Non-random overshoots of Lévy processes and a fluctuation result for random walks

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Lévy processes and fluctuation theory

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Definition (Lévy process)

A continuous-time $\mathbb{F}$-adapted stochastic process X with state space $\mathbb{R}$ is a *Lévy process* on the stochastic basis $(\mathbb{F}, \mathsf{P})$, if it starts at 0 a.s.-P, is continuous in probability, $X_{t-s} \sim X_t - X_s \perp \mathcal{F}_s$ (stationary independent increments) and is càdlàg off a P-null set.

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- Characterized by the Lévy triplet (σ^2, λ, μ), which features in the characteristic exponent $\mathsf{E}[e^{ipX_t}] = e^{t\Psi(p)}$ $(p \in \mathbb{R})$. Example: compound Poisson processes.
- Fluctuation theory: studies first passage times, supremum/infimum processes, excursions from the maximum etc.
- Important results: Wiener-Hopf factorization, two-sided exit problem.

(cont.)

Definition (First passage times)

For $x \in \mathbb{R}$: $T_x := \inf\{t \geq 0 : X_t \geq x\}$ (resp. $\hat{T}_x := \inf\{t \geq 0 : X_t > x\}$) is the *first entrance time* of X to $[x, \infty)$ (resp. (x, ∞)).

- Overshoots ($x \geq 0$): $R_x := X(\hat{T}_x) - x$ on $\{\hat{T}_x < \infty\}$.
- Miscellaneous: $\mathbb{Z}_h := h\mathbb{Z}$.

Definition (Spectrally negative Lévy process)

A Lévy process is called *spectrally negative* if it has no positive jumps a.s.-P and does not have monotone paths.

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- Answer: yes and we can characterize precisely the class of Lévy processes for which this is true.
- Loosely speaking: for the overshoots of a Lévy process to be (conditionally on the process going above the level in question) almost surely constant quantities, it is both necessary and sufficient that *either* the process has no positive jumps (a.s.) *or* for some $h > 0$, it is compound Poisson, living on the lattice $\mathbb{Z}_h := h\mathbb{Z}$, and can only jump up by h.

Formal result

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Definition (Upwards-skip-free Lévy chain)

A Lévy process X is an *upwards-skip-free Lévy chain* if it is a compound Poisson process, and for some $h > 0$, $\mathrm{supp}(\lambda) \subset \mathbb{Z}_h$ and $\mathrm{supp}(\lambda|_{\mathcal{B}((0,\infty))}) = \{h\}$.

(Discrete time right-continuous random walk embedded into continuous time as a Lévy process.)

Definition (P-triviality)

A random variable R is said to be P-*trivial* on an event $A \in \mathcal{F}$ if there exists $r \in \mathbb{R}$ such that $R = r$ a.s.-P on A. R may only be defined on some $B \supset A$.

(i.e. R is a.s.-P constant conditionally on A.)

(cont.)

Theorem (Non-random position at first passage time)

The following are equivalent:

(a) *For some* $x > 0$, $X(T_x)$ *is* P*-trivial on* $\{T_x < \infty\}$.

(b) *For all* $x \in \mathbb{R}$, $X(T_x)$ *is* P*-trivial on* $\{T_x < \infty\}$.

(c) *For some* $x \geq 0$, $X(\hat{T}_x)$ *is* P*-trivial on* $\{\hat{T}_x < \infty\}$ *and a.s.-*P *positive thereon (in particular the latter obtains if* $x > 0$*).*

(d) *For all* $x \in \mathbb{R}$, $X(\hat{T}_x)$ *is* P*-trivial on* $\{\hat{T}_x < \infty\}$.

(e) *Either* X *has no positive jumps, a.s.-*P *or* X *is an upwards-skip-free Lévy chain.*

If so, then outside a P*-negligible set, for each* $x \in \mathbb{R}$, $X(T_x)$ *(resp.* $X(\hat{T}_x)$*) is constant on* $\{T_x < \infty\}$ *(resp.* $\{\hat{T}_x < \infty\}$*), i.e. the exceptional set in (b) (resp.* (d)*) can be chosen not to depend on* x.

Idea of proof

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1. First observe the following:

Lemma (Continuity of the running supremum)

The supremum process $\overline{X}$ is continuous (P-a.s.) iff X has no positive jumps (P-a.s). In particular, if $X(T_x) = x$ a.s.-P on $\{T_x < \infty\}$ for each $x > 0$, then X has no positive jumps, a.s.-P.

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2. Then show:

Proposition (P-triviality of $X(T_x)$)

$X(T_x)$ on $\{T_x < \infty\}$ is a P-trivial random variable for each $x > 0$ iff either one of the following mutually exclusive conditions obtains:

(a) *X has no positive jumps (P-a.s.) (equivalently: $\lambda((0, \infty)) = 0$).*

(b) *X is compound Poisson and for some $h > 0$, $\mathrm{supp}(\lambda) \subset \mathbb{Z}_h$ and $\mathrm{supp}(\lambda|_{\mathcal{B}((0,\infty))}) = \{h\}$.*

by appealing to lemma in order to get (a) and then treating separately the CP case;

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3. Generalize from above proposition to the full setting of the theorem via the Strong Markov property, namely (introducing $Q^x := X(T_x)_\star P(\cdot \cap \{T_x < \infty\})$, the (possibly subprobability) law of $X(T_x)$ on $\{T_x < \infty\}$ under P on the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$)
   1. establish the intuitively appealing identity $Q^b(A) = \int dQ^c(x_c) Q^{b-x_c}(A - x_c)$ for Borel sets A and $c \in (0, b)$ (where Q^c must be completed).
   2. use this to show that $\mathcal{A} := \{x \in \mathbb{R} : Q^x \text{ is a weighted (possibly by 0) } \delta\text{-distribution}\}$ is dense in the reals, whenever $\mathcal{A} \cap (0, \infty) \neq \emptyset$.
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Note. With reference to our original motivation: a full fluctuation theory for upwards-skip-free Lévy chains can be developed with results which are essentially (but not entirely) analogous to the spectrally negative case.

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$$\mathcal{L}(\beta) := \sum_{i\in\mathbb{Z}} p_i \beta^{-i}$$
for $\beta \in [-1,1]\backslash\{0\}$ and observe that $\mathcal{L}(1) = 1$ and $\mathcal{L}$ is strictly convex on restriction to $(0,1]$. Moreover, $\lim_{0+} \mathcal{L} = +\infty$. Other than 1 there is hence at most one root of $\mathcal{L} - 1$ on $(0,1]$, we denote it by $\alpha(1)$. It is clear that $\mathcal{L}|_{(0,\alpha(1)]} : (0,\alpha(1)] \to [1,\infty)$ is a decreasing bijection, and we let α be its inverse.

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- Introduce $T_n := \inf\{k \geq 0 : W_k \geq n\}$.

(cont.)

(a) For any $\gamma \geq 1$ with $\alpha(\gamma) < 1$, we have for all $n \in \mathbb{N} \cup \{0\}$:

$$\mathrm{E}[\gamma^{-T_n} \mathbb{1}(T_n < \infty)] = \frac{1 - \tilde{\lambda}_-(\gamma)}{\tilde{\lambda}_+(\gamma) - \tilde{\lambda}_-(\gamma)} \tilde{\lambda}_+(\gamma)^{n+1} - \frac{1 - \tilde{\lambda}_+(\gamma)}{\tilde{\lambda}_+(\gamma) - \tilde{\lambda}_-(\gamma)} \tilde{\lambda}_-(\gamma)^{n+1}$$

where $\tilde{\lambda}_\pm(\gamma)$ is the unique root of $\mathcal{L} - \gamma$ on $\pm(0,1)$. Further, $\tilde{\lambda}_+(\gamma) = \alpha(\gamma)$ and the following inequalities hold: $-\tilde{\lambda}_+(\gamma) < \tilde{\lambda}_-(\gamma) < 0 < \tilde{\lambda}_+(\gamma) < 1$.

When $\alpha(\gamma) = 1$ and hence $\gamma = 1$, then for each $n \in \mathbb{N} \cup \{0\}$, $\mathrm{P}(T_n < \infty) = 1$.

(b) For each $n \geq 0$: $\mathrm{P}(W(T_n) = n, T_n < \infty) = \frac{1}{\tilde{\lambda}_+(1) - \tilde{\lambda}_-(1)}(\tilde{\lambda}_+(1)^{n+1} - \tilde{\lambda}_-(1)^{n+1})$ and $\mathrm{P}(W(T_n) = n+1, T_n < \infty) = \frac{-\tilde{\lambda}_+(1)\tilde{\lambda}_-(1)}{\tilde{\lambda}_+(1) - \tilde{\lambda}_-(1)}(\tilde{\lambda}_+(1)^n - \tilde{\lambda}_-(1)^n)$.

Here $\tilde{\lambda}_+(1) = \alpha(1)$ is the smallest root of $\mathcal{L} - 1$ on $(0,1]$ and $\tilde{\lambda}_-(1)$ is the unique root of $\mathcal{L} - 1$ on $(-1,0)$. It holds: $-\tilde{\lambda}_+(1) < \tilde{\lambda}_-(1) < 0 < \tilde{\lambda}_+(1) \leq 1$.

(c) We have:

$$\lim_{n \to \infty} \frac{\mathrm{P}(W(T_n) = n+1 | T_n < \infty)}{\mathrm{P}(W(T_n) = n | T_n < \infty)} = -\tilde{\lambda}_-(1) \in (0,1).$$

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- Interesting at least to ask what happens in the setting when jumps are allowed upwards up to a certain treshold $N \in \mathbb{N}$.
- Analysis extends naturally, but recurrence relations no longer seem to admit a tractable solution.

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& a happy New Year!