Scaling Limits in Computational Bayesian Inversion

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THE UNIVERSITY OF WARWICK

SAM

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Swiss Federal Institute of Technology Zurich

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Outline

Inverse Problem

Physical Model

$$G(u) \to \delta$$

- u parameter vector / parameter function
- G the forward map modelling the physical process
- δ result / observations

Forward Problem

Find the output δ for given parameters u

$\to$ **well-posed**

Inverse Problem

Find the parameters u from (noisy) observations δ

$\to$ **ill-posed**

Inverse Problem

Physical Model

$$\mathcal{G}(u) \to \delta$$

- u parameter vector / parameter function
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- $\mathcal{G}$ forward response operator

Forward Problem

Find the output δ for given parameters u

$\rightarrow$ **well-posed**

Inverse Problem

Find the parameters u from (noisy) observations δ

$\rightarrow$ **ill-posed**

Inverse Problem

Find the unknown data $u \in X$ from noisy observations

$$\delta = \mathcal{G}(u) + \eta$$

Deterministic optimization problem

$$\min_u \frac{1}{2} \|\delta - \mathcal{G}(u)\|^2 + R(u)$$

- $\|\delta - \mathcal{G}(u)\|$ potential / data misfit
- R regularization term

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Deterministic optimization problem

$$\min_u \frac{1}{2} \|\delta - \mathcal{G}(u)\|^2 + R(u)$$

- **Large-scale, deterministic optimization problem**
- **No quantification of the uncertainty in the unknown $\mathbf{u}$**
- **Proper choice of the regularization term $\mathbf{R}$**

Inverse Problem

Find the unknown data $u \in X$ from noisy observations

$$\delta = \mathcal{G}(u) + \eta$$

Bayesian inverse problem

- u, η, δ random variables / fields
- Goal of computation: moments of system quantities under the posterior w.r. to noisy data δ

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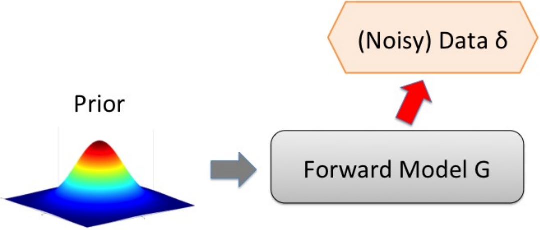
Bayesian inverse problem



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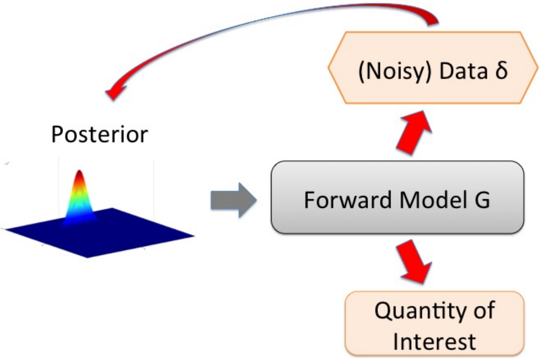

Inverse Problem

Find the unknown data $u \in X$ from noisy observations

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Bayesian inverse problem

- **Quantification of uncertainty in $\mathbf{u}$ and system quantities**
- **Well-posedness of the inverse problem**
- **Incorporation of prior knowledge on the uncertain data $\mathbf{u}$**
- **Need of efficient approximations of the posterior**

Inverse Problem

Find the unknown data $u \in X$ from noisy observations

$$\delta = \mathcal{G}(u) + \eta$$

Bayesian inverse problem

- MCMC methods
- MAP estimators
- Ad hoc algorithms (EnKF, SMC, GP,...)
- Sparse deterministic approximations of posterior expectations

Bayesian Inverse Problems (Stuart 2010)

Find the unknown data $u \in X$ from noisy observations

$$\delta = \mathcal{G}(u) + \eta,$$

Goal: Efficient estimation of system quantities from noisy observations

- Infinite-dimensional parameter space
- Fast convergence by exploiting sparsity of the underlying forward problem
- Suitable for application to a broad class of forward problems

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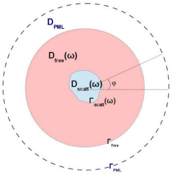
UQ in Nano Optics



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UQ in Biochemical Networks

- Infinite-dimensional parameter space
- Fast convergence by exploiting sparsity of the underlying forward problem
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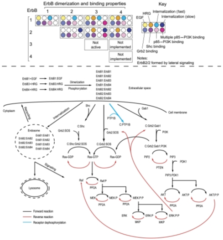

Source: Chen et al.

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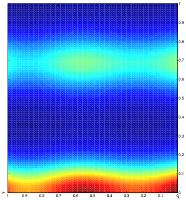
UQ in Groundwater Flow



Bayesian Inverse Problems (Stuart 2010)

Find the unknown data $u \in X$ from noisy observations

$$\delta = \mathcal{G}(u) + \eta,$$

- X separable Banach space
- $G : X \mapsto \mathcal{X}$ the forward map

Abstract Operator Equation

$$\text{Given } u \in X, \text{ find } q \in \mathcal{X} : \quad A(u;q) = F(u) \quad \text{in} \quad \mathcal{Y}'$$

with $A(u;\cdot) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$, $F : X \mapsto \mathcal{Y}'$, $\mathcal{X}$, $\mathcal{Y}$ reflexive Banach spaces,
$\mathfrak{a}(v, w) := {}_{\mathcal{Y}}\langle w, Av\rangle_{\mathcal{Y}'}$ $\forall v \in \mathcal{X}, w \in \mathcal{Y}$ corresponding bilinear form

- $\mathcal{O} : \mathcal{X} \mapsto \mathbb{R}^K$ bounded, linear observation operator
- $\mathcal{G} : X \mapsto \mathbb{R}^K$ uncertainty-to-observation map, $\mathcal{G} = \mathcal{O} \circ G$
- $\eta \in \mathbb{R}^K$ the observational noise ($\eta \sim \mathcal{N}(0, \Gamma)$)

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Model Parametric Elliptic Problem

$$\text{Given } u \in X, \text{ find } q \in \mathcal{X}: \quad -\nabla \cdot (u \nabla q) = f \quad \text{in} \quad D, \qquad q = 0 \text{ in } \partial D$$

with weak formulation $\int_D u(x)\nabla q(x) \cdot \nabla w(x) dx = {}_{\mathcal{X}}\langle w, f\rangle_{\mathcal{X}'}$ for all $w \in \mathcal{X}$,
$\mathcal{X} = \mathcal{Y} = H_0^1(D)$, $D \subset \mathbb{R}^d$ bounded Lipschitz domain with Lipschitz boundary ∂D

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Least squares potential $\Phi : X \times \mathbb{R}^K \to \mathbb{R}$

$$\Phi(u; \delta) := \frac{1}{2} \left((\delta - \mathcal{G}(u))^\top \Gamma^{-1} (\delta - \mathcal{G}(u)) \right)$$

Reformulation of the forward problem with unknown stochastic input data as an infinite dimensional, parametric deterministic problem

Bayesian Inverse Problems (Stuart 2010)

Parametric representation of the unknown u

$$u = u(\boldsymbol{y}) := \langle u \rangle + \sum_{j \in \mathbb{J}} y_j \psi_j \in X$$

- $\boldsymbol{y} = (y_j)_{j \in \mathbb{J}}$ iid sequence of real-valued random variables $y_j \sim \mathcal{U}[-1, 1]$
- $\langle u \rangle, \psi_j \in X$
- $\mathbb{J}$ finite or countably infinite index set

Prior measure on the uncertain input data

$$\mu_0(d\boldsymbol{y}) := \bigotimes_{j \in \mathbb{J}} \frac{1}{2} \lambda_1(dy_j) \; .$$

- $(U, \mathcal{B}) = \left([-1, 1]^{\mathbb{J}}, \; \bigotimes_{j \in \mathbb{J}} \mathcal{B}^1[-1, 1] \right)$ measurable space

Bayesian Inverse Problem

Theorem (Schwab and Stuart 2011)

Assume that $\mathcal{G}(u)\Big|_{u=\langle u\rangle+\sum_{j\in\mathbb{J}} y_j\psi_j}$ is bounded and continuous.

Then $\mu^\delta(d\boldsymbol{y})$, the distribution of $\boldsymbol{y}\in U$ given δ, is absolutely continuous with respect to $\mu_0(d\boldsymbol{y})$, and

$$\frac{d\mu^\delta}{d\mu_0}(\boldsymbol{y}) = \frac{1}{Z}\Theta(\boldsymbol{y})$$

with the parametric Bayesian posterior Θ given by

$$\Theta(\boldsymbol{y}) = \exp\big(-\Phi(u;\delta)\big)\Big|_{u=\langle u\rangle+\sum_{j\in\mathbb{J}} y_j\psi_j},$$

and the normalization constant

$$Z = \int_U \Theta(\boldsymbol{y})\mu_0(d\boldsymbol{y}) \; .$$

Bayesian Inverse Problem

Expectation of a *Quantity of Interest* $\phi : X \to S$

$$\mathbb{E}^{\mu^\delta}[\phi(u)] = Z^{-1} \int_U \exp\big(-\Phi(u;\delta)\big)\phi(u)\Big|_{u=\langle u\rangle+\sum_{j\in\mathbb{J}} y_j\psi_j} \mu_0(d\mathbf{y}) =: Z'/Z$$

with $Z = \int_U \exp(-\frac{1}{2}\left((\delta - \mathcal{G}(u))^\top \Gamma^{-1}(\delta - \mathcal{G}(u))\right))\mu_0(d\mathbf{y})$.

- Reformulation of the forward problem with unknown stochastic input data as an infinite dimensional, parametric deterministic problem
- Parametric, deterministic representation of the derivative of the posterior measure with respect to the prior μ_0
- Approximation of Z' and Z to compute the expectation of QoI under the posterior given data δ

Efficient algorithm to approximate the conditional expectations given the data with dimension-independent rates of convergence

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Exploiting sparsity in the parametric operator equation

- Parameters belonging to a specified sparsity class
- Analytic regularity of the parametric, deterministic Bayesian posterior
- Parametric, deterministic Bayesian posterior belongs to the same sparsity class

$\to$ **Sparsity of generalized pce + dimension-independent convergence rates for Smolyak integration algorithms**

Model Parametric Elliptic Problem

$$\int_D u(x)\nabla q(x)\cdot\nabla w(x)dx = \ _{\mathcal{X}}\langle w,f\rangle_{\mathcal{X}'} \quad \text{for all} \quad w\in\mathcal{X}$$

with $D\subset\mathbb{R}^d$ bounded Lipschitz domain with Lipschitz boundary ∂D, $\mathcal{X}=H_0^1(D), f\in L^2(D)$.

Structural Assumptions

$$u(x,\boldsymbol{y}) = \langle u\rangle(x) + \sum_{j\in\mathbb{J}} y_j\psi_j(x), \quad x\in X$$

with $\langle u\rangle \in L^\infty(D)$ and $(\psi_j)_{j\in\mathbb{J}}\subset L^\infty(D), \boldsymbol{y}=(y_1,y_2,\dots)\in U=[-1,1]^{\mathbb{J}}$

Assumption 1 (UEA)

$$0<u_{\min}\le u(x,y)\le u_{\max}<\infty \quad \forall x\in D, y\in U$$

with $0<u_{\min}\le u_{\max}<\infty$.

Assumption 2

$$\sum_{j\in\mathbb{J}}\|\psi_j\|_{L^\infty(D)} \le \frac{\kappa}{1+\kappa}\langle u\rangle_{\min}$$

with $\langle u\rangle_{\min} = \min_{x\in D}\langle u\rangle(x) > 0$ and $\kappa>0$

Sparse Polynomial Chaos Approximations

Sparsity in the unknown coefficient function u,

i.e. if $\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D)}^p < \infty$ for $0 < p < 1$,

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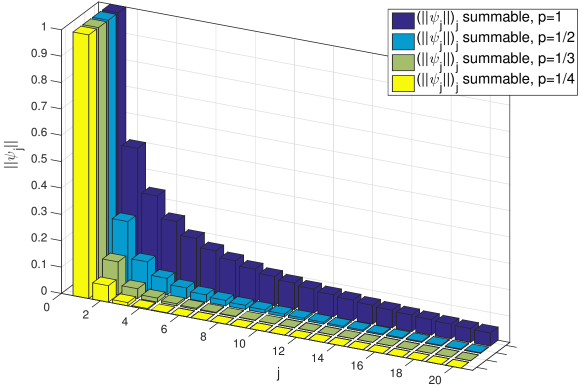

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i.e. if $\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D)}^p < \infty$ for $0 < p < 1$,

implies p-sparsity in the solution's Taylor expansion,

$\forall y \in U : \quad q(\boldsymbol{y}) = \sum_{\nu \in \mathcal{F}} \tau_\nu \boldsymbol{y}^\nu \ , \quad \tau_\nu := \frac{1}{\nu!} \partial_{\boldsymbol{y}}^\nu q(\boldsymbol{y}) \ |_{\boldsymbol{y}=0} \ , \quad \mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty\}$:

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There exists a p-summable, monotone envelope $\mathbf{t} = \{\mathbf{t}_\nu\}_{\nu \in \mathcal{F}}$

i.e. $\mathbf{t}_\nu := \sup_{\mu \geq \nu} \|\tau_\nu\|_{\mathcal{X}}$ with $C(p, \boldsymbol{q}) := \|\mathbf{t}\|_{\ell^p(\mathcal{F})} < \infty$.

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imp

$\forall y \in$ … $\infty\}$:

The

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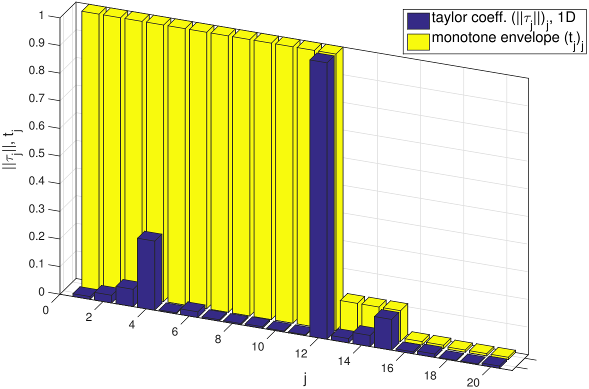

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and monotone $\Lambda_N^T \subset \mathcal{F}$ corresponding to the N largest terms of $\mathbf{t}$ with

$$\sup_{\boldsymbol{y} \in U} \left\| q(\boldsymbol{y}) - \sum_{\nu \in \Lambda_N^T} \tau_\nu \boldsymbol{y}^\nu \right\|_{\mathcal{X}} \leq C(p, \mathbf{t}) N^{-(1/p-1)} \ .$$

A. Cohen, R. DeVore and Ch. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs, *Analysis and Applications*, 2010.

Sparse Polynomial Chaos Approximations

Idea of Proof

Holomorphic Extension

$$z \mapsto q(z)\;, \qquad z \in \mathcal{U}_\rho := \otimes_{j\geq 1}\{z_j \leq \rho_j\}$$

holomorph with bound $\|q(z)\|_{\mathcal{X}} \leq C_\delta$ for any positive sequence $\rho = \{\rho_j\}_{j\geq 1}$ such that $\sum_{j\geq 1}\rho_j|\psi_j(x)| \leq \langle u\rangle(x) - \delta,\ x \in D$ for some $\delta > 0$.

Estimate on the Taylor Coefficients

$$\|\tau_\nu\|_{\mathcal{X}} \leq C_\delta \inf\{\rho^{-\nu} : \sum_{j\geq 1}\rho_j|\psi_j(x)| \leq \langle u\rangle(x) - \delta,\ x \in D\}$$

by recursive application of Cauchy's formula

Construction of a particular ρ

$$\{\|\psi_j\|_{L^\infty}\}_{j\geq 1} \in \ell^p(\mathbb{N}) \Rightarrow \{\mathbf{t}_\nu\}_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$$

Stechkin

$$\sup_{\mathbf{y}\in U}\left\|q(\mathbf{y}) - \sum_{\nu\in\Lambda_N^T}\tau_\nu \mathbf{y}^\nu\right\|_{\mathcal{X}} \leq \sum_{\nu\notin\Lambda_N^T}\|\mathbf{t}_\nu\|_{\mathcal{X}} \leq CN^{-\frac{1}{p}+1}$$

Sparsity of the Posterior

Theorem (CIS and Schwab 2013)

Assume that the unknown coefficient function u is p-sparse for $0 < p < 1$. Then, the posterior density's Taylor expansion is p-sparse with the same p.

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N-term Approximation Results

$$\sup_{\boldsymbol{y}\in U} \left\| \Theta(\boldsymbol{y}) - \sum_{\nu\in\Lambda_N^P} \Theta_\nu^P P_\nu(\boldsymbol{y}) \right\|_{\mathcal{X}} \leq N^{-s} \|\boldsymbol{\theta}^P\|_{\ell_m^p(\mathcal{F})}, \quad s := \frac{1}{p} - 1 \, .$$

Adaptive Smolyak quadrature algorithm with convergence rates depending only on the summability of the parametric operator

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Examples

- Parametric initial value ODEs (Hansen & Schwab; Vietnam J. Math. 2013)
- Affine-parametric, linear operator equations (CIS & Schwab; 2013)
- Semilinear elliptic PDEs (Hansen & Schwab; Math. Nachr. 2013)
- Elliptic multiscale problems (Hoang & Schwab; Analysis and Applications 2012)
- Nonaffine holomorphic-parametric, nonlinear problems (Cohen, Chkifa & Schwab; 2013)

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Goal of computation

Expectation of a *Quantity of Interest* $\phi : X \to S$

$$\mathbb{E}^{\mu^\delta}[\phi(u)] = Z^{-1} \int_U \exp\big(-\Phi(u;\delta)\big)\phi(u)\Big|_{u=\langle u\rangle+\sum_{j\in\mathbb{J}} y_j\psi_j} \mu_0(d\boldsymbol{y}) =: Z'/Z$$

with $Z = \int_U \exp(-\frac{1}{2}\left((\delta - \mathcal{G}(u))^\top \Gamma^{-1}(\delta - \mathcal{G}(u))\right)\mu_0(d\boldsymbol{y})$.

Sparse Quadrature Operator

For any finite monotone set $\Lambda \subset \mathcal{F}$, the quadrature operator is defined by

$$\mathcal{Q}_\Lambda = \sum_{\nu \in \Lambda} \Delta_\nu = \sum_{\nu \in \Lambda} \bigotimes_{j \geq 1} \Delta_{\nu_j}$$

with associated collocation grid

$$\mathcal{Z}_\Lambda = \cup_{\nu \in \Lambda} \mathcal{Z}^\nu \, .$$

- $(Q^k)_{k \geq 0}$ sequence of univariate quadrature formulas
- $\Delta_j = Q^j - Q^{j-1}$, $j \geq 0$ univariate quadrature difference operator
- $\mathcal{Q}_\nu = \bigotimes_{j \geq 1} Q^{\nu_j}$, $\Delta_\nu = \bigotimes_{j \geq 1} \Delta_{\nu_j}$ tensorized multivariate operators

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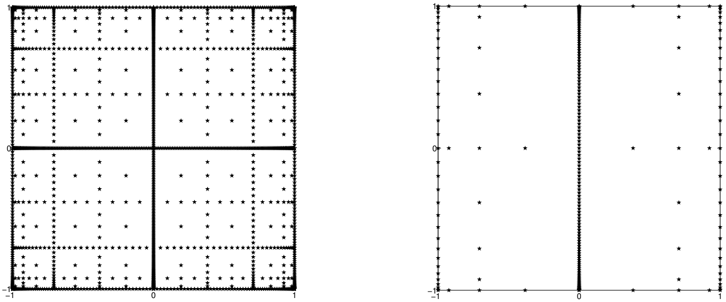


Convergence Rates for Adaptive Smolyak Integration

Theorem

Assume that the unknown coefficient function u is p-sparse for $0 < p < 1$.

Then there exists a sequence $(\Lambda_N)_{N\geq 1}$ of monotone index sets $\Lambda_N \subset \mathcal{F}$ such that $\#\Lambda_N \leq N$ and

$$|Z - \mathcal{Q}_{\Lambda_N}[\Theta]| \leq C^1 N^{-\frac{1}{p}+1},$$

with $Z = \int_U \Theta(\boldsymbol{y})\mu_0(d\boldsymbol{y})$, $\Theta(\boldsymbol{y}) = \exp\big(-\Phi(u;\delta)\big)\Big|_{u=\langle u\rangle+\sum_{j\in\mathbb{J}} y_j\psi_j}$ and,

$$\|Z' - \mathcal{Q}_{\Lambda_N}[\Psi]\|_{\mathcal{X}} \leq C^2 N^{-\frac{1}{p}+1},$$

with $Z' = \int_U \Psi(\boldsymbol{y})\mu_0(d\boldsymbol{y})$, $\Psi(\boldsymbol{y}) = \Theta(\boldsymbol{y})\phi(u)\Big|_{u=\langle u\rangle+\sum_{j\in\mathbb{J}} y_j\psi_j}$ $C^1, C^2 > 0$ independent of N.

Remark: SAME index sets Λ_N for BOTH, Z' and Z.

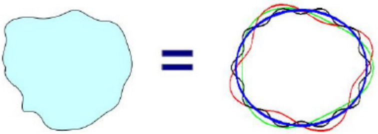
ClS and Schwab Sparsity in Bayesian Inversion of Parametric Operator Equations, 2013.

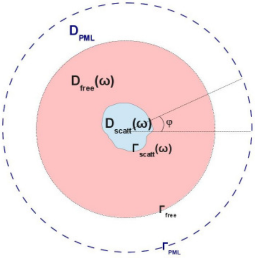
Uncertainty Quantification in Nano Optics

Goal: Quantification of the influence of defects in fabrication process on the optical response of nano structures

- Propagation of plane wave and its interaction with scatterer described by Helmholtz equation (2D).
- Stochastic shape of the scatterer

$$0 < \rho_{\min} \leq \rho(\omega, \phi) \leq \rho_{\max}, \quad \omega \in \Omega,\ \phi \in [0, 2\pi)$$





Collaboration with Ralf Hiptmair, Laura Scarabosio

SAM ETH Eidgenössische Technische Hochschule Zürich Swiss Federal Institute of Technology Zurich

Uncertainty Quantification in Nano Optics

Forward Problem

$$
\begin{aligned}
-\Delta u - k_1^2 u &= 0 \quad \text{in } D_1(\omega) \\
-\Delta u - k_2^2 u &= 0 \quad \text{in } D_2(\omega) \\
u|_+ - u|_- &= 0 \quad \text{on } \Gamma(\omega) \\
\left.\frac{\partial u}{\partial \mathbf{n}}\right|_+ - \mu \left.\frac{\partial u}{\partial \mathbf{n}}\right|_- &= 0 \quad \text{on } \Gamma(\omega)
\end{aligned}
$$

$$
\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial}{\partial |x|} - ik_1\right)(u(\omega) - u_i)(x) = 0, \qquad u_i(x) = e^{ik_1 \cdot x}
$$

Parametrization of the shape

$$
r(\omega,\varphi) = r_0(\varphi) + \sum_{k=1}^{64} \frac{1}{k^{\zeta}} y_{2k}(\omega)\cos(k\varphi) + \frac{1}{k^{\zeta}} y_{2k+1}(\omega)\sin(k\varphi), \quad \varphi \in [0, 2\pi)
$$

Numerical Results of Sparse Interpolation

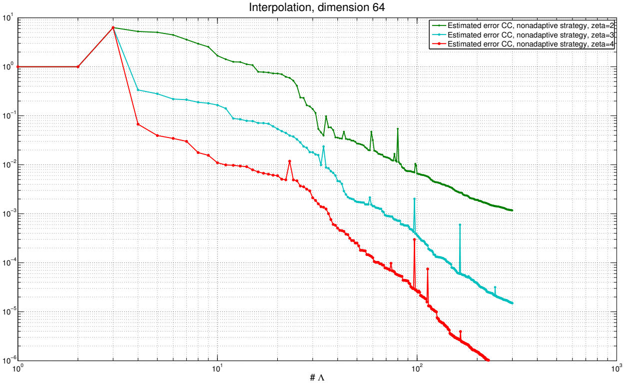


Figure: Estimated error curves of the interpolation error w.r. to the cardinality of the index set Λ_N based on the sequences CC with nonadaptive refinement, uniform distribution of the parameters, dimension of the parameter space 64.

Concentration Effect of the Posterior for $0 < \Gamma \ll 1$

Model parametric elliptic problem

$$-\mathrm{div}(u\nabla q) = f \quad \text{in } D := [0,1]\,,\; q = 0 \quad \text{in } \partial D\,,$$

with $f(x) = 100 \cdot x$ and

$$u(x,y) = 0.15 + y_1\psi_1 + y_2\psi_2\,,$$

with $J = 2$, $\mathbb{J} = \{1,2\}$, $\psi_1(x) = 0.1\sin(\pi x)$, $\psi_2(x) = 0.025\cos(2\pi x)$ and with $y_j \sim \mathcal{U}[-1,1]$, $j \in \mathbb{J}$.

Concentration Effect of the Posterior for $0 < \Gamma \ll 1$

For given (noisy) data δ,

$$\delta = \mathcal{G}(u) + \eta\,,$$

we are interested in the behavior of the posterior

$$\Theta(\boldsymbol{y}) = \exp\big(-\Phi_{\Gamma_{obs}}(u;\delta)\big)\Big|_{u=\sum_{j=1}^{2} y_j\psi_j},$$

with

$$\Phi_{\Gamma_{obs}}(u;\delta) = \frac{1}{2}\Big((\delta - \mathcal{G}(u))^{\top}\Gamma^{-1}(\delta - \mathcal{G}(u))\Big)\ .$$

and variance of the noise $\Gamma = \lambda \cdot id$

- QoI ϕ is the solution of the forward problem at $x = 0.5$.
- Observation operator $\mathcal{O}$ consists of 2 system responses at $x = 0.25$ and $x = 0.75$.

Concentration Effect of the Posterior for $0 < \Gamma \ll 1$

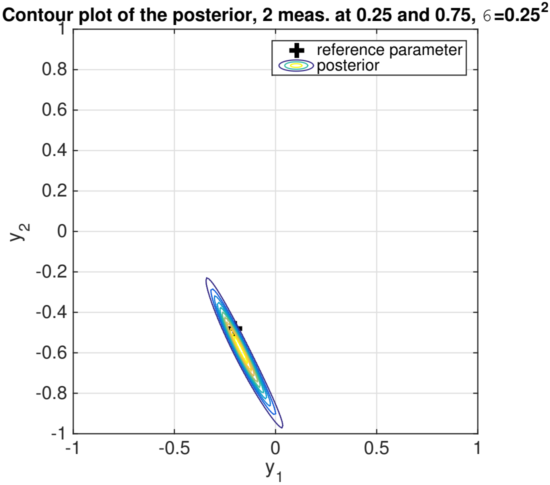


Figure: Contour plot of the posterior with observational noise $\lambda = 0.25^2$.

Concentration Effect of the Posterior for $0 < \Gamma \ll 1$

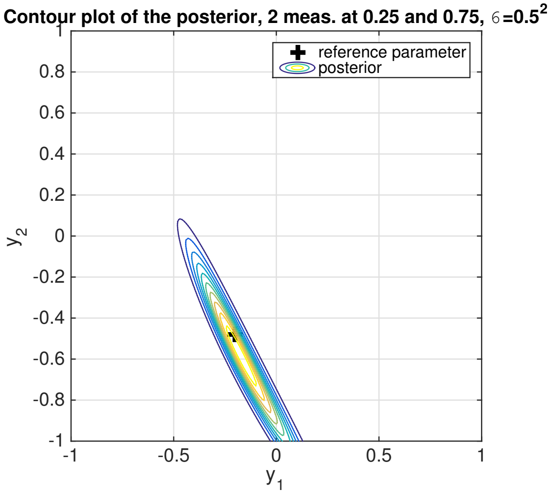


Figure: Contour plot of the posterior with observational noise $\lambda = 0.5^2$.

Concentration Effect of the Posterior for $0 < \Gamma \ll 1$

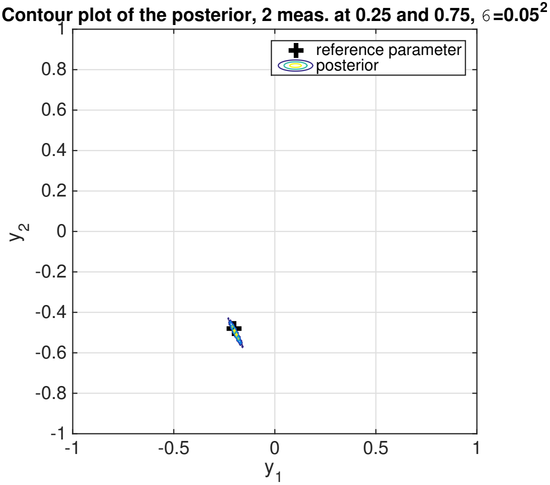


Figure: Contour plot of the posterior with observational noise $\lambda = 0.05^2$.

Asymptotic Analysis

Theorem (CIS and Schwab 2014)

Assume that the parameter space is finite (possibly after dimension - truncation) and $\mathcal{G}(\cdot)$, δ are such that the assumptions of Laplace's method hold; in particular, the minimum $\boldsymbol{y}_0$ of

$$S(y) = \frac{1}{2}\Big((\delta - \mathcal{G}(u))^\top(\delta - \mathcal{G}(u))\Big)$$

is nondegenerate.
Then, as $\Gamma \downarrow 0$, the Bayesian estimate admits an asymptotic expansion

$$\mathbb{E}^{\mu^\delta}[\phi] = \frac{Z'_\Gamma}{Z_\Gamma} \sim a_0 + a_1\lambda + a_2\lambda^2 + ...$$

where $a_0 = \phi(\boldsymbol{y}_0)$.

Curvature Rescaling Regularization

Removing the degeneracy in the integrand function

- The maximizer $\boldsymbol{y}_0$ of the posterior measure $\Theta(\boldsymbol{y})$ is computed by minimizing the potential

$$\frac{1}{2}\big((\delta - \mathcal{G}(u))^\top(\delta - \mathcal{G}(u))\big)\Big|_{u=\sum_{j=1}^{2} y_j\psi_j} .$$

  using a trust-region Quasi-Newton approach with SR1 updates.

- Diagonalize the approximated Hessian $H_{SR1} = QMQ^\top$ and regularize the integrand by the curvature rescaling transformation

$$\boldsymbol{y}_0 + \Gamma^{1/2}QM^{-1/2}\boldsymbol{z}\,, \quad \boldsymbol{z} \in \mathbb{R}^J .$$

Numerical Experiments

Curvature rescaling

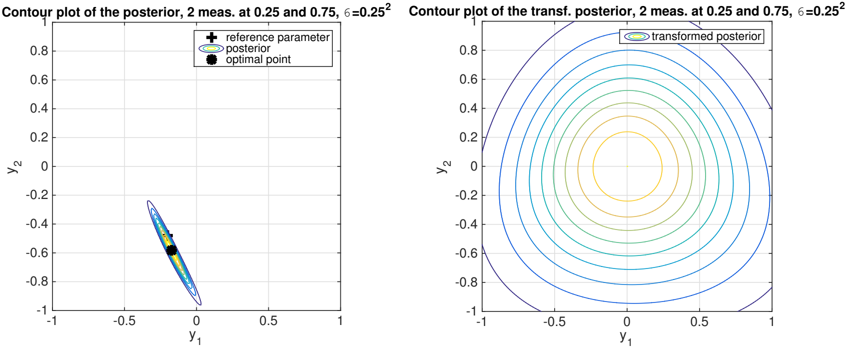


Figure: Contour plot of the posterior with observational noise $\Gamma_{obs} = 0.25^2 \cdot Id$ (left) and contour plot of the transformed posterior (right).

Numerical Experiments

Curvature rescaling

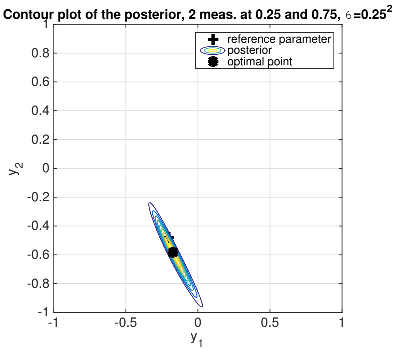


Figure: Contour plot of the posterior with observational noise $\Gamma_{obs} = 0.25^2 \cdot Id$ (left) and and truncated domain of integration of the rescaled Smolyak approach in the original coordinate system (right).

Numerical Experiments

Curvature rescaling

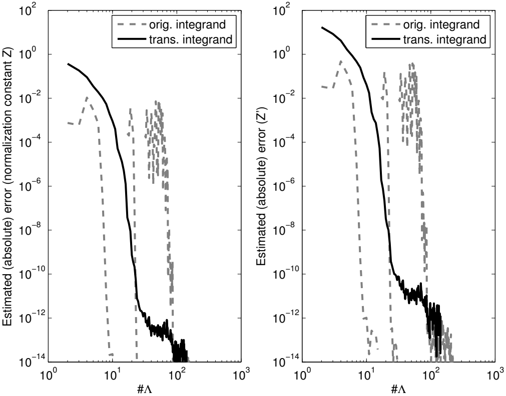


Figure: Comparison of the estimated (absolute) error curves using the Smolyak approach for the original integrand (gray) and the transformed integrand (black) for the computation of $Z_{\Gamma_{obs}}$ (left) and $Z'_{\Gamma_{obs}}$ (right) with observational noise $\Gamma_{obs} = 0.25^2 \cdot Id$.

Numerical Experiments

Curvature rescaling

Extension to lognormal case

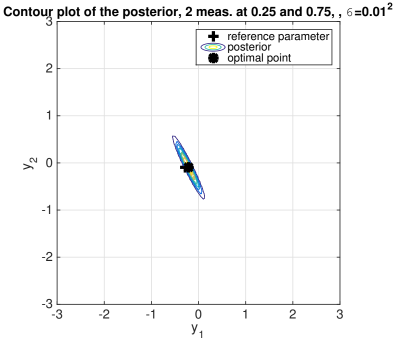
$$\ln(u(x,y)) = 0.15 + y_1\psi_1 + y_2\psi_2\,,$$

with $J = 2$, $\mathbb{J} = \{1, 2\}$, $\psi_1(x) = 0.1\sin(\pi x)$, $\psi_2(x) = 0.025\cos(2\pi x)$ and with $y_j \sim \mathcal{N}(0, 1),\ j \in \mathbb{J}$.

Numerical Experiments

Curvature rescaling

Extension to lognormal case



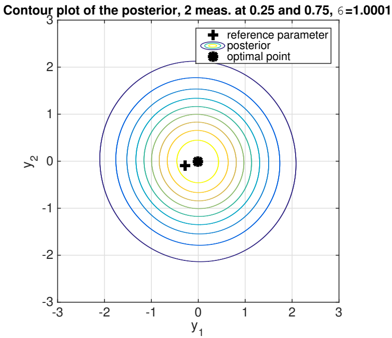


Figure: Contour plot of the posterior density with observational noise $\Gamma_{obs} = 0.01^2 \cdot Id$ (left), and contour plot of the transformed posterior (right).

Numerical Experiments

Curvature rescaling

Extension to lognormal case

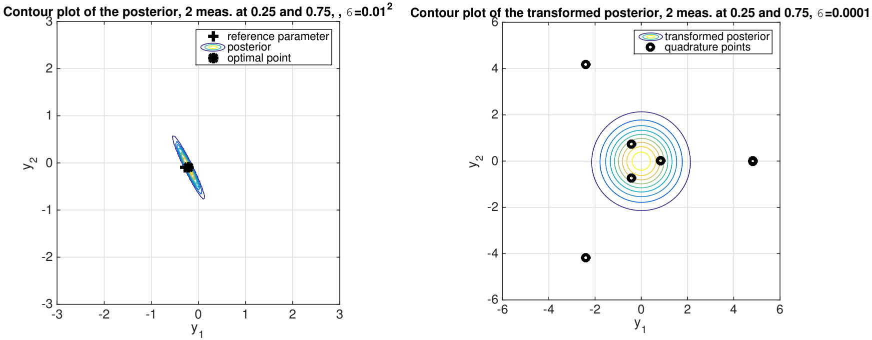


Figure: Contour plot of the posterior density with observational noise $\Gamma_{obs} = 0.01^2 \cdot Id$ (left), and contour plot of the transformed posterior (right).

Conclusions and Outlook

- New class of sparse, adaptive quadrature methods for Bayesian inverse problems for a broad class of operator equations
- Dimension-independent convergence rates depending only on the summability of the parametric operator
- Numerical confirmation of the predicted convergence rates

Conclusions and Outlook

- New class of sparse, adaptive quadrature methods for Bayesian inverse problems for a broad class of operator equations
- Dimension-independent convergence rates depending only on the summability of the parametric operator
- Numerical confirmation of the predicted convergence rates

- Efficient treatment of small observation noise variance Γ
- Development of preconditioning techniques to overcome the convergence problems in the case $\Gamma \downarrow 0$
- Combination of optimization and sampling techniques

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