Sequences and series

Section 2: Arithmetic sequences and series

Notes and Examples

In this section you learn about arithmetic sequences and series (sometimes called arithmetic progressions or A.P.s), which were introduced in the last section.

These notes contain subsections on

- **Formulae for arithmetic series**
- **Worked examples**
- **Harder examples**

Formulae for arithmetic series

All you need to know to answer a question on arithmetic sequence are the two general formulae for any arithmetic sequence:

- the formula for a_k, the kth term of the sequence

$$a_k = a + (k-1)d$$ *Proof of this formula*

- the formula for S_n, the sum of the first n terms of the sequence

$$S_n = \tfrac{1}{2}n[2a + (n-1)d]$$ *Proof of this formula*

where a is the first term of the sequence and d is the common difference (the difference between successive terms).

S_n, the sum of the first n terms of the sequence, can also be calculated as

$$S_n = \tfrac{1}{2}n[\text{first term} + \text{last term}]$$ *Proof of this formula*

To solve a problem, all you need to do is to substitute the information given in the question into the appropriate formula, and solve the resulting equation. Sometimes this may involve solving a quadratic equation or simultaneous equations.

The **Arithmetic series spreadsheet** shows you graphs of the terms of arithmetic series and the sum of n terms. Try varying the values of a and d, looking in particular at what happens when d is negative.

Worked examples

Example 1 shows a straightforward application of the formulae.

Example 1

An arithmetic sequence which has 16 terms starts 2, 5, 8 …

(i) Find the last term.

(ii) Find the sum of the terms of the sequence.

Solution

(i) The first term, a, is 2 and the common difference, d, is 3.

Use the formula $a_k = a + (k-1)d$

To find the 16^{th} term, substitute $k = 16$, $a = 2$ and $d = 3$

$$16^{th}\text{ term} = 2 + (16 - 1)\times 2$$
$$= 2 + 15\times 2$$
$$= 30$$

(ii) Use the formula $S_n = \frac{1}{2}n[2a + (n-1)d]$.

Substitute as before:

$$\text{Sum} = \tfrac{1}{2}\times 16[2\times 2 + (16-1)\times 3]$$
$$= 8[4 + 15\times 3]$$
$$= 8\times 49$$
$$= 392$$

To see further examples, use the Flash resources ***n*th terms of an AP** and **Sum of an AP**.

For practice in finding a particular term of an arithmetic series, as in Example 1(i), try the interactive questions **Finding terms in arithmetic series**.

For practice in finding the sum of an arithmetic series, as in Example 1(ii), try the interactive questions **Finding the sum of an arithmetic series**.

Example 2

An arithmetic series has first term 3 and the sum of the first 20 terms is 288.

Find the common difference.

Solution

$$S_n = \tfrac{1}{2}n\left[2a + (n-1)d\right]$$
$$288 = \tfrac{1}{2}\times 20\left[(2\times 3) + 19d\right]$$
$$= 10\left[6 + 19d\right]$$
$$= 60 + 190d$$
$$190d = 228$$
$$d = 1.2$$

Substituting $n = 20$, $a = 3$, and $S_n = 288$

For practice in questions like the one above, try the interactive questions
Finding the common difference in an A.P.

Example 3
An arithmetic series has common difference -0.5 and the sum of the first 25 terms is 350.
Find the first term.

Solution

$$S_n = \tfrac{1}{2}n\left[2a + (n-1)d\right]$$
$$350 = \tfrac{1}{2} \times 25\left[2a + (24 \times -0.5)\right]$$
$$= \tfrac{1}{2} \times 25\left[2a - 12\right]$$
$$2a - 12 = 28$$
$$2a = 40$$
$$a = 20$$

Substituting $n = 25$, $d = -0.5$, and $S_n = 350$

For practice in questions like the one above, try the interactive questions
Finding the first term of an A.P.

Harder examples

Example 4 involves solving simultaneous equations.

Example 4
The 5th term of an arithmetic sequence is 24 and the 9th term is 4.
(i) Find the first term and the common difference.
(ii) The last term of the sequence is –36. How many terms are in the sequence?

Solution
(i) Using the formula $a_k = a + (k-1)d$

For the 5th term: $24 = a + (5 - 1)d$
$24 = a + 4d$ ①
For the 9th term: $4 = a + (9 - 1)d$
$4 = a + 8d$ ②

$24 = a + 4d$ ①
$4 = a + 8d$ ②
Subtracting: $20 = -4d$
$d = -5$

As the 9th term is smaller than the 5th term, you would expect the common difference to be negative.

Substituting $d = -5$ into either equation gives $a = 44$.

(ii) Substituting a = 44 and d = -5 into the formula $a_k = a + (k-1)d$:

For the last term: $-36 = 44 + (k - 1)\times -5$

$-80 = -5(k - 1)$

$16 = k - 1$

$k = 17$

There are 17 terms.

In the next example, you are given the sum of n terms and you need to find the value of n. You need to solve a quadratic equation to find n.

Example 5

The sum of the terms of an arithmetic sequence with first term 5 and common difference 6 is 658. How many terms are there in the sequence?

Solution

Substituting a = 5 and d = 5 into the formula $S_n = \frac{1}{2}n[2a + (n-1)d]$ gives

$$658 = \tfrac{1}{2}n[2\times 10 + (n-1)\times 6]$$

$$658 = \tfrac{1}{2}n(20 + 6n - 6)$$

$$658 = \tfrac{1}{2}n(14 + 6n)$$

$$658 = 7n + 3n^2$$

$$3n^2 + 7n - 658 = 0$$

$$(3n + 47)(n - 14) = 0$$

Since n must be positive, n = 14

The sequence has 14 terms.

For practice in questions like the one above, try the interactive questions **Finding the number of terms in an A.P.**