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Pricing Multivariate Currency Options with Copulas

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INTRODUCTION

Multivariate options are widely used when there is a need to hedge against a number of risks simultaneously; such as when there is an exposure to several currencies or the need to provide cover against an index such as the FTSE100, or indeed any portfolio of assets. In the case of a basket option the payoff depends on the value of the entire portfolio or basket of assets where the basket is some weighted average of the underlying assets. The principal reason for using basket options is that they are cheaper to use for portfolio insurance than a corresponding portfolio of plain vanilla options on the individual assets. This cost saving depends on the correlation structure between the assets; the lower the correlation between currency pairs in a currency portfolio for instance, the greater the cost saving.

However, the accurate pricing of basket options is a non-trivial task when, as is generally the case, there is no accurate analytic expression of the distribution of the weighted sum of the underlying assets in the basket. Apart from using Monte Carlo methods, basket options are often priced by assuming the basket or index is a single underlying and then applying standard option pricing theory based on the Black-Scholes (1973) framework. However, a weighted sum of log-normals is not itself log-normally distributed and potentially significant errors are introduced through this approximation by ignoring the distributional characteristics of the individual underlying assets and the nature of their dependencies beyond simple correlation. Recent surveys of pricing multiple contingent claims can be found for instance in Carmona and Durrleman (2003 and 2006).

In this paper we exploit recent developments in the use of copula methods by Hurd, Salmon and Schleicher (2005) to price multivariate currency options and in doing so we extend related approaches put forward in the limited literature in this area – for instance by Bennett and Kennedy (2004), Taylor and Wang (2005), Beneder and Baker (2005), van den Goorbergh, Genest and Werker (2005), and Cherubini and Luciano (2002). One property of copulas is that they split a complex task (modelling a joint-distribution) into two simpler tasks (modelling the margins and the dependence pattern). This property makes it substantially easier to construct multivariate distributions in general and hence to accurately price multivariate options as we demonstrate below.

In the next section we describe the approach we have taken to derive the prices for basket, spread and best of two options following the general procedure developed by Hurd, Salmon and Schleicher (2005). We first describe the theoretical argument for deriving the risk neutral measure consistent estimation of the implied joint density. Hurd et al. (2005) were unable to find suitable parametric copulas that closely fitted the data. We therefore use the Bernstein copula, which exhausts the space of all possible copula functions, as a general approximation procedure for copulas before turning to the application and drawing some conclusions.

THE METHODOLOGY

Our methodology builds on earlier unpublished work by Bikos (2000), who uses one-parameter copulas such as the Gaussian and the Frank copula to model the joint distribution of the dollar-sterling and euro-sterling exchange rates. The marginal distributions are given by univariate risk-neutral densities estimated using the Malz (1997) method and the parameter of the copula function is chosen in such a way that the empirical correlation coefficient (computed from the variances of the two bilateral exchange rates and the cross-rate) equals the implied correlation coefficient (computed from ATM volatilities). A very similar approach has been taken in a recent contribution by Taylor and Wang (2005), who also fit to the implied correlation coefficient, but use a more refined setup which ensures that the implied joint density belongs to a common risk-neutral numeraire measure. Both studies (Bikos, and Taylor and Wang) suggest that one-parameter copulas provide a reasonable fit to the data but essentially use one observation to fit a single parameter.[1]

A more general approach is taken by Bennett and Kennedy (2004), who use copulas in conjunction with a triangular no-arbitrage condition to price quanto FX options, i.e. FX options whose payout is in a third currency. Similar to Bikos and Taylor and Wang, they use option-implied densities as margins for the bivariate distribution. However, they estimate their copula function by fitting an entire set of option contracts in the third bilateral (over different strike prices) instead of fitting just the implied correlation coefficient. This additional information enables them to use a Gaussian copula which is perturbed by a cubic spline and which therefore allows for a more flexible dependence structure between the three currency pairs. In the context of the quanto pricing problem this approach is appealing because the perturbation function indicates the extent of departure from the standard Black Scholes model corresponding to a joint lognormal distribution.

Estimating copulas consistent with triangular no-arbitrage

We extend these previous methods by estimating a joint distribution that is consistent with the option-implied marginal distribution of the third bilateral *over its entire support*. In order to do this we proceed in the following steps:

Step 1 Let $S_t^{i,j}$ denote the price of one unit of currency j in terms of currency i at time t and $M_{t_1,t_2}^{i,j}$ the forward exchange rate at time t_1 with maturity at time $t_2 \geq t_1$. Next we define $z_{0,t,T}^{a,b}$, $z_{0,t,T}^{c,a}$, $z_{0,t,T}^{c,b}$ to be the logarithmic deviations of three triangular exchange rates $S_t^{a,b}$, $S_t^{c,a}$, $S_t^{c,b}$ from their respective forward rates $M_{0,T}^{a,b}$, $M_{0,T}^{c,a}$, $M_{0,T}^{c,b}$, i.e.

$$z_{0,t,T}^{i,j} \equiv \log S_t^{i,j} - \log M_{0,T}^{i,j} = \log \frac{S_t^{i,j}}{M_{0,T}^{i,j}}. \qquad (1.1)$$

For ease of notation we will usually write $z^{i,j}$ instead of $z_{0,t,T}^{i,j}$, unless the time-subscripts are necessary to avoid ambiguity. Hurd et al. (2005) show that at any time $t \leq T$ the relationship between the univariate PDF of $z^{a,b}$ under the risk-neutral measure $\mathbb{Q}_a$[2] and the bivariate PDF of $z^{c,a}$ and $z^{c,b}$ under the risk-neutral measure $\mathbb{Q}_c$

is given by

$$f^{\mathbb{Q}_a}_{z^{a,b}}(s) = \int_{-\infty}^{\infty} f^{\mathbb{Q}_c}_{z^{c,a},z^{c,b}}(u, u+s)e^u du. \tag{1.2}$$

The additional term e^u is necessary, because the left hand side and the right hand side of equation (1.2) are expressed under different measures. Note also that triangular arbitrage implies that

$$z^{a,b} = z^{c,b} - z^{c,a}. \tag{1.3}$$

Step 2 By Sklar's theorem there exists a copula $C(\cdot)$ with density $c(\cdot)$ which allows us to write the bivariate distribution of $z^{c,a}_T$ and $z^{c,b}_T$ in its canonical representation

$$f^{\mathbb{Q}_c}_{z^{c,a},z^{c,b}}(u, v) = c\left(F^{\mathbb{Q}_c}_{z^{c,a}}(u), F^{\mathbb{Q}_c}_{z^{c,b}}(v)\right) f^{\mathbb{Q}_c}_{z^{c,a}}(u) f^{\mathbb{Q}_c}_{z^{c,b}}(v). \tag{1.4}$$

Step 3 We then estimate a parametric representation, $\hat{c}(\cdot;\hat{\theta})$, of the copula density by minimizing the L^2-distance between the option-implied third bilateral $f^{\mathbb{Q}_a}_{z^{a,b}}$ and its copula-implied counterpart $\hat{f}^{\mathbb{Q}_a}_{z^{a,b}}(\cdot;\hat{\theta})$, where

$$\hat{\theta} = \text{arginf}_{\theta}\left[\int_{-\infty}^{\infty}\left(f^{\mathbb{Q}_a}_{z^{a,b}}(s) - \hat{f}^{\mathbb{Q}_a}_{z^{a,b}}(s, \hat{\theta}(s;\theta))\right)^2 ds\right]^{\frac{1}{2}}, \tag{1.5}$$

and

$$\hat{f}^{\mathbb{Q}_a}_{z^{a,b}}(s;\hat{\theta}) = \int_{-\infty}^{\infty} \hat{c}\left(F^{\mathbb{Q}_c}_{z^{c,a}}(u), F^{\mathbb{Q}_c}_{z^{c,b}}(u+s);\hat{\theta}\right) f^{\mathbb{Q}_c}_{z^{c,a}}(u) f^{\mathbb{Q}_c}_{z^{c,b}}(u+s)e^u du \tag{1.6}$$

is the distribution of the third bilateral implied by the estimated parameters $\hat{\theta}$.

The Bernstein copula

The underlying idea of the Bernstein copula is to define a function $\alpha(\omega)$ on a set of grid points and then use a polynomial expansion to extend the function to all points in the unit square. In our application we use an evenly spaced grid of $(m+1)^2$ points, $\omega = \frac{k}{m} \times \frac{l}{m}$, $k, l = 0, ..., m$. The bivariate Bernstein copula or Bernstein(m) copula is then defined as

$$C^B(u, v) = \sum_{k=0}^{m}\sum_{l=0}^{m} \alpha\left(\frac{k}{m}, \frac{l}{m}\right) P_{k,m}(u) P_{l,m}(v), \tag{1.7}$$

where

$$P_{j,m}(x) = \begin{pmatrix} m \\ j \end{pmatrix} x^j (1-x)^{m-j}$$

is the j-th Bernstein polynomial of order m (for $j = 0, ..., m$). Sancetta and Satchell (2004) show that this function will be a copula as long as $\alpha(\omega)$ satisfies the basic three conditions of a copula (grounded, consistent with margins and two increasing[3]) for all points on the grid.

Similarly, the density of the bivariate Bernstein copula is given by

$$c^B(u,v) = m^2 \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \beta\left(\frac{k}{m}, \frac{l}{m}\right) P_{k,m-1}(u) P_{l,m-1}(v), \tag{1.8}$$

where

$$\begin{aligned} \beta\left(\frac{k}{m}, \frac{l}{m}\right) &= \alpha\left(\frac{k+1}{m}, \frac{l+1}{m}\right) - \alpha\left(\frac{k+1}{m}, \frac{l}{m}\right) \\ &\quad -\alpha\left(\frac{k}{m}, \frac{l+1}{m}\right) + \alpha\left(\frac{k}{m}, \frac{l}{m}\right). \end{aligned}$$

Note that the two-increasing property of α ensures that the density is non-negative.

The Bernstein copula allows us to compute the third marginal distribution in equation (1.2) as a linear combination of basis functions

$$\begin{aligned} f^{\mathbb{Q}_a}_{z^{a,b}}(s;\theta) &= \int_{-\infty}^{\infty} c\left(F^{\mathbb{Q}_c}_{z^{c,a}}(u), F^{\mathbb{Q}_c}_{z^{c,b}}(u+s); \theta\right) \\ &\quad \times f^{\mathbb{Q}_c}_{z^{c,a}}(u) f^{\mathbb{Q}_c}_{z^{c,b}}(u+s) e^u du \\ &= \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \theta_{k,l} \psi_{k,l}(s), \end{aligned} \tag{1.9}$$

where $\theta_{k,l} = \beta\left(\frac{k}{m}, \frac{l}{m}\right)$ and

$$\begin{aligned} \psi_{k,l}(s) &= m^2 \int_{-\infty}^{\infty} P_{k,m-1}\left(F^{\mathbb{Q}_c}_{z^{c,a}}(u)\right) P_{l,m-1}\left(F^{\mathbb{Q}_c}_{z^{c,b}}(u+s)\right) \\ &\quad \times f^{\mathbb{Q}_c}_{z_T^{c,a}}(u) f^{\mathbb{Q}_c}_{z^{c,b}}(u+s) e^u du. \end{aligned} \tag{1.10}$$

These basis functions have the property that $\psi_{k,l}(\cdot) \geq 0$ and $\int_{-\infty}^{\infty} \psi_{k,l}(s) ds = 1$, for all $k, l = 0, ..., m-1$.

Due to the properties of α, the coefficients $\theta_{k,l}$ satisfy the following restrictions

$$\begin{aligned} \theta_{k,l} &\geq 0, \quad k,l=0,...,m-1, & (1.11)\\ \sum_{k=0}^{m-1}\theta_{k,l} &= \frac{1}{m}, \quad l=0,...,m-1, \quad \text{and} & (1.12)\\ \sum_{l=0}^{m-1}\theta_{k,l} &= \frac{1}{m}, \quad k=0,...,m-1. & (1.13)\end{aligned}$$

These restrictions also imply that the sum of all coefficients equals unity.

The optimization problem (1.5) can be restated as

$$\begin{gathered} \inf_{\{\theta_{k,l}\}_{k,l=0}^{m-1}} \int_{-\infty}^{\infty}\left(\sum_{k=0}^{m-1}\sum_{l=0}^{m-1}\theta_{k,l}\psi_{k,l}(s)-f_{z^{a,b}}^{\mathbb{Q}_a}(s;\theta)\right)^2 ds \\ \text{subject to restrictions on } \{\theta_{k,l}\}_{k,l=0}^{m-1}, \end{gathered} \quad (1.14)$$

which can be simplified to

$$\inf_\theta \ \theta'\mathbf{H}\theta-2\mathbf{g}\theta, \qquad \text{subject to } \ \mathbf{R}_1\theta\leq\mathbf{q}_1,\ \ \mathbf{R}_2\theta=\mathbf{q}_2, \quad (1.15)$$

where

$$\mathbf{H}=\int_{-\infty}^{\infty}\psi(s)\psi'(s)ds, \qquad \mathbf{g}=\int_{-\infty}^{\infty}f_z^{\mathbb{Q}_a}(s)\psi'(s)ds,$$

$$\begin{aligned} \theta &= [\theta_{0,0},...,\theta_{0,m-1},\theta_{1,0},...,\theta_{1,m-1},...,\theta_{m-1,0},...,\theta_{m-1,m-1}]', \\ \psi(s) &= [\psi_{0,0}(s),...,\psi_{0,m-1}(s),\psi_{1,0}(s),...,\psi_{1,m-1}(s),..., \\ &\qquad \psi_{m-1,0}(s),...,\psi_{m-1,m-1}(s)]', \end{aligned}$$

and the matrices $\mathbf{R}_j$ and vectors $\mathbf{q}_j$ impose the equality $(j=1)$ and inequality $(j=2)$ constraints (1.11) to (1.13). Expression (1.15) is a standard quadratic programming problem that can be solved using a Lagrangian approach (see e.g. Greene (1993)).

PRICING MULTIVARIATE CURRENCY OPTIONS

Our empirical examples focus on options that depend on the relative performance of different currencies and for this purpose we define the gross-return of a currency as the ratio of the spot rate over the forward-rate fixed at some time 0:

$$Z_{0,t,T}^{a,b} \equiv e^{z_{0,t,T}^{a,b}} = \frac{S_t^{a,b}}{M_{0,T}^{a,b}}. \tag{1.16}$$

With some abuse of notation we abbreviate this as $Z_t^{a,b}$. We then consider call options with strike price K and European exercise with payout $G(Z_T^{c,a}, Z_T^{c,b}, K)$ denominated in currency c. We consider three different options, given by the following payoff profiles:

$$G_1(Z_T^{c,a}, Z_T^{c,b}, K) = \max\left\{(Z_T^{c,a})^{\omega_a}(Z_T^{c,b})^{\omega_b} - K, 0\right\} \tag{1.17}$$

$$G_2(Z_T^{c,a}, Z_T^{c,b}, K) = \max\left\{\omega_a Z_T^{c,a} + \omega_b Z_T^{c,b} - K, 0\right\} \tag{1.18}$$

$$G_3(Z_T^{c,a}, Z_T^{c,b}, K) = \max\left\{\max\left(Z_T^{c,a}, Z_T^{c,b}\right) - K, 0\right\} \tag{1.19}$$

The first ($G_1(\cdot)$) represents an option on a geometric index. When $(\omega_a, \omega_b) = (1, -1)$ it becomes an option on a ratio. The second ($G_2(\cdot)$) corresponds to basket options which include the spread option ($(\omega_a, \omega_b) = (1, -1)$) as a special case. Finally, $G_3(\cdot)$ is the payoff of a best-of-two-assets option.

Under the assumption of a non-stochastic discount rate for currency c, any of these options can be valued using the Feynman-Kaç formula

$$V_0 = e^{-r_c T} \int_0^\infty \int_0^\infty G(u, v) f^{\mathbb{Q}_c}_{Z_T^{c,a}, Z_T^{c,b}}(u, v) du dv. \tag{1.20}$$

The bivariate returns distribution $f^{\mathbb{Q}_c}_{Z_T^{c,a}, Z_T^{c,b}}$ can be recovered from $f^{\mathbb{Q}_c}_{z_T^{c,a}, z_T^{c,b}}$ (equation (1.2)) by using the same copula and transforming the margins as

$$f^{\mathbb{Q}_c}_{Z_T^{c,a}}(s) = f^{\mathbb{Q}_c}_{z_T^{c,a}}(e^s)e^s. \tag{1.21}$$

Estimating the margins and the copula

For our empirical examples we use over-the-counter (OTC) quotes from the 13th of January 2006 provided by a major market maker. These data are described in Table 1 and contain at-the-money (ATM) contracts as well as 25 and 10 delta risk-reversals and butterflies for the three bilateral currencies JPY/EUR, JPY/USD and USD/EUR. The table also includes the discount rates for the three currencies. A positive sign on the risk-reversal indicates that the base currency is favored.

Table 1 One-month contracts for January 13, 2006.

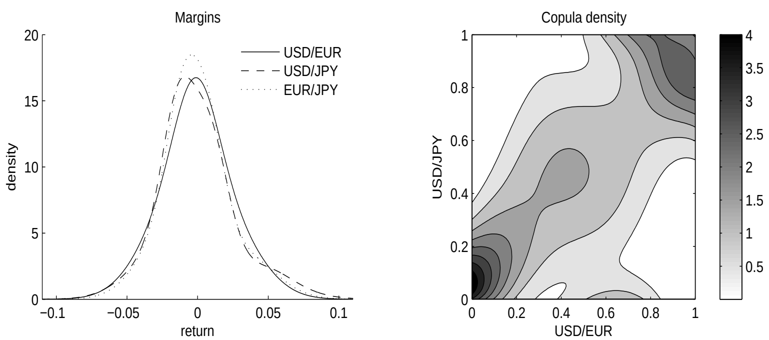
	JPY/EUR	**JPY/USD**	**USD/EUR**
ATM	9.30	9.15	8.95
25D RR	-0.70	-1.05	0.18
10D RR	-1.20	-1.75	0.28
25D Fly	0.20	0.20	0.15
10D Fly	0.65	0.80	0.40
	EUR	**JPY**	**USD**
discount rate	2.4811	0.0506	4.6171

Our method is independent of the way in which the margins are estimated. For example, we could use a mixture of log-normals (as in Bennett and Kennedy (2004), and Taylor and Wang (2005)) or the smoothing spline-method of Bliss and Panigirtzoglou (2002). Here we follow Hurd et al. (2005) and use an extension of Malz's (1997) smile interpolation method which is specifically tailored to the FX OTC market. Malz models the volatility smile as a function of delta by fitting a quadratic function to the three most liquid contracts (the ATM and 25 delta risk-reversal and butterfly). We include the additional two 10 delta contracts, which are liquid for major bilaterals at short horizons, by fitting a spline consisting of two cubics (in the intervals between 0.1 and 0.25 and 0.75 and 0.9) and a quartic (in the interval between 0.25 and 0.75). We impose the restriction that the first three derivatives are continuous. The marginal distributions are then obtained easily by converting the smile into the call-price function and taking the second derivative with respect to the strike price (Breeden and Litzenberger, 1978).

The left panel of Figure 1 shows the three margins $f^{\mathbb{Q}_{USD}}_{z^{USD,EUR}}$, $f^{\mathbb{Q}_{JPY}}_{z^{USD,JPY}}$, and $f^{\mathbb{Q}_{EUR}}_{z^{EUR,JPY}}$.[4] The width of the three distributions is very similar, however, the two yen-bilaterals are more lepotkurtic and exhibit a marked skew towards yen appreciation. This is a reflection of the larger (absolute) value of yen-butterflies and risk-reversals.

We then apply the method described in the previous section to link the two dollar-bilaterals using a Bernstein copula. We find that we need at least an order of $m = 11$ for the Bernstein expansion to obtain a good fit for the EUR/JPY margin. The estimated Bernstein(11) copula is shown in the right-hand panel of Figure 1. It clearly exhibits the characteristics of positive dependence in the sense that most probability mass is concentrated near the (0,0) and

Figure 1 Marginal distributions of currency returns (left panel) and the estimated Bernstein(11) copula density (right panel).



(1,1) corners. However, there is a notable degree of asymmetry: First, large appreciations of the dollar against the euro and the yen are more likely to occur than large depreciations. Second, there is a third local peak of the density near (0.65,0) corresponding to a situation where the dollar appreciates strongly against the yen but moves little against the euro.

Options on geometric indexes: smiles and frowns

We first look at options on a geometric index (payoff function $G_1(\cdot)$), because a simple modification of the standard Black (1976) formula exists for this particular payoff.[5] The Black-model is based on the assumption of joint (log)normality and takes as an input only the three (ATM) volatilities $\sigma_{c,a}$, $\sigma_{c,b}$, and $\sigma_{a,b}$. In Figure 2 we compare the familiar oval-shaped normal density assumed by the Black-model with the bivariate distribution of the option-implied margins linked by the Bernstein(11) copula. The distributions are drawn such that each line represents a decile. Both distributions clearly represent random variables with overall positive association, but the copula-based density differs in several aspects:

1. It has less probability mass in the center of the distribution.
2. There is little indication of positive association for small movements – the contour of the first decile is roughly circular, while that of the normal distribution is oval-shaped.

Figure 2 Multivariate densities corresponding to the Black model (a), the Bernstein copula model (b), and their difference (c).

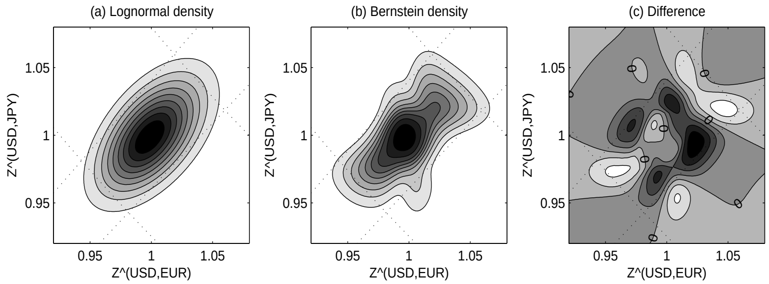
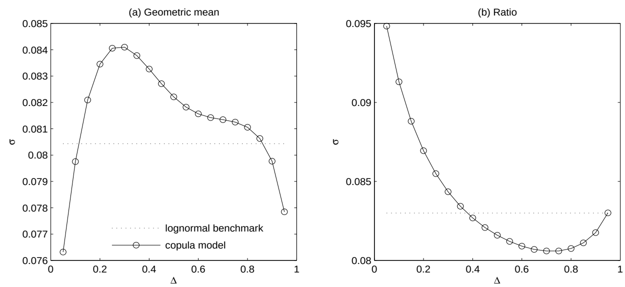


Figure 3 Smiles of an index option (weights $\omega_a = \omega_b = 0.5$) and a ratio option (weights $\omega_a = 1$, $\omega_b = -1$).



3. The copula-based density gives more probability to events in which either the euro or the yen can undergo large movements versus the dollar but changes little against the other currency.

We then use numerical evaluation of the Feynman-Kaç formula to obtain the prices of an index option with weights $\omega_a = \omega_b = 0.5$ over a range of strikes. We compare these prices to the standard model by computing the Black-model implied volatilities which are shown in the first panel of Figure 3. We find that for most strikes, except those with deltas close to 0 and 1, the copula-based model predicts a higher payoff than the Black-model. Options with strikes far from the current level of the index are relatively cheap, however, leading to an implied-volatility “frown”. To understand the cause of this inverted smile we superimpose the loci corresponding to 5 and 95 delta contracts on the bivariate densities in Figure 2 (downward-sloping dotted lines). We see that the integration regions for 5 delta puts (bottom line) and 5 delta calls (top line) both fall outside the areas where the Bernstein density has higher mass than the bivariate normal.

We then look at prices for an index option with weights $\omega_a = 1$ and $\omega_b = -1$, which corresponds to a ratio of cross returns. Here the implied volatility smile has a more usual convex shape (right panel in Figure 3) and for deltas larger than 0.35 the copula model yields lower option prices than the log-normal model. The loci of the 5 and 95 delta contracts are represented by the upward-sloping dotted lines in Figure 2. For put options that are out-of-the money or near-the-money, the Bernstein-distribution has lower probability mass over the integration region (north-west of the strike). For out-of-the-money calls, on the other hand, the integration region includes the protuberance around the (1.1,0.95) outcome, and they are therefore relatively expensive compared to the Black-model.

Baskets, spreads and best-of-two-assets

Next we check whether our results for options with geometric payoff (G_1) also hold for the more common basket and spread options (G_2). In Table 2 we compare the prices of the copula model and the Black model for out-of-the-money (OTM), near-the-money (NTM) and in-the-money (ITM) calls. We find that options based on the arithmetic payoff follow a very similar pattern to those based on a geometric payoff, in the sense that the differences between the prices implied by the copula model and the log-normal benchmark always have the same sign. In general, the magnitude of the difference tends to be larger for baskets and spreads, indicating that smile effects are more pronounced. The only exception is the OTM spread call, for which the two models yield a very similar price (in contrast to the ratio option).

Finally we briefly look at best-of-two-asset options (payoff G_3). We find, similar to the case of ratios and spreads, that the ITM and NTM contracts are over-priced by the Black-model, while the OTM contract is underpriced.

Table 2 Option prices.

	Strike	**Black-model**	**Copula-model**	Difference
Index	0.98	2.2293	2.2339	-0.0046
(G_1, $w_a = w_b = 0.5$)	1.00	0.9191	0.9393	-0.0202
	1.02	0.2541	0.2785	-0.0244
Basket	0.98	2.2287	2.2395	-0.0108
(G_2, $w_a = w_b = 0.5$)	1.00	0.9132	0.9430	-0.0298
	1.02	0.2489	0.2807	-0.0318
Ratio	0.98	2.2796	2.2623	0.0173
(G_1, $w_a = 1, w_b = -1$)	1.00	0.9674	0.9505	0.0169
	1.02	0.2828	0.3132	-0.0304
Spread	-0.02	2.2880	2.2458	0.0422
(G_2, $w_a = 1, w_b = -1$)	0.00	0.9878	0.9352	0.0526
	0.02	0.2950	0.2996	-0.0046
Best-of-two-assets	0.98	3.1001	3.0465	0.0536
(G_3)	1.00	1.5365	1.5144	0.0221
	1.02	0.5556	0.5985	-0.0429

CONCLUSIONS

In this chapter we have presented a methodology for computing prices for bivariate currency options that are consistent with the observed quotes of univariate instruments on three triangular bilateral exchange rates. We first establish a relationship between the bivariate distribution of the two bilateral exchange rates involving the payout currency and the univariate distribution of the cross-rate. We then express this relationship, which constitutes a no-arbitrage condition, in terms of three option-implied margins and a Bernstein copula. The Bernstein copula has the important feature that it exhausts the space of all possible copula functions. We estimate the "copula-parameters" by minimizing the L^2-distance between the option-implied distribution of the cross-rate and the distribution implied by the copula. We then apply the bivariate Feynman-Kaç formula to compute the price of options with particular payoff functions corresponding to basket, spread and best of two options.

Compared to other copula-based approaches our method has the advantage that it uses *all* available information from the univariate contracts. The method is also flexible in the sense that it works independently of the way in which the margins are computed. Since the Bernstein copula may assume the shape corresponding to any possible dependence function, a failure to find a good fit to the third distribution implies that the three margins violate triangular no-arbitrage in terms of higher moments.[6]

1 Rosenberg (2003) follows a different route by using a nonparametric method and a copula which is estimated from *historical* exchange rate movements.

2 More precisely the risk-neutral measure $\mathbb{Q}_j$ is the equivalent martingale measure associated with a discount bond in currency j.

3 See Schmidt (2006) for details.

4 In the notation used so far, we have USD $= c$, EUR $= a$, and JPY $= b$.

5 By simple application of Itô's lemma to the bivariate geometric Brownian motion $[dZ_t^{c,a}, dZ_t^{c,b}]'$ the Black-price for an option on a geometric index is given by

$$V_0^{BS}(M_{0,T}^I, K, \sigma_I, T) = e^{-rc}\left(M_{0,T}^I \Phi(d_1) - K\Phi(d_2)\right), \tag{1.22}$$

where M^I and σ_I are the forward price and the volatility of the index

$$\begin{aligned} M^I &= \exp(0.5(\omega_a(\omega_a - 1)\sigma_{c,a}^2 + \omega_b(\omega_b - 1)\sigma_{c,b}^2 + \omega_a\omega_b(\sigma_{c,a}^2 + \sigma_{c,b}^2 - \sigma_{a,b}^2))), \\ \sigma_I &= \omega_a^2\sigma_{c,a}^2 + \omega_b^2\sigma_{c,b}^2 + \omega_a\omega_b(\sigma_{c,a}^2 + \sigma_{c,b}^2 - \sigma_{a,b}^2), \end{aligned}$$

d_1 and d_2 are defined as usual as

$$d_1 = \frac{\log \frac{M_{0,T}^I}{K} - 0.5\sigma_I^2 T}{\sigma_I\sqrt{T}}, \qquad d_2 = d_1 - \sigma_I\sqrt{T},$$

and $\sigma_{i,j}$ is the volatility of currency pair $S^{i,j}$.

6 A simple example is the case where the three margins are log-normally distributed and the implied volatilities violate the Schwarz-inequality:

$$|\sigma_{a,b}^2 - \sigma_{c,a}^2 - \sigma_{c,b}^2| > 2\sigma_{c,a}\sigma_{c,b}.$$

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