Week 2: Selected probabilistic notions and results

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1 Generated σ-algebra

- Recall: given a set Ω and $\mathcal{A} \subset 2^\Omega$ we denote by $\sigma(\mathcal{A})$ (note: reference to Ω omitted!), the smallest σ-algebra on Ω containing the set $\mathcal{A}$. Important: smallest = smallest in the sense of inclusion and *not* size (i.e. cardinality)! (Although, as a consequence, $\sigma(\mathcal{A})$ also has the smallest cardinality among all σ-algebras containing $\mathcal{A}$.)

- To be completely unambiguous, note that an arbitrary intersection of σ-algebras on the set Ω is again a σ-algebra on the same set (this is elementary to argue ...). Thus:

$$\sigma(\mathcal{A}) := \cap\{\mathcal{F} : \mathcal{F}\ \sigma - \text{algebra on } \Omega \text{ and } \mathcal{A} \subset \mathcal{F}\}.$$

  Remark: 2^Ω is always a σ-algebra, containing $\mathcal{A}$, therefore above is well-defined.

- Note:

  (i) $\sigma(\mathcal{A})$ is a σ-algebra on Ω.

  (ii) $\mathcal{A} \subset \sigma(\mathcal{A})$.

  (iii) For every σ-algebra $\mathcal{G}$ on Ω, containing $\mathcal{A}$, $\mathcal{G} \supset \sigma(\mathcal{A})$.

  In this precise sense $\sigma(\mathcal{A})$ is smallest (with ref. to property (iii)) σ-algebra (with ref. to (i)) containing $\mathcal{A}$ (with ref. to property (ii)).

- How do we "cook-up" $\sigma(\mathcal{A})$ in a sufficiently simple situation. Recipe:

  1. First use properties of σ-algebras (closure under complementation, countable unions and intersections, having $\emptyset$ and Ω as elements etc.) to get a candidate set $\mathcal{F}$, which *must* be contained in every σ-algebra which contains $\mathcal{A}$ and then hope you've found enough sets to make $\mathcal{F}$ into a σ-algebra.
  2. Check properties (i-iii) above for $\mathcal{F}$ in place of $\sigma(\mathcal{A})$. (ii) and (iii) will be immediate by construction. Only left to verify $\mathcal{F}$ is a σ-algebra.

3. If so, then $\mathcal{F} = \sigma(\mathcal{A})$.

- Example: problem 3c(i&ii) from the exam.

2 "Almost surely", "with probability 1" and" almost every" qualifications

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space.

- We say an event A is *almost sure*, if $\mathsf{P}(A) = 1$.

- We say a proposition $P(\omega)$ about the elements of the sample space $\omega \in \Omega$ holds almost surely (or with probability one, or for almost every $\omega \in \Omega$), if (technicality) the set $A(P) := \{\omega \in \Omega : P(\omega) \text{ holds}\} \in \mathcal{F}$ and (crucially) the event $A(P)$ is almost sure.

- Example: SLLN: if $X_1, X_2, \ldots$ is an iid sequence and $\mu := E[|X_1|] < \infty$, then with probability one, i.e. almost surely, $S_n \to \mu$ as $n \to \infty$. Here $S_n := (X_1 + \ldots + X_n)/n$ are the sample means ($n \geq 1$). This means: $\mathsf{P}(\{\omega \in \Omega : S_n(\omega) \to \mu \text{ as } n \to \infty\}) = 1$.

- Example: Consider a sequence of coin tosses of a biased coin whose probability of landing heads is $p \in (0, 1)$. Then, almost surely, we will see a head eventually (but there is an element of the sample space when we never see a head)!

- Example: consider throwing darts. Suppose we are sufficiently skilled to always hit the dartboard, but we are otherwise complete amateurs and will score according to the uniform distribution. The probability measure is (hence) the normalized restriction of the 2D Lebesgue measure to (say) the unit circle. Then, almost surely, we will hit any subset of the unit circle which has full Lebesgue measure.

3 The events "infinitely often" and "always, except for finitely many times"; the Borel-Cantelli lemmas

Let $(A_i)_{i \geq 1}$ be a sequence of events.

- Recall the definition of $\limsup A_i$ and $\liminf A_i$. These are events "infinitely many A_i happen" and "all but finitely many events A_i happen", respectively. Remark: get one from the other by taking complements.

- Recall the Borel-Cantelli lemmas: (i) if $\sum \mathsf{P}(A_i) < \infty$ then $\mathsf{P}(\limsup A_i) = 0$ (ii) If A_i are independent and $\sum \mathsf{P}(A_i) = \infty$ then $\mathsf{P}(\limsup A_i) = 1$.

- Example: rolling a die (Klenke). Suppose we throw a die again and again and observe whether or not we get 6 on each roll. The event that we see infinitely many 6's has probability 1 by BC(ii). Alternatively: suppose we take $A_1 = A_2 = \ldots = \{\text{first roll comes up 6}\}$. Then $\mathsf{P}(\limsup A_i) = \mathsf{P}(A_1) = 1/6$, which shows that in BC(ii), independence is indispensible!

- Example (Klenke). Let $0 < \lambda_n \leq \Lambda$ for a fixed Λ and let $X_i \sim \text{Pois}(\lambda_i)$. Then $\mathsf{P}(X_n \geq n \text{ for infinitely many } n) = 0$. Here we apply BC(i) and exchange the order of summation ...

4 Types of convergence

- *almost surely*: e.g. SLLN.

- *in probability*: Consider the canonical space $([0,1], \mathcal{B}([0,1]), \lambda)$ where λ denotes Lebesgue measure. Consider further the sequence of RVs defined as follows (for $s \in [0,1]$): $X_0(s) = s + \mathbb{1}_{[0,1]}(s)$, $X_1(s) = s + \mathbb{1}_{[0,1/2]}(s)$, $X_2(s) = s + \mathbb{1}_{[1/2,1]}(s)$, $X_3(s) = s + \mathbb{1}_{[0,1/3]}(s)$, $X_4(s) = s + \mathbb{1}_{[1/3,2/3]}(s)$, $X_5(s) = s + \mathbb{1}_{[2/3,1]}(s)$ and so on. Then $X_n \to \text{id}_{[0,1]}$ in probability, but not almost surely!

- *in distibution/weakly/in law*: CLT.

- *in p-mean*: Let X_n be a sequence of RVs with $\mathsf{P}(X_n = \pm 1/n) = 1/2$. Then $X_n \to 0$ in p-mean for all $p > 0$.

- Remark & exercise: Almost sure convergence and convergence in probability determine the limit up to almost sure equality.

$$\begin{array}{ccccc} \xrightarrow{L^s} & \underset{s>r\geq 1}{\Rightarrow} & \xrightarrow{L^r} & & \\ & & \Downarrow & & \\ \xrightarrow{a.s.} & \Rightarrow & \xrightarrow{p} & \Rightarrow & \xrightarrow{d} \end{array}$$

Figure 1: **Various types of convergences.** And implications between them. Ref.: Wikipedia.