Testing Conditional Factor Models Using Completion Portfolios[1]

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Abstract

In this paper, we use the notion of completion portfolios to construct a test of asset pricing models in the presence of conditioning information, extending the analysis of Shanken (1985) and Lehmann (1987).

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The purpose of this note is to provide an alternative test of conditional factor models, using completion portfolios. A conditional factor model is one in which the factor loadings are allowed to be time-varying functions of some conditioning information. A given set of factors is a true asset pricing model if and only if the unconditionally efficient frontier spanned by the factor-mimicking portfolios touches the efficient frontier spanned by the traded assets. One method of testing this is to measure the minimum difference in Sharpe ratios of these two frontiers. In the absence of a risk-free asset, we consider the 'generalized' Sharpe ratio relative to a given zero-beta rate, and find the rate for which the difference between asset and factor Sharpe ratios is minimized. The model prices all traded assets conditionally correctly if and only if this minimum difference is zero.

Our main result is to express the Sharpe ratio difference as the expectation of a particular completion portfolio, and thus express the test statistic as a simple moment condition. Moreover, even if the test is rejected, the completion portfolio can be interpreted as the 'missing factor' in the sense that adding it to the existing factors completes the model into a viable asset pricing model. Our analysis thus extends that of Shanken (1985) to the case with conditioning information, and also the results of Lehmann (1987).

1 Set-Up and Notation

Consider a financial market where trading takes place in discrete time and information flow is described by a filtration $(\mathcal{F}_t)_t$. Consider the period beginning at time $t-1$ and ending at time t. Denote by L_t^2 the space of square-integrable, $\mathcal{F}_t$-measurable random variables. There are n traded risky assets, indexed $k = 1 \ldots n$. We denote the gross return of the k-th asset by $r_t^k \in L_t^2$. Let X_t be the space of all elements $x_t \in L_t^2$ that can be written in the form,

$$x_t = \sum_{k=1}^{n} r_t^k \, \theta_{t-1}^k, \tag{1}$$

for $\mathcal{F}_{t-1}$-measurable functions θ_{t-1}^k. To simplify notation, we write (1) as $x_t = \tilde{R}_t' \, \theta_{t-1}$, where $\tilde{R}_t := (\, r_t^1 \ldots r_t^n \,)'$ the n-vector of asset returns. While L_t^2 describes the space of all (not necessarily attainable) state-contingent pay-offs, X_t is the space of pay-offs that are attainable by forming *managed* strategies in the base assets, with *weights* θ_{t-1}^k that are functions of the conditioning information $\mathcal{F}_{t-1}$.

By construction, the *price* of a strategy $x_t \in X_t$ is given by $\Pi_{t-1}(\, x_t \,) = e'\theta_{t-1}$, where e is an n-vector of 'ones'. Denote by $R_t = \Pi_{t-1}^{-1}\{\, 1 \,\}$ the set of *returns* in X_t, and by $Z_t = \Pi_{t-1}^{-1}\{\, 0 \,\}$ the space of *excess* (i.e. zero cost) returns.

Conditional Factor Models

To construct asset pricing models, we take as given a set of m *factors* $F_t^i \in L_t^2$, indexed $i = 1 \ldots m$. We say that the given set of factors gives rise to a viable asset pricing model if and only if there exist $\mathcal{F}_{t-1}$-measurable *factor loadings* a_{t-1} and b_{t-1}^i so that

$$m_t = a_{t-1} + \sum_{i=1}^{m} b_{t-1}^i F_t^i \tag{2}$$

is an admissible *stochastic discount factor* (SDF) in the sense that m_t prices all managed strategies correctly, $E_{t-1}(\, m_t x_t \,) = \Pi_{t-1}(\, x_t \,)$ for all $x_t \in X_t$. It is easy to show that necessary and sufficient for m_t to be an admissible SDF is $E(\, m_t r_t \,) = 1$ for all $r_t \in R_t$. In other words, we can use an *unconditional* pricing relation to test a *conditional* factor model. This fact is central to our analysis.

If the factors are not themselves (portfolios of) traded assets, we associate with each factor F_t^i its unique *factor-mimicking portfolio*[1] (FMP) $f_t^i \in R_t$.

In analogy to the preceding section, denote by X_t^F the space of all pay-offs that can be written in the form (1), with the traded assets r_t^k replaced by the FMPs f_t^i. Similarly, we define $R_t^F = X_t^F \cap R_t$ and $Z_t^F = X_t^F \cap Z_t$. For what follows, we will omit the time subscript and write r instead of r_t, etc.

2 Intersection and Completion Portfolios

The question is, when does a given set of factors F^i give rise to a viable asset pricing model? Basu and Stremme (2005) show that this is the case if and only if a managed portfolio of the FMPs exists that is unconditionally efficient in the augmented asset return space R. In other words, a given set of factors is a true asset pricing model if and only if the efficient frontiers spanned by managed portfolios of the traded assets and the FMPs, respectively, intersect. Motivated by this observation, we define the distance measure

$$\delta_*^2 = \inf_{\nu \in \mathbb{R}} \lambda_*^2(\nu) - \lambda_F^2(\nu), \qquad \text{where} \quad \lambda_*(\nu) = \sup_{r \in R} \frac{E(r) - 1/\nu}{\sigma(r)}, \tag{3}$$

and $\lambda_F(\nu)$ is defined analogously for R^F. In other words, $\lambda_*(\nu)$ and $\lambda_F(\nu)$ are the maximum Sharpe ratios in R and R^F, respectively, relative to the zero-beta rate ν. Because the frontier spanned by the FMPs is contained within the asset frontier, we always have $\delta_*^2 \geq 0$. Obviously, the two frontiers touch if and only if $\delta_*^2 = 0$.

Hansen and Richard (1987) show that the efficient frontier in the presence of conditioning information is spanned by the return $r^* \in R$ with minimal second moment, and a canonically chosen (orthogonal) excess return $z^* \in Z$.

[1] An explicit construction of the FMPs can be found in Basu and Stremme (2005). It can be shown that, if the model is indeed a true asset pricing model, the expression reduces to that derived by Ferson, Siegel, and Xu (2005).

We modify this construction and choose as benchmark $r^0 \in R$ the return with minimum variance[2] (GMV). In analogy with the Hansen and Richard (1987) construction, we choose the canonical excess return $z^0 \in Z$ as the Riesz representation of the expectation functional on Z with respect to the covariance inner product[3], i.e. $\mathrm{cov}(z^0, z) = E(z)$ for all $z \in Z$.

We denote by γ_1 and γ_2 the mean and variance of r^0, respectively, and set $\gamma_3 = E(z^0)$. Note that γ_1 and γ_2 describe the *location* of the efficient frontier in mean-standard deviation space, while its *curvature* is given by $1/\gamma_3$. Abhyankar, Basu, and Stremme (2005) show that the maximum zero-beta Sharpe ratio admits a decomposition of the form,

$$\lambda_*^2(\nu) = \lambda_0^2(\nu) + \gamma_3, \qquad \text{where} \quad \lambda_0^2(\nu) = \frac{(\gamma_1 - 1/\nu)^2}{\gamma_2} \tag{4}$$

Similarly, the frontier in R^F generated by the FMPs is spanned by corresponding elements $r_F^0 \in R^F$ and $z_F^0 \in Z^F$. We denote the corresponding moments by γ_1^F, γ_2^F and γ_3^F, respectively. Of course, a decomposition analogous to (4) also holds true for the factor Sharpe ratio $\lambda_F(\nu)$, with all quantities replaced by their respective counterparts in R^F.

Theorem 2.1 *The minimum Sharpe ratio difference can be expressed as,*

$$\delta_*^2 = \lambda_*^2(\nu^*) - \lambda_F^2(\nu^*) = (\gamma_3 - \gamma_3^F) - \frac{(\gamma_1 - \gamma_1^F)^2}{(\gamma_2^F - \gamma_2)} \tag{5}$$

This minimum is attained at the zero-beta rate $\nu^* = (\gamma_2 - \gamma_2^F)/(\gamma_1^F\gamma_2 - \gamma_1\gamma_2^F)$,

Note that, as the factor frontier is contained in the asset frontier, we always have $\gamma_2^F \geq \gamma_2$ (the factor GMV has higher variance than the asset GMV). Similarly, $\gamma_3^F \leq \gamma_3$ (the factor frontier has higher curvature). As a consequence, (5) is the difference of two positive terms.

Our main result can now be stated; its proof is immediate.

[2] Note that, by the first-order condition of the variance minimization, r^0 can also be characterized as the return that has zero covariance with the space Z of excess returns.

[3] In the absence of a risk-free asset (i.e. constant pay-off), the covariance functional is indeed positive definite and hence a well-defined inner product.

Theorem 2.2 *The Sharpe ratio distance can be expressed as* $\delta_*^2 = \sigma^2(\, z^\delta\,) = E(\, z^\delta\,)$, *where the* completion excess return $z^\delta \in Z$ *can be written as,*

$$z^\delta = [\, z^0 - z_F^0\,] - \frac{\gamma_1 - \gamma_1^F}{\gamma_2^F - \gamma_2} \cdot [\, r^0 - r_F^0\,] \tag{6}$$

Testing a conditional factor model thus reduces to simply testing whether the unconditional mean $E(\, z^\delta\,)$ is zero. Moreover, because z^δ has the property that its mean equals its variance, the corresponding T-test statistic reduces simply to the standard deviation of z^δ.

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