/\eta_2$ are the mean sizes of the upward and downward jumps, respectively.

The double exponential distribution makes possible the derivation of the characteristic function for the mean-reverting processes under consideration. In addition, it enables capturing the empirically observed both upward and downward jumps in implied volatility, as well; the two types of jumps have the same intensity but their amplitudes are drawn from different (exponential) distributions[7]. Previous researchers (see e.g., Duffie et al., 2000, Bakshi and Cao, 2004, Broadie et al., 2004, Eraker, 2004) have assumed that the jump size of volatility is distributed exponentially and hence they allow only for up jumps. This ensures that volatility remains positive but it comes at the cost that it cannot account for the down jumps in volatility that have been observed empirically.

The Geometric Brownian Motion Process augmented by Jumps (GBMPJ)

Conditional on a Poisson event, the logarithm of the jump size y is distributed normally with mean and variance γ and δ^2, respectively. In the GBMPJ process case, Press (1967) has shown that the probability density function of the log-returns $x_t = \log\left(\frac{V(t+\tau)}{V(t)}\right)$ is described as a discrete Poisson mixture of j normal probability density functions where j tends to infinity, i.e.

$$f(\log(V(t+\tau)) | \log(V(t)), \Theta) = \sum_{j=0}^{\infty} \frac{(\lambda\tau)^j e^{-\lambda\tau}}{j!} \frac{1}{\sqrt{2\pi(\sigma^2\tau + j\delta^2)}} \exp\left(-\frac{(x_t - \alpha\tau - j\gamma)^2}{2(\sigma^2\tau + j\delta^2)}\right) \quad (10)$$

where j is the number of jumps, τ is the sampling frequency, and $a = \mu - \frac{1}{2}\sigma^2$. For a sample $\{V(t)\}_{t=1}^{T}$ the log-likelihood function to be maximised is given by:

$$\Im = \max_{\{\mu,\sigma,\lambda,\gamma,\delta\}} \left(-\mathrm{T}\lambda\tau - \frac{\mathrm{T}}{2}\log(2\pi) + \sum_{t=1}^{T-\tau} \log\left[\sum_{j=0}^{\infty} \frac{(\lambda\tau)^j}{j!} \frac{1}{\sqrt{(\sigma^2\tau + j\delta^2)}} \exp\left(-\frac{(x_t - \alpha\tau - j\gamma)^2}{2(\sigma^2\tau + j\delta^2)}\right)\right]\right) \quad (11)$$

To perform MLE in the GBMPJ case, two numerical issues have to be dealt with. The infinite sum in equation (11) is truncated to j=10; Ball and Torous (1985) have found that this truncation provides accurate ML estimates (see also Jorion, 1998, for an application of this approach to the forex and stock market). Second, the mixture density (10) has the property that a global maximum of the log-likelihood does not exist; in some cases the log-likelihood may become infinite (singularity problem, see Hamilton, 1994,

[7] Alternatively, the assumption of a normally distributed jump size would allow capturing both upward and downward jumps (see Das, 2002). However, in this case, the estimation process is much more time consuming since the characteristic function for the mean-reverting processes can only be evaluated numerically.

page 689). In this case, “strange” estimates (e.g., negative σ and/or δ^2) come up as a warning (see Honoré, 1998, and the references therein). However, the singularity problem can be avoided provided that the numerical maximization algorithm converges to a local maximum. This can be achieved by using a new starting value in the case where the algorithm becomes stuck (see Hamilton, 1994). We feel comfortable with our estimated values since we have followed this route, and no “strange estimates” were encountered.

The Mean-Reverting Gaussian Process augmented by Jumps (MRGPJ)

The MRGPJ and the MRSRPJ processes do not have a known density regardless of the assumption on the jump size distribution. In the MRGPJ where the jump size follows an asymmetric double exponential distribution, the parameters *A* and *B* in the formula of the characteristic function [equation (4)] are given by[8]

$$B(\tau;s) = ise^{-k\tau} \tag{12}$$

and

$$\begin{aligned} A(\tau;s) = {} & is\theta(1-e^{-k\tau}) - s^2\sigma^2\left(\frac{1-e^{-2k\tau}}{4\kappa}\right) \\ & +\frac{\lambda}{k}\left(ip\left(ArcTan\left(\frac{s}{\eta_1}\right) - ArcTan\left(\frac{e^{-k\tau}s}{\eta_1}\right)\right)\right. \\ & \left.+i(1-p)\left(ArcTan\left(\frac{e^{-k\tau}s}{\eta_2}\right) - ArcTan\left(\frac{s}{\eta_2}\right)\right)\right) \\ & +\frac{\lambda}{2k}\left(p\left(\log\left(\eta_1^2+e^{-2k\tau}s^2\right)+\log\left(\eta_1^2+s^2\right)\right)\right. \\ & \left.+(1-p)\left(\log\left(\eta_2^2+e^{-2k\tau}s^2\right)+\log\left(\eta_2^2+s^2\right)\right)\right) \end{aligned} \tag{13}$$

The Mean-Reverting Square Root Process augmented by Jumps (MRSRPJ)

In the MRSRPJ where the jump sizes are drawn from an asymmetric double exponential distribution with up and down jumps, the parameters *A* and *B* in equation (4) are given by[9]

$$B(\tau;s) = \frac{ksie^{-k\tau}}{k-\frac{1}{2}i\sigma^2 s\left(1-e^{-k\tau}\right)} \tag{14}$$

$$A(\tau;s) = a(\tau,s) + z_1(\tau,s) + z_2(\tau,s) \tag{15}$$

where

[8] Das (2002) derives the characteristic function for the case where $\eta_1=\eta_2$, the so-called Bernoulli signed exponential distribution.

[9] Bakshi and Cao (2004) derive the characteristic function for the case of upward only exponential jumps.

$$a(\tau;s) = -\frac{2k\theta}{\sigma^2} \times log\left(\frac{k - \frac{1}{2}i\sigma^2 s\left(1-e^{-k\tau}\right)}{k}\right), \quad z_1(\tau;s) = \frac{2\lambda p}{2k - \eta_1\sigma^2} \times log\left(\frac{k - \frac{1}{2}i\sigma^2 s + is\left(\frac{\sigma^2}{2} - \frac{k}{\eta_1}\right)e^{-k\tau}}{k - \frac{isk}{\eta_1}}\right), \text{ and}$$

$$z_2(\tau;s) = \frac{2\lambda(1-p)}{2k + \eta_2\sigma^2} \times log\left(\frac{k - \frac{1}{2}i\sigma^2 s + is\left(\frac{\sigma^2}{2} + \frac{k}{\eta_2}\right)e^{-k\tau}}{k + \frac{isk}{\eta_2}}\right)$$

In general, to perform the maximization of the conditional log-likelihood function [equation (3)] initial values are required for the parameters to be estimated. In the case of the MRSRPJ and MRGPJ processes, the starting values of the parameters are obtained from the Bernoulli mixture of normal densities introduced by Ball and Torous (1983, see also Das, 2002, for an application to interest rates of a more general version of the algorithm). The numerical integration is performed by the Gauss-Legendre quadrature method.

5. MLE Results

In this Section, first the parameters of the proposed implied volatility processes are estimated and the results are discussed. Then, the robustness of the results is checked. A subtle point should be noticed. The parameters of the GBMP/GBMPJ are estimated by using the density of the log-returns while the parameters of the other processes have been estimated using the density of the level of volatility. This does not allow direct comparison of the maximized log-likelihood values across the estimated processes for any given data set. In addition, for any given process, the maximized log-likelihood values cannot be compared directly across the various data sets since the sample sizes are different. To deal with the first issue, the following Proposition is developed.

Proposition 1. Let $x_{t+\tau} = \log\left(\frac{V_{t+\tau}}{V_t}\right)$, $g\left(x_{t+\tau} \mid V_t, \Theta\right)$, $f\left(V_{t+\tau} \mid V_t, \Theta\right)$ be the conditional probability density functions of the log-returns and levels of volatility, respectively and $\Im_R = \max_{\Theta} \sum_{t=1}^{T} \log[g\left(x_{t+\tau} \mid V_t, \Theta\right)]$. Then,

$$\Im_R = \sum_{t=1}^{T} \log(V_{t+\tau}) + \max_{\Theta} \sum_{t=1}^{T} \log[f\left(V_{t+\tau} \mid V_t, \Theta\right)] \quad (16)$$

Proof. It follows directly from the rule of change of variables for probability density functions.

To address the second issue, the maximised log-likelihoods are standardised by dividing with (T-1).

5.1 Results & Discussion

Tables 3-10 show the MLE results for VIX, VXO, VXN, VXD, VSTOXX, VDAX, VX1 and VX6, respectively. Within the jump diffusion processes, we also consider the cases where only up jumps are allowed since this was a common assumption in the previous literature. Hence, we study the MRGPJ and the MRSRPJ augmented by only up jumps (MRGPUJ and MRSRPUJ, respectively). For each one of the processes under scrutiny, the estimated parameters, the t-statistics (within the parentheses), the Akaike Information Criterion (AIC), the Bayes Information Criterion (BIC), and the maximised log-likelihood values $\Im$ (unstandardised and standardised with the number of observations) are reported; $\Im$ is reported in terms of the density of the levels of volatility by applying Proposition 1. The likelihood ratio test (LRT) is also used to compare the goodness-of-fit of nested models. The LRT results support the ranking obtained from the other criteria; they are not reported due to space limitations.

We can see a number of points with interesting implications[10]. First, all parameters are significant at a 1% level of significance; the only exception appears in the VXN case where most of the estimated parameters are not statistically significant. This may be attributed either to the nature of the Nasdaq100 index (technology index), or to the fact that the VXN sample size that is employed for the MLE is the smallest among all samples; the distribution of the t-statistic as well as the properties of the ML estimators hold asymptotically.

Second, we find that the best model is the GBMPJ. The worst model is the MRGP. Depending on the data set, the MRSRPJ model is either the second or third best model despite the fact that the mean reversion and the up and down jumps are statistically significant. These results are confirmed by all statistical criteria. Interestingly, the standardised $\Im$ value shows that the best fit of the GBMPJ is obtained for VXO and the worst for VX1.

[10] Our results cannot be compared directly with those found in the earlier literature on the properties of the instantaneous volatility. This is because the latter is developed in a two-dimensional (stochastic volatility option pricing) context where the underlying asset price and volatility are modeled jointly; the impact of the joint estimation on the estimated parameters of the volatility process cannot be filtered out.

Third, the addition of mean-reversion decreases the goodness-of-fit despite the fact that the mean reversion parameter is statistically significant per se; this holds for both the diffusion/jump diffusion models. Fourth, all jump diffusion models outperform their diffusion counterparts[11]. This implies that the implied volatility presents jumps, i.e. it has large movements that cannot be explained by diffusion models. In particular, in the mean reverting jump diffusion processes, these jumps are both upwards and downwards and they are asymmetric; in all cases the mean size and the probability of the up jump is greater than those of the down jump. Along these lines, it is also informative to compare the estimated parameters in the cases where only up jumps are allowed (MRGPUJ and MRSRPUJ) with the cases where down jumps are also allowed (MRGPJ and MRSRPJ). In the former case, we can see that the intensity λ is much smaller. The mean jump sizes, the speed of mean reversion and the volatility of volatility are greater. These results also suggest the presence of down jumps in the data, as well. This is due to the following reason. Suppose that the occurrence of down jumps is not modelled, yet down jumps do occur. Then, conditional on an upward jump, the mean reversion has to increase so as the process to revert to its long run mean. By the same token, the volatility of volatility has to increase so as to capture the down jumps.

Fifth, there is interplay between jumps, the characteristics of mean-reversion (speed and long-run average), and the volatility of volatility. In particular, the introduction of jumps decreases the volatility of volatility (see also Das, 2002, for similar result in the interest rate literature). This implies that jumps account for a substantial component of σ, as expected intuitively. This also explains why the GBMP model provides implausible values of volatility (in most of the cases above 70%). On the other hand, the incorporation of jumps increases the speed of mean reversion (κ), and decreases the long-run average volatility (θ). Therefore, jumps cannot substitute the speed of mean-reversion in implied volatilities. This is in contrast to the results found in the interest rate literature where the incorporation of jumps decreases the speed of mean reversion (see e.g., Das, 2002).

Finally, within each class of models several interesting points also arise. Within the class of diffusion processes, the MRSRP performs better than the MRGP. Similarly, the MRSRPPJ performs better than the MRGPJ. Interestingly, the intensity (λ) increases

[11] Exceptions occur in the cases of VXO, VXN, VXD, VDAX, and VX1 where the MRSRP performs better than the MRGPUJ.

dramatically in the MRGPJ case compared with the MRSRPPJ; for instance, in the case of VIX the intensities are 256 and 34 per year, respectively. This again implies that the MRGPJ model is mis-specified since it cannot disentangle the continuous from the abnormal movements of the implied volatility indices. Therefore, within the class of jump diffusion models, a more complex volatility structure can account for the heavy tails of the volatility distribution in preference to the jump intensity. The out-performance of the square-root processes in both the diffusion and jump-diffusion cases can be attributed to the empirical fact that the variability of implied volatility depends on the level of implied volatility (see Jones, 2003).

The general patterns that have been discussed above appear in all implied volatility indices despite the fact that these cover different time periods. Therefore, our results are robust in the sense that they do not depend on a particular data set and the time period under scrutiny.

5.2 Robustness of the MLE Results

To ensure that the obtained parameters correspond to a global rather than a local maximum of the log-likelihood, three rounds of estimation were conducted for each series with different starting values. Moreover, the accuracy of the obtained MRSRPJ/MRSRPUJ ML estimates was examined by Monte-Carlo simulation. To this end, the obtained ML estimates were used as the true parameters. For each one of the two processes, 500 simulation runs were conducted with the same stream of random numbers. A number of implied volatility observations equal to the size of the sample where the original MLE was implemented were simulated. On each simulated path, MLE was performed. Then, the 500 obtained estimates were averaged and the standard deviation was calculated. A *t*-statistic tested the null hypothesis that the true parameter is equal to the (average) estimated parameter. The null hypothesis could not be rejected for either of the two processes. Due to space limitations these results are not reported. Therefore, the accuracy of the obtained ML MRSRPJ/MRSRPUJ estimates is confirmed. Finally, the issue of the possible existence of structural breaks in the data generating process was addressed for each index by performing the MLE for various sub-samples and investigating whether the ranking of the processes had been altered. No structural break was detected.

6. Ranking the Processes: Evidence from the Volatility Futures

In this Section, the alternative implied volatility processes are ranked according to a financial criterion. To this end, we use the data from the CBOE volatility futures on VIX. A volatility futures pricing model corresponds to each process. Then, the processes are ranked according to the pricing performance of the corresponding model in the volatility futures market. Given that data on the VIX volatility futures are available from 26/03/2004 onwards, this application may also be viewed as a test of the out-of-sample performance of the econometrically estimated models on VIX.

It should be noticed that the pricing of volatility futures is not model-free since VIX is not a tradable asset; Carr and Wu (2004a) have derived arbitrage bounds to the price of volatility futures by assuming that the index futures price has continuous paths. This is in contrast to the case of volatility/variance swaps that can be replicated by trading in standard European options, assuming a general jump diffusion process for the evolution of the index futures price (see Carr and Wu, 2004b, 2004c, and Carr and Lee, 2005). Therefore, an assumption about the implied volatility process needs to be made in order to develop a volatility futures pricing model.

6.1 The Volatility Futures Pricing Models

Let $G_t(V,T)$ denote the futures price at time t for a futures contract on V with maturity T. Under the risk-adjusted equivalent martingale measure Q, $G_t(V,T)$ equals the conditional expected value of V_T at time T; the expected value is conditional on the information up to time t, i.e.

$$G_t(V,T) = E_t^Q(V_T), \qquad t < T \tag{17}$$

Therefore, the corresponding expected value of volatility is required in order to price volatility futures under the alternative diffusion/jump diffusion processes. In the case of the MRGPJ and the MRSRPJ processes, the expected value of volatility is derived from the characteristic function since the conditional density function is not known in closed form. This is done by differentiating the characteristic function once with respect to s and then evaluating the derivative at s=0.

Table 11 shows the expected value of volatility of each one of the eight processes under scrutiny. Interestingly, the expected value of volatility is the same under the MRGP and the MRSRP. Similarly, the expected value of volatility is the same under the MRGPJ and the MRSRPJ. The expected value is also the same in the cases where only up jumps are considered. This is because the above-mentioned pairs of processes have the same drift; the expected value depends only on the characteristics of the drift (this is not the case for the higher order moments).

The expected value under the GBMP resembles the cost-of-carry relationship for futures written on a tradable asset. The expected value under the MRSRP is the Grünbichler and Longstaff (1996) volatility futures model. A direct comparison between the diffusion and the jump diffusion formulae is possible; this is valid under the assumption that the estimates of the parameters that appear in both processes are the same. The comparison of the volatility futures model derived under the GBMPJ with that derived under the GBMP shows that the difference between the two model prices depends on the sign of the mean of the log jump (γ). In the case where γ>0 (γ<0), the GBMPJ futures price will be greater (smaller) than the GBMP price. In the case of the mean-reverting diffusion/jump diffusion processes, we recall that $E(y) = \frac{p}{\eta_1} - \frac{(1-p)}{\eta_2}$. Therefore, if *E(y)>0*, then the volatility futures price under the MRGPJ/MRSRPJ will be greater than the price delivered by its diffusion counterpart (MRGP/MRSRP).

Finally, the pricing performance of the various models is expected to depend on the remaining time-to-maturity. As the time-to-maturity increases the futures price that corresponds to the mean-reverting processes (non-mean-reverting processes) tends to a constant (grows exponentially). Therefore, the “mean-reverting” volatility futures models are expected to perform worse as the time-to-maturity increases since they cannot capture the stochastic evolution of volatility.

6.2 Pricing Performance: Results & Discussion

The assessment of the pricing performance of the volatility futures models is done as follows. The parameters of the pricing models are the risk-adjusted parameters. These can be obtained either by calibration or by making an assumption about the market price of volatility/jump risk and then apply Girsanovs’ theorem (see Runggaldier, 2003, for the jump diffusion version). In the case of calibration, a natural approach would be to calibrate each model to the market futures prices of the shortest futures series. Then, the pricing performance of each model would be investigated for the remaining futures series.

To this end, a generalized non-linear least squares regression method was used (see also Bates, 1996, 2000). However, calibration is an ill-posed problem in the case of the jump-diffusion processes under consideration. This is because there are more parameters to be estimated than the number of equations delivered by the first order conditions in the optimization method; these equations collapse to one equation (under-identified problem).

Therefore, we have to resort to an assumption about the market price of the volatility/jump risk. Let $G(T)_{t,i}^{M}$ and $G(T)_{t}^{A}$ be the T-expiry i-model and market volatility futures prices, respectively (T=1,2,3, i=1,…,8) on date t. Let also $(\frac{G(T)_{t,i}^{M} - G(T)_{t}^{A}}{G(T)_{t}^{A}})^2$ be the squared percentage pricing error on date t. For any given futures pricing model, the T-expiry mean squared percentage pricing error is calculated for the shortest, second shortest, and third shortest contract; the averaging of the daily percentage pricing errors is done over the available observations for each one of the futures series. Table 12 shows the mean squared percentage pricing errors for each one futures pricing model that corresponds to the processes of implied volatility under consideration. The results are reported for the shortest, second shortest, and third shortest contract. The market prices of the volatility/jump risk are assumed to be zero. Hence, the pricing performance of the various models is examined by using as inputs the estimated parameters obtained from the MLE under each process.

We can see that under the financial metric the GBMPJ process performs best (for all three maturities) with the (driftless) GBMP following closely just as was the case with the econometric analysis of VIX. Moreover, the pricing performance of the models depends on the maturity of the contract; it decreases as we move to the more distant expiries. To confirm the robustness of our results to the choice of the value of the market price of volatility risk, pricing errors were calculated for a range of negative values of the volatility risk premium; negative values were chosen so as to be consistent with the empirical evidence that the volatility risk premium is negative (see e.g., Bakshi and Kapadia, 2003). We found that the ranking of the processes does not depend on the chosen values. This is in accordance with Daouk and Guo (2004) and Psychoyios and Skiadopoulos (2006) who found that the choice of the value of the volatility risk premium does not affect the pricing and hedging results, respectively.

7. Conclusions

The accurate modeling of implied volatility in continuous time is important for option pricing and risk management. Yet, this topic has received limited attention, to date. The objective of this paper is to identify the process that describes well the evolution of implied volatility in continuous time. To this end, we have examined alternative affine specifications of the process that drives the dynamics of implied volatility in continuous time. Various diffusion and jump diffusion processes have been compared. A rich data set has been employed from the European and U.S. volatility derivatives markets; eight implied volatility indices and the recently introduced CBOE volatility futures were used. The alternative processes have been examined under both an econometric (conditional maximum likelihood) and a financial metric (CBOE volatility futures market). This research approach has enabled us to decide on whether the results are robust across different implied volatility indices, time periods, and metrics and therefore should constitute a solid basis for continuous-time financial applications.

We found that under both metrics, the simplest jump diffusion model à la Merton (1976) performed best followed closely by the simplest diffusion specification (geometric Brownian motion). The econometric results hold regardless of the implied volatility index under scrutiny.

The study has at least three implications. First, a diffusion model does not suffice to describe the dynamics of implied volatility. The addition of jumps is necessary. On the other hand, the mean reversion in implied volatilities is of second order importance. Second, the pattern of the behavior of implied volatility indices is the same across different European and US markets. It is not affected by the time horizon that they refer to, the time period under consideration, and the construction method that is used. Third, naïve models perform better than more complex models for volatility futures pricing purposes. This is in accordance with the results found in the standard options literature where the pricing models that are based on simpler assumptions about the evolution of the price of the underlying asset perform better than more "elegant" models (see e.g., Bakshi et al., 1997, Dumas et al., 1998).

Future research should look at the presented specification of the implied volatility process by using alternative econometric techniques, e.g., regime switching models (see e.g., Daouk and Guo, 2004); yet, our approach suggests that there is no structural break in the data generating process of volatility. In a second step, it is also worth investigating more complex specifications of the implied volatility process. For example, non-linear specifications of the drift/volatility structure in the spirit of Bakshi et al. (2005) could be examined in the presence of jumps in volatility. It may be the case that the presence of jumps removes any such non-linearities, as found in the interest rates literature (see e.g., Das, 2002). However, the non-linear specification makes the affine structure to be lost and it does not make possible the derivation of the characteristic function. This calls for an alternative econometric methodology. The jump intensity could also be allowed to depend on the level of volatility rather than being constant (see e.g., Wu, 2005). Alternative financial metrics (e.g., Value-at-Risk) should also be considered in order to rank the alternative implied volatility processes.

Implied Volatility Index	Option Pricing model	Underlying Asset	Options used	Represents
VXO	Merton (1973)	S&P 100	4 puts and calls of 2 nearest to 30 days expiries, with 2 strikes around an at-the-money (ATM) point.	The implied volatility of an ATM option with constant 30 calendar days to expiry.
$VIX, VXN, VXD, VSTOXX$	Independent of model	S&P 500, Nasdaq 100, Dow Jones 100, DJ EURO STOXX 50	Out-of-the-money (OTM) puts and calls of 2 nearest to 30 days expiries, covering a wide range of strikes.	The square root of implied variance across options of all strikes, with constant 30 calendar days to expiry.
$VDAX$	Black's model (1976)	DAX	8 pairs of puts and calls of 2 nearest to 45 days expiries, with 4 strikes around an ATM point.	The implied volatility of an ATM option with constant 45 calendar days to expiry.
$VX1, VX6$	Merton (1973)	CAC 40	4 calls of 2 nearest to 31 (185) days expiries, with 2 strikes around an ATM point.	The implied volatility of an ATM option with a constant 31 (VX1) and 185 (VX6) calendar days to expiry.

Table 1: Synopsis of the methods that are used to construct the implied volatility indices under consideration.

Panel A: Implied Volatility Indices - Summary Statistics (Levels)

	VIX	VXO	VXN	VXD	VSTOXX	VDAX	VX1	VX6
Starting Date	2/1/1990	2/1/1986	2/2/2001	6/10/1997	4/1/1999	2/1/1992	14/10/97	14/10/97
# Observations	3586	4608	787	1625	1327	3070	1614	1613
Mean	0.20	0.21	0.42	0.24	0.30	0.22	0.26	0.25
Std. Dev.	0.06	0.07	0.12	0.06	0.09	0.09	0.08	0.06
Skewness	0.85	1.16	0.08	0.78	1.34	1.15	1.20	0.95
Kurtosis	3.62	5.18	2.05	3.45	4.11	4.08	4.24	4.69

Panel B: Implied Volatility Indices – Summary Statistics (First Differences)

	VIX	VXO	VXN	VXD	VSTOXX	VDAX	VX1	VX6
Mean	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Std. Dev.	0.01	0.01	0.02	0.01	0.02	0.01	0.04	0.02
Skewness	0.55	2.15	0.19	0.58	1.40	0.76	0.68	0.71
Kurtosis	8.78	39.77	6.93	6.95	17.34	10.42	15.19	19.12
First Order Autocorrelation	-0.038*	-0.078*	0.029	-0.016	-0.038	-0.058*	-0.373*	-0.384*

Panel C: CBOE Volatility Futures: Summary Statistics

	Levels			First Differences		
	Shortest Series	**Second Shortest**	**Third Shortest**	**Shortest Series**	**Second Shortest**	**Third Shortest**
# Observations	307	306	268	306	305	267
Mean	0.15	0.17	0.17	0.00	0.00	0.00
Std. Dev.	0.02	0.02	0.02	0.00	0.00	0.00
Skewness	0.35	0.28	0.20	1.77	0.64	0.20
Kurtosis	2.23	1.91	1.75	11.09	7.83	6.73
First-Order Autocorrelation				-0.006	0.035	-0.137*

Table 2: Summary statistics: Implied Volatility indices (levels and daily differences) and CBOE Volatility futures data on VIX (shortest, second shortest, third shortest series – levels and daily differences). The data for the implied volatility indices have been collected from the introduction of each one of the indices up to 24/03/2004. The data for the volatility futures data on VIX have been collected from 26/03/2004 up to 17/06/2005. The asterisk denotes statistical significance at a 5% confidence level.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	**GBMP**	**MRGP**	**MRSRP**	**GBMPJ**	**MRGPJ**	**MRGPUJ**	**MRSRPJ**	**MRSRPUJ**
μ	0.4083 (1.6903)	-	-	-0.8333 (-3.3155)	-	-	-	-
k	-	4.9297 (5.8680)	4.7457 (5.8359)	-	7.0253 (7.9735)	8.4433 (30.4745)	6.5453 (7.9786)	7.4405 (9.2933)
θ	-	0.2018 (18.6286)	0.2010 (19.6533)	-	0.1374 (7.5593)	0.1210 (17.0505)	0.1497 (17.7136)	0.1538 (21.6877)
σ	0.8927 (84.6468)	0.2014 (83.7275)	0.4145 (83.8146)	0.6662 (13.0338)	0.0885 (8.3235)	0.1315 (77.7601)	0.3268 (39.4030)	0.3537 (60.3589)
λ	-	-	-	74.5825 (1.9569)	256.4492 (3.5268)	71.1278 (7.9360)	34.7677 (4.5514)	18.9503 (4.3467)
γ	-	-	-	0.0143 (2.4456)	-	-	-	-
δ	-	-	-	0.0659 (6.2621)	-	-	-	-
p	-	-	-	-	0.5435 (7.0056)	1	0.7944 (10.4569)	1
$1/\eta 1$	-	-	-	-	0.0086 (8.8749)	0.0105 (13.2825)	0.0151 (8.8639)	0.0173 (7.9531)
$1/\eta 2$	-	-	-	-	0.0066 (5.4747)	-	0.0143 (4.3912)	-
AIC	-22,296	-21,200	-21,946	-22,570	-21,946	-21,696	-22,266	-22,228
BIC	-22,283	-21,181	-21,927	-22,539	-21,903	-21,665	-22,223	-22,197
$\Im$	11,150	10,603	10,976	11,290	10,980	10,853	11,140	11,119
$\Im/(n-1)$	3.1102	2.9576	3.0616	3.1492	3.0628	3.0273	3.1074	3.1015

Table 3: VIX: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 2/01/1990 to 24/03/2004.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	**GBMP**	**MRGP**	**MRSRP**	**GBMPJ**	**MRGPJ**	**MRGPUJ**	**MRSRPJ**	**MRSRPUJ**
μ	0.4192 (1.9524)	-	-	-0.7710 (-4.3054)	-	-	-	-
k	-	5.3942 (6.9467)	4.9225 (6.7125)	-	6.0211 (12.8327)	8.5424 (43.2010)	6.9146 (10.0023)	8.4111 (12.1524)
θ	-	0.2145 (20.7327)	0.2138 (21.6975)	-	0.1620 (15.2341)	0.1501 (22.7930)	0.1574 (21.8533)	0.1693 (29.9071)
σ	0.9174 (95.9681)	0.2387 (94.8588)	0.4540 (94.9907)	0.6178 (27.3371)	0.0895 (27.7992)	0.1604 (105.8486)	0.3002 (44.7147)	0.3743 (63.0857)
λ	-	-	-	75.2667 (5.2891)	201.1133 (11.5511)	33.4097 (8.5509)	55.2801 (7.5369)	13.9803 (5.5614)
γ	-	-	-	0.0154 (3.7403)	-	-	-	-
δ	-	-	-	0.0754 (13.7406)	-	-	-	-
p	-	-	-	-	0.5134 (15.5880)	1	0.6953 (14.4348)	1
$1/\eta 1$	-	-	-	-	0.0113 (18.7457)	0.0192 (11.7861)	0.0169 (12.4523)	0.0277 (9.0053)
$1/\eta 2$	-	-	-	-	0.0089 (13.2364)	-	0.0161 (8.1050)	-
AIC	-27,862	-25,698	-27,104	-29,156	-27,692	-27,018	-28,040	-27,830
BIC	-27,850	-25,679	-27,085	-29,125	-27,649	-26,987	-27,997	-27,799
$\Im$	13,933	12,852	13,555	14,583	13,853	13,514	14,027	13,920
$\Im/(n-1)$	3.8876	3.5859	3.7821	4.0689	3.8652	3.7706	3.9138	3.8839

Table 4: VXO: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 2/01/1986 to 24/03/2004.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	**GBMP**	**MRGP**	**MRSRP**	**GBMPJ**	**MRGPJ**	**MRGPUJ**	**MRSRPJ**	**MRSRPUJ**
μ	-0.0750 (-0.2314)	-	-	-0.5426 (-0.5206)	-	-	-	-
k	-	2.2147 (1.8060)	1.8302 (1.7748)	-	2.1608 (1.5384)	3.4241 (2.8459)	1.9376 (1.7542)	2.6763 (2.4332)
θ	-	0.3806 (5.3794)	0.4178 (6.4142)	-	0.3449 (2.0891)	0.3468 (6.8152)	0.3695 (5.1615)	0.3477 (6.7435)
σ	0.5727 (39.6270)	0.2616 (39.4201)	0.3798 (39.4604)	0.4820 (2.8535)	0.2150 (10.2045)	0.2443 (34.1544)	0.3496 (31.5595)	0.3640 (36.8908)
λ	-	-	-	99.9997 (0.2587)	40.0174 (0.7383)	5.6682 (1.0473)	7.7714 (1.4256)	2.7800 (1.1275)
γ	-	-	-	0.0045 (0.6672)	-	-	-	-
δ	-	-	-	0.0309 (0.8892)	-	-	-	-
p	-	-	-	-	0.5490 (1.2664)	1	0.4378 (1.3714)	1
$1/\eta 1$	-	-	-	-	0.0168 (1.2978)	0.0293 (1.8510)	0.0351 (1.7453)	0.0386 (1.6814)
$1/\eta 2$	-	-	-	-	0.0163 (1.9840)	-	0.0252 (1.7453)	-
AIC	-4,414	-4,224	-4,352	-5,347	-4,288	-4,265	-4,372	-4,369
BIC	-4,405	-4,210	-4,338	-5,323	-4,255	-4,242	-4,340	-4,345
$\Im$	2,209	2,115	2,179	2,678	2,151	2,138	2,193	2,189
$\Im/(n-1)$	2.8106	2.6910	2.7723	3.4075	2.7366	2.7195	2.7902	2.7855

Table 5: VXN: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 2/02/2001 to 24/03/2004.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	**GBMP**	**MRGP**	**MRSRP**	**GBMPJ**	**MRGPJ**	**MRGPUJ**	**MRSRPJ**	**MRSRPUJ**
μ	0.2956 (0.9456)	-	-	-1.5370 (-2.5647)	-	-	-	-
k	-	6.6559 (4.5528)	6.7265 (4.6803)	-	9.8184 (6.4060)	9.1150 (12.7875)	9.0769 (6.3722)	9.4792 (6.7368)
θ	-	0.2385 (19.6561)	0.2376 (21.3593)	-	0.1957 (11.6560)	0.1766 (10.6615)	0.1970 (16.9805)	0.1979 (18.1749)
σ	0.7957 (56.9679)	0.2050 (56.1566)	0.3995 (56.1979)	0.5094 (3.9600)	0.1438 (13.6866)	0.1624 (35.1637)	0.3418 (32.6790)	0.3468 (38.7023)
λ	-	-	-	212.2382 (1.1475)	100.7100 (2.1742)	49.9911 (2.8281)	27.4933 (2.4940)	24.7722 (2.6124)
γ	-	-	-	0.0084 (1.4158)	-	-	-	-
δ	-	-	-	0.0415 (3.7760)	-	-	-	-
p	-	-	-	-	0.6105 (3.4103)	1	0.9502 (9.2680)	1
$1/\eta 1$	-	-	-	-	0.0119 (5.9444)	0.0123 (5.2884)	0.0153 (5.0355)	0.0156 (4.9948)
$1/\eta 2$	-	-	-	-	0.0079 (3.2803)	-	0.0166 (1.0178)	-
AIC	-9,844	-9,555	-9,758	-10,677	-9,762	-9,737	-9,858	-9,861
BIC	-9,833	-9,539	-9,742	-10,650	-9,724	-9,710	-9,821	-9,834
$\Im$	4,924	4,780	4,882	5,343	4,888	4,874	4,936	4,935
$\Im/(n-1)$	3.0320	2.9436	3.0062	3.2902	3.0099	3.0009	3.0395	3.0390

Table 6: VXD: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 6/10/1997 to 24/03/2004.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	**GBMP**	**MRGP**	**MRSRP**	**GBMPJ**	**MRGPJ**	**MRGPUJ**	**MRSRPJ**	**MRSGPUJ**
	0.4351			-1.7871				
μ	(0.9277)	-	-	(-4.1537)	-	-	-	-
		4.8028	5.3184		13.2346	15.6751	9.6566	10.6469
k	-	(3.3372)	(3.3946)	-	(11.6877)	(16.0767)	(6.5836)	(7.0949)
		0.3000	0.2996		0.1720	0.2095	0.2046	0.2183
θ	-	(9.5409)	(10.9136)	-	(9.2511)	(16.1404)	(9.9314)	(13.8151)
	0.8375	0.2822	0.4764	0.5394	0.1128	0.1843	0.3067	0.3697
σ	(56.1398)	(55.6805)	(55.6992)	(13.5302)	(12.7196)	(25.7743)	(21.7294)	(39.3090)
				99.8906	233.1253	78.0693	85.6988	40.4363
λ	-	-	-	(2.9256)	(5.3390)	(5.4009)	(4.3990)	(3.6958)
				0.0202				
γ	-	-	-	(3.3463)	-	-	-	-
				0.0576				
δ	-	-	-	(8.1031)	-	-	-	-
					0.7826		0.8285	
p	-	-	-	-	(17.6612)	1	(11.5063)	1
					0.0119	0.0173	0.0159	0.0195
$1/\eta_1$	-	-	-	-	(9.5065)	(9.3060)	(8.1062)	(6.8248)
					0.0110	-	0.0165	
$1/\eta_2$	-	-	-	-	(6.4732)		(3.8801)	-
AIC	-7,378	-6,942	-7,226	-7,573	-7,386	-7,272	-7,466	-7,448
BIC	-7,367	-6,926	-7,210	-7,547	-7,350	-7,246	-7,430	-7,422
$\Im$	3,691	3,474	3,616	3,792	3,700	3,641	3,740	3,729
$\Im/(n-1)$	2.7834	2.6199	2.7270	2.8594	2.7903	2.7459	2.8205	2.8122

Table 7: VSTOXX: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 4/01/1999 to 24/03/2004.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	GBMP	MRGP	MRSRP	GBMPJ	MRGPJ	MRGPUJ	MRSRPJ	MRSRPUJ
μ	0.3120 (1.5284)	-	-	-1.0467 (-5.2494)	-	-	-	-
k	-	2.2772 (3.7390)	2.1753 (3.7031)	-	5.1353 (87.1684)	4.1091 (95.7415)	4.3868 (7.3239)	5.8575 (10.2716)
θ	-	0.2296 (9.5908)	0.2229 (10.4636)	-	0.1576 (22.1287)	0.0862 (20.6334)	0.1423 (13.3190)	0.1486 (19.2789)
σ	0.7121 (78.3108)	0.1900 (77.8723)	0.3563 (77.9313)	0.4496 (20.0012)	0.0978 (72.0869)	0.0930 (179.0919)	0.2329 (29.2241)	0.2779 (57.9741)
λ	-	-	-	100.0000 (4.8889)	107.6290 (9.1897)	67.0833 (13.6009)	62.5958 (6.7281)	29.8565 (6.4975)
γ	-	-	-	0.0129 (3.9813)	-	-	-	-
δ	-	-	-	0.0522 (13.1137)	-	-	-	-
p	-	-	-	-	0.5484 (8.9674)	1	0.6856 (11.8399)	1
$1/\eta 1$	-	-	-	-	0.0160 (16.5864)	0.0126 (20.4242)	0.0142 (11.4697)	0.0167 (10.2020)
$1/\eta 2$	-	-	-	-	0.0093 (22.5484)	-	0.0121 (7.0980)	-
AIC	-19,950	-18,477	-19,422	-21,269	-19,487	-19,086	-19,889	-19,779
BIC	-19,938	-18,459	-19,404	-21,239	-19,445	-19,056	-19,847	-19,749
$\Im$	9,977	9,242	9,714	10,639	9,751	9,548	9,952	9,895
$\Im/(n-1)$	3.2508	3.0112	3.1652	3.4667	3.1771	3.1111	3.2426	3.2240

Table 8: VDAX: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 2/01/1992 to 24/03/2004.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	**GBMP**	**MRGP**	**MRSRP**	**GBMPJ**	**MRGPJ**	**MRGPUJ**	**MRSRPJ**	**MRSRPUJ**
μ	1.7636 (2.3723)	-	-	-0.2336 (-1.5694)	-	-	-	-
k	-	24.8323 (8.4796)	24.9986 (8.5560)	-	15.8909 (8.7761)	20.9334 (13.6715)	16.3983 (8.4947)	21.9061 (26.1470)
θ	-	0.2572 (27.9724)	0.2570 (31.3129)	-	0.1852 (15.3561)	0.2096 (23.6205)	0.2025 (20.2394)	0.1939 (33.5075)
σ	1.8743 (56.7805)	0.5774 (54.0784)	1.0296 (54.0360)	1.0135 (28.8598)	0.1786 (22.1648)	0.3715 (78.9908)	0.4564 (23.4670)	0.5052 (27.9841)
λ	-	-	-	47.6852 (7.3783)	152.7927 (9.2828)	22.9904 (10.8861)	85.6176 (7.0320)	42.1697 (9.9335)
γ	-	-	-	0.0429 (0.9634)	-	-	-	-
δ	-	-	-	0.2416 (13.6018)	-	-	-	-
p	-	-	-	-	0.6530 (14.6191)	1	0.6555 (13.0744)	1
$1/\eta 1$	-	-	-	-	0.0266 (11.8912)	0.0551 (6.9328)	0.0355 (8.4002)	0.0491 (25.7780)
$1/\eta 2$	-	-	-	-	0.0306 (8.1942)	-	0.0408 (7.9158)	-
AIC	-6,840	-6,263	-6,668	-7,890	-7,254	-6,550	-7,356	-6,692
BIC	-6,830	-6,247	-6,652	-7,863	-7,216	-6,523	-7,318	-6,665
$\mathfrak{I}$	3,422	3,135	3,337	3,950	3,634	3,280	3,685	3,351
$\mathfrak{I}/(n-1)$	2.1216	1.9434	2.0688	2.4487	2.2529	2.0335	2.2846	2.0775

Table 9: VX1: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 14/10/1997 to 24/03/2004.

	Diffusion Processes			Jump Diffusion Processes				
Parameter	**GBMP**	**MRGP**	**MRSRP**	**GBMPJ**	**MRGPJ**	**MRGPUJ**	**MRSRPJ**	**MRSRPUJ**
μ	0.7321 (1.5266)	-	-	-0.5185 (-2.5193)	-	-	-	-
k	-	17.4189 (7.4984)	16.8043 (7.4145)	-	6.6523 (5.2821)	7.8561 (59.5406)	7.3739 (5.5122)	7.4687 (69.9909)
θ	-	0.2429 (33.8459)	0.2427 (34.7921)	-	0.1999 (17.1465)	0.2031 (26.0354)	0.1982 (19.5780)	0.9951 (39.6537)
σ	1.2700 (59.5305)	0.3315 (57.4619)	0.6372 (57.5411)	0.4724 (17.5460)	0.0873 (17.0578)	0.0958 (40.6159)	0.2046 (17.6625)	0.1805 (315.7771)
λ	-	-	-	91.2083 (7.9672)	150.8600 (10.2950)	55.3563 (12.7496)	122.7977 (9.2603)	52.3661 (14.8853)
γ	-	-	-	0.0126 (1.2223)	-	-	-	-
δ	-	-	-	0.1158 (14.8516)	-	-	-	-
p	-	-	-	-	0.5410 (14.7022)	1	0.5640 (14.1569)	1
$1/\eta 1$	-	-	-	-	0.0176 (12.0346)	0.0214 (14.3580)	0.0193 (12.2437)	0.0223 (21.9923)
$1/\eta 2$	-	-	-	-	0.0168 (11.3449)	-	0.0189 (9.7596)	-
AIC	-8,470	-8,802	-9,034	-10,206	-10,044	-9,516	-10,092	-9,763
BIC	-8,460	-8,786	-9,018	-10,179	-10,006	-9,489	-10,054	-9,736
$\Im$	4,237	4,404	4,520	5,108	5,029	4,763	5,053	4,887
$\Im/(n-1)$	2.6350	2.7388	2.8109	3.1765	3.1275	2.9621	3.1424	3.0389

Table 10: VX6: Parameter estimation and their corresponding *t*-statistic (in parentheses) for the diffusion and jump diffusion processes. The Information Criterion and (AIC), the Bayes Information Criterion (BIC), and the Log-Likelihood are also reported. The estimation period is from 14/10/1997 to 24/03/2004.

<table>
<tr><th>Implied Volatility Process</th><th>Expected Value</th></tr>
<tr><td>$dV_t = \mu V_t dt + \sigma V_t dW_t$</td><td>$E_t\left(V_T\right) = V_t e^{\mu(T-t)}$</td></tr>
<tr><td>$dV_t = k\left(\theta - V_t\right)dt + \sigma dW_t$</td><td rowspan="2">$E_t\left(V_T\right) = V_t e^{-k(T-t)} + \theta[1 - e^{-k(T-t)}]$</td></tr>
<tr><td>$dV_t = k\left(\theta - V_t\right)dt + \sigma\sqrt{V_t}W_t$</td></tr>
<tr><td>$dV_t = V_t\mu + \sigma V_t dW_t + yV_t dq_t$</td><td>$E_t\left(V_T\right) = V_t\, \mathrm{e}^{\mu(T-t)+\lambda(T-t)[\exp(\gamma)-1]}$</td></tr>
<tr><td>$dV = k\left(\theta - V_t\right)dt + \sigma dW_t + ydq_t$</td><td rowspan="2">$E_t\left(V_T\right) = V_t e^{-k(T-t)} + \theta(1 - e^{-k(T-t)}) + \frac{\lambda}{k}(1 - e^{-k(T-t)})[\frac{p}{\eta_1} - \frac{(1-p)}{\eta_2}]$</td></tr>
<tr><td>$dV_t = k\left(\theta - V_t\right)dt + \sigma\sqrt{V_t}dW_t + ydq_t$</td></tr>
</table>

Table 11: Implied Volatility Processes and the corresponding expected value of volatility $E_t(V_T)$ at time T formed at time t ($t<T$).

	GBMP	MRGP	MRSRP	GBMPJ	MRGPJ	MRGPUJ	MRSRPJ	MRSRPUJ
Shortest	0.0086	0.0094	0.0086	0.0063	0.0148	0.0292	0.0131	0.0161
Second Shortest	0.0213	0.0184	0.0169	0.0131	0.0264	0.0512	0.0238	0.0282
Third Shortest	0.0335	0.0326	0.0305	0.0145	0.0363	0.0630	0.0344	0.0372

Table 12: Ranking of the Alternative Processes according to their pricing performance in the CBOE Volatility futures Market. The mean squared percentage pricing error $\frac{1}{N}\sum_{t=1}^{N}(\frac{G(T)_{t,i}{}^{M} - G(T)_t{}^{A}}{G(T)_t{}^{A}})^2$ over the period 26/03/2004-17/06/2005 is reported where $G(T)_{t,i}{}^{M}$ and $G(T)_t^{A}$ be the T-expiry i-model and market volatility futures prices, respectively (i=1,…,8), and N is the number of observations. Zero volatility/jump risk premia have been assumed.

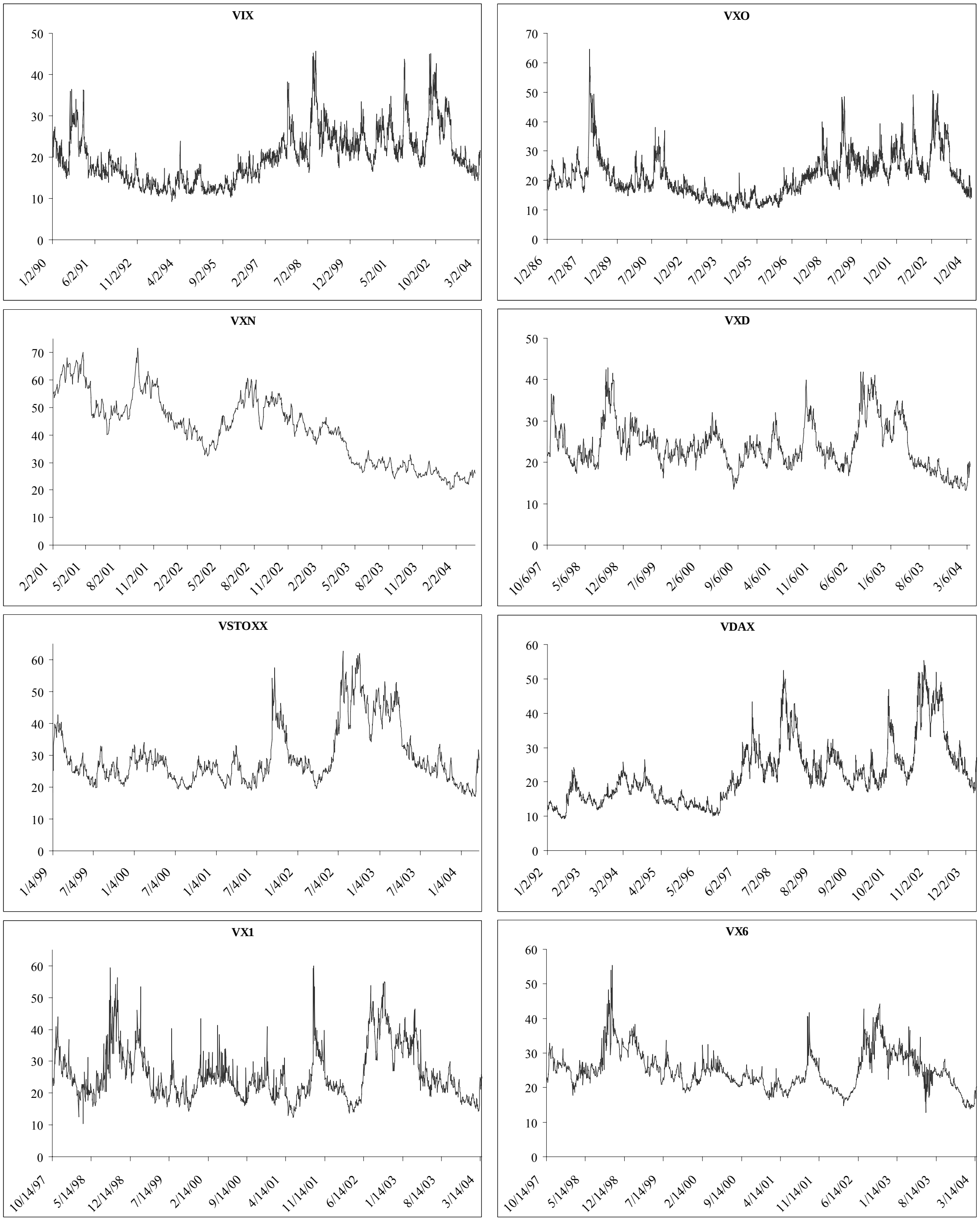

Figure 1: Evolution of the implied volatility indices VIX, VXO, VXN, VXD, VDAX, VSTOXX, VX1, VX6 over time (from the introduction of the index up to 24/03/2004).

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Appendices

Appendix A1: Derivation of the characteristic function for the MRGPJ

Das (2002) shows that the conditional characteristic function $F(V_t, \tau; s) = E(e^{isV_{t+\tau}} \left| V_t; \Theta \right.)$ of the MRGPJ must satisfy the following Kolmogorov backward differential equation

$$0 = \frac{\partial F}{\partial V_t} k(\theta - V_t) + \frac{1}{2}\frac{\partial^2 F}{\partial V_t^2}\sigma^2 - \frac{\partial F}{\partial \tau} + \lambda \mathrm{E}\left[F(V_t + y) - F(V_t)\right] \tag{18}$$

subject to the boundary condition

$$F(V_t, \tau = 0; s) = e^{isV_t} \tag{19}$$

where $i = \sqrt{-1}$. Differentiating the characteristic function given by equation (4) yields

$$\begin{aligned} F_V &= BF \\ F_{VV} &= B^2 F \\ F_\tau &= F\left(A_\tau + VB_\tau\right) \end{aligned} \tag{20}$$

where the subscripts denote the corresponding partial derivatives. Replacing equations (20) in equation(18) and rearranging yields

$$V_t\left(-kB - B_\tau\right) + \left(k\theta B + \frac{1}{2}\sigma^2 B^2 - A_\tau + \lambda \mathrm{E}\left[e^{yB} - 1\right]\right) = 0 \tag{21}$$

where *E(·)* is the expectation operator over the asymmetric double exponential distribution. Since $V_t \neq 0$, the expressions in the parentheses in equation (21) must equal zero. Therefore we obtain the following ordinary differential equations (ODEs)

$$-kB - B_\tau = 0 \tag{22}$$

$$k\theta B + \frac{1}{2}\sigma^2 B^2 - A_\tau + \lambda \mathrm{E}\left[e^{yB} - 1\right] = 0 \tag{23}$$

Also,

$$\mathrm{E}\left[e^{yB} - 1\right] = p\int_0^{+\infty} \eta_1 e^{-\eta_1 y} e^{yB} dy + q\int_0^{+\infty} \eta_2 e^{\eta_2 y} e^{yB} dy - 1 = p\frac{\eta_1}{\eta_1 - \mathrm{B}} + q\frac{\eta_2}{\eta_2 + \mathrm{B}} - 1 \tag{24}$$

Equations (12) and (13) are the solutions of the ODEs (22) and (23), respectively. The ODEs are solved subject to the boundary conditions $A(\tau = 0; s) = 0$, $B(\tau = 0; s) = is$ that are implied by equation (19).

Appendix A2: Derivation of the characteristic function for the MRSRPJ

Bakshi and Cao (2004) show that the characteristic function of the MRSRPJ must satisfy the following Kolmogorov backward differential equation

$$\frac{\partial F}{\partial V_t} + k(\theta - V_t) + \frac{1}{2}\frac{\partial^2 F}{\partial V_t^2} V_t \sigma^2 - \frac{\partial F}{\partial \tau} + \lambda \mathrm{E}\left[F(V_t + y) - F(V_t)\right] = 0 \tag{25}$$

subject to the boundary condition given by equation (19). Following the same steps as in Appendix A1 yields the following ODEs

$$-kB - B_\tau + \frac{1}{2}\sigma^2 B^2 = 0 \tag{26}$$

$$k\theta B - A_\tau + \lambda \mathrm{E}\left[e^{yB} - 1\right] = 0 \tag{27}$$

Using equation (24), equations (14), and (15) are the solutions of the ODEs (26) and (27), respectively. The ODEs are solved subject to the boundary conditions $A(\tau = 0; s) = 0$, and $B(\tau = 0; s) = is$.