# The 3D Euler & Navier-Stokes equations : how much do we know?

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# Summary of this talk

## Themes

1. Some informal introductory remarks on the 3D incompressible Euler and Navier-Stokes equations w.r.t. vortex stretching.
2. Regularity of solutions of 3D Euler equations : the Beale-Kato-Majda theorem & variations.
3. The search for singularities : a list of numerical experiments on 3D Euler.
4. Onsager's conjecture (1949) and "wild solutions" of 3D Euler.
5. Ertel's Theorem and its uses : the Hessian matrix of the pressure. Other models : Restricted Euler ; surface QG equations.
6. Remarks on the 3D Navier-Stokes equations :
   - Regularity issues : past & recent history ; singular set
   - Accumulation on 'thin sets' ;
   - Nonlinear depletion.
7. Outstanding issues.

## Compressible Euler

With $\boldsymbol{u}(\boldsymbol{x},\,t)$ as the fluid velocity field and $\rho(\boldsymbol{x},\,t)$ the fluid density

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p \qquad \text{with} \qquad \frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\boldsymbol{u} = 0\,, \qquad \frac{D}{Dt} = \partial_t + \boldsymbol{u}\cdot\nabla\,.$$

## Incompressible Euler

For an incompressible inviscid fluid ($\mathrm{div}\,\boldsymbol{u} = 0$), the Euler equns are (Majda & Bertozzi 2001, Constantin 2007)

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p\,.$$

## Incompressible Navier-Stokes

For an *incompressible* viscous fluid ($\mathrm{div}\,\boldsymbol{u} = 0$), the Navier-Stokes equns are :

$$\frac{D\boldsymbol{u}}{Dt} = \nu\Delta\boldsymbol{u} - \nabla p$$

## Effect of the pressure field

The div-free condition div $\boldsymbol{u} = 0$ for both E & NS enforces the condition

$$-\Delta p = u_{i,j}u_{j,i} \qquad \text{or} \qquad \Delta p = \tfrac{1}{2}\omega^2 - Tr\, S^2\,.$$

$S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ is the rate of strain matrix ; the vorticity is $\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$

## Vorticity formulation & vortex stretching in 3D Euler/NS

$$\partial_t \boldsymbol{u} - \boldsymbol{u} \times \omega = \nu\Delta\boldsymbol{u} - \nabla\left(p + \tfrac{1}{2}u^2\right)$$

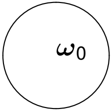
$$\partial_t\, \boldsymbol{\omega} = \text{curl}\left(\boldsymbol{u} \times \boldsymbol{\omega}\right) + \nu\Delta\boldsymbol{\omega}\,,$$

$$\begin{aligned} \frac{D\boldsymbol{\omega}}{Dt} &= \nu\Delta\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} \\ &= \nu\Delta\boldsymbol{\omega} + S\,\boldsymbol{\omega} \end{aligned}$$

Alignment of $\omega$ with e-vectors of $S$ leads to “stretching & folding” in the $\omega$-field – **produces the fine-scale “crinkles” in the vorticity field.**

The existence of weak solutions was proved by **Yudovich (1963)**. Euler vortex patches are such that when $\boldsymbol{u}$ is smooth, $\omega$ is a particular solution of the 2D incompressible Euler equations ($\omega = \text{curl}\,\boldsymbol{u}$)

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\omega = 0\,, \qquad\qquad \omega\cdot\nabla\boldsymbol{u} = 0\,.$$




In the interior of $\Omega(t)$, $\omega = \omega_0 = \text{const}$ for all $t > 0$ : in the exterior $\omega = 0$. The **boundary**, initially $\Gamma_0$, evolves as $\Gamma_t$.

**Chemin (1993)** proved that if the boundary $\Gamma_t$ is initially smooth ($\Gamma_0$ is $C^r$ for $r > 1$) then it remains smooth for all $t > 0$. The theoretical bounds for the parametrization of $\Gamma_t$ are $\exp\exp t$.

- $\exists$ a different version on this proof by **Bertozzi & Constantin (1993)**.
- See also Chapter 9 of the book by **Majda & Bertozzi (2001)**.

## Three invariants of Euler :

- **Energy :** $\int_V |\boldsymbol{u}|^2\, dV$ **Circulation :** $\oint_C \boldsymbol{u} \cdot d\boldsymbol{r}$ – Kelvin's Theorem
- **Helicity :** $\int_V \boldsymbol{\omega} \cdot \boldsymbol{u}\, dV$ : for a 3D vector field this is the standard measure of the extent to which the field lines coil around one another. The helicity is most studied when the fluid is allied to a magnetic field (MHD).

**(i) Arnol'd & Khesin,** Topological Methods in Hydrodynamics, 1998.
**(ii) Moffatt (1969,78,85/6,92), and Ricca & Moffatt (1992).**

## Some exact solutions :

1. **Hill's spherical vortex (1895) :** $\omega_\theta \neq 0$ inside a sphere but $\boldsymbol{\omega} = 0$ outside. The velocity field is continuous across the boundary.
2. **Vortex rings :** Norbury (1973) showed $\exists$ a family of vortex rings close to HSV.
3. **Burgers' vortex tubes :** $\omega_\theta = 0$ with $\omega_z$ growing exponentially.
4. **Taylor Green vortex.**

### Theorem (BKM theorem)

*There exists a global solution of the 3D incompressible Euler equations* $\boldsymbol{u} \in C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1})$ *for* $s > 5/2$ *if, for every* $T > 0$,

$$\int_0^T \|\boldsymbol{\omega}(\tau)\|_{L^\infty(\mathbb{R}^3)}\, d\tau < \infty\,.$$

*Conversely, suppose* $\exists$ *a time* $T^*$ *such that the solution cannot be continued to* $t = T^*$, *then if* $T^*$ *is the first such time*

$$\int_0^{T^*} \|\boldsymbol{\omega}(\tau)\|_{L^\infty(\mathbb{R}^3)}\, d\tau = \infty\,,$$

Ponce (1985) has proved $\int_0^t \|\mathbf{S}(\cdot\,,\tau)\|_\infty\, d\tau < \infty$ controls solutions. Ferrari (1993) & Shirota & Yanagisawa (1993) have used BCs $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$. Kozono & Taniuchi (2000) in BMO.

### Corollary

*If a singularity* $\|\omega(\cdot\,,t)\|_\infty \sim (T^* - t)^{-\beta}$ *is observed in a numerical experiment, for the singularity to be genuine & not an artefact of the numerics then* $\beta \geq 1$.

*An up-dated version of a list compiled by JDG (2007) & Rainer Grauer*

**1) Morf, Orszag & Frisch (1980) :** studied complex time singularities of the 3D Euler equations using Padé-approximants. **Singularity; yes.**

**2) Chorin (1982) :** Vortex–method. **Singularity; yes.**

**3) Brachet, Meiron, Nickel, Orszag & Frisch (1983) :** Taylor Green i.c.s : saw production of vortex sheets. **Singularity; no.**

**4) Siggia 1984 :** Vortex–filament method. **Singularity; yes.**

**5) Ashurst & Meiron (1987) :** Finite-difference methods. **Singularity; yes.**

**6) Kerr & Pumir (1987) :** Pseudo-spectral method. **Singularity; no.**

**7) Pumir & Siggia (1990) :** adaptive grid method showed evidence for quasi-2D structures with exponential growth of vorticity. **Singularity; no.**

**8) Bell & Marcus (1992) :** projection method with $128^3$ mesh points ; amplification of vorticity by 6. **Singularity; yes.**

**9) Brachet, Meneguzzi, Vincent, Politano & Sulem (1992) :** Pseudospectral code, Taylor–Green vortex, with a resolution of $864^3$. Amplification of vorticity by 5. **Singularity; no.**

**10) Kerr (1993) :** Chebyshev polynomials with anti–parallel initial conditions ; resolution $512^2 \times 256$. Amplification of vorticity by 18. Observed vorticity growth $\|\omega\|_{L^\infty(\Omega)} \sim (T^* - t)^{-1}$. **Singularity; yes.**

**11) Grauer and Sideris (1991) :** 3D axisymm swirling flow. **Singularity; yes.** However, both E & Henderson separately concluded : **Singularity; no.**

**12) Boratav & Pelz (1994,95), Pelz & Gulak (1997), and Pelz (1997,2002) :** series of $1024^3$ grid-point simulations under Kida's high symmetry condition. **Singularity; yes.**

**13) The FDR-memorial issue for Pelz (2005) :**

1. Cichowlas and Brachet : **Singularity; no.**
2. Gulak and Pelz : **Singularity; yes.**
3. Pelz and Ohkitani,: **Singularity; no.**
4. Kerr : **Singularity; yes.**

**14) Grauer, Marliani & Germaschewski (1998) :** $2048^3$ adaptive mesh refinement of the Bell-Marcus i.c. Amplification of 21. **Singularity; yes.**

**15) Hou & Li (2006) :** a $1536 \times 1024 \times 3072$ pseudo-spectral calculation agreed with Kerr (1993) until the final stage but growth slowed; vorticity grew double exponentially. **Singularity; no.**

**16) Orlandi & Carnevale (2007) :** $1024^3$ finite difference calculation with Lamb dipoles as i.cs. & 2 symmetry planes. Found a period of rapid growth of vorticity consistent with $\|\omega\|_\infty \sim (t^* - t)^{-1}$: **Singularity; yes.**

**17) Kerr & Bustamante (2008) :** Found growth of vorticity consistent with $\|\omega\|_{L^\infty(\Omega)} \sim (T - t)^{-\beta}$ for $\beta > 1$ : **Singularity; yes.**

**18) Bustamante & Brachet 2012 :** Complex singularities & the analyticity strip method : **Singularity; yes.**

**19) Kerr (2013) :** $1024^2 \times 2048$ pseudo-spectral calculation : late change to double exp-growth. **Singularity; no.**

**20) Luo & Hou (2013) :** Symmetric-flow in cylinder with $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ : $10^8$-fold increase in $\|\omega\|_\infty$ : ArXiv 1310.0497v1 **Singularity; yes.**

**21) JDG-Titi (2013) :** 3D Euler + passive scalar $\theta$ on a finite domain $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ : singularity if $\mathcal{B} = \nabla q \times \nabla\theta$ has a zero where $q = \omega \cdot \nabla\theta$ : **Singularity; maybe.**

**Constantin, Fefferman & Majda (1996)** discussed the idea of vortex lines being "smoothly directed" in a region of greatest curvature : no singularity at time $T$ if $\|\boldsymbol{u}\|_\infty$ is finite in a ball ($B_{4\rho}$) &

$$\lim_{t\to T}\sup_{\boldsymbol{W}_0}\int_0^t \|\nabla\boldsymbol{\xi}(\cdot\,,\tau)\|^2_{L^\infty(B_{4\rho})}\,d\tau < \infty$$

where $\boldsymbol{\xi}(\boldsymbol{x},t) = \boldsymbol{\omega}(\boldsymbol{x},t)/|\boldsymbol{\omega}(\boldsymbol{x},t)|$.

**Theorem (Deng-Hou-Yu 2006)**

*Take the arc length $L(t)$ of a vortex line with unit normal $\hat{\boldsymbol{n}}$ & curvature $\kappa$. No blow-up can occur at time $T$ if*

1. $U_\xi(t) + U_{\hat{n}}(t) \lesssim (T-t)^{-A} \qquad A+B=1,$
2. $M(t)L(t) \leq const > 0$
3. $L(t) \gtrsim (T-t)^B$ .

$U_\xi(t) = \max_{\boldsymbol{x},\boldsymbol{y}}\left[(\boldsymbol{u}\cdot\boldsymbol{\xi})(\boldsymbol{x},t) - (\boldsymbol{u}\cdot\boldsymbol{\xi})(\boldsymbol{y},t)\right]$ & $U_{\hat{\boldsymbol{n}}}(t) = \max_{L_t}\left[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}}\right]$ &

$$M(t) \equiv \max\left(\|\nabla\cdot\boldsymbol{\xi}\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)}\right)$$

Consider the energy $\frac{1}{2}\int_{\mathcal{V}}|\boldsymbol{u}|^2 dV$ for 3D Euler :

$$\tfrac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}|\boldsymbol{u}|^2 dV + \int_{\mathcal{V}}\boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p)\,dV = 0\,.$$

subject to div $\boldsymbol{u} = 0$.

## Onsager's conjecture 1949

1. $C^{0,\alpha}$ Hölder continuous solutions of 3D Euler are energy conservative when $\alpha > \frac{1}{3}$.
2. There exist dissipative solutions with $C^{0,\alpha}$ regularity for any $\alpha \leq \frac{1}{3}$. This is known as "anomalous dissipation".

(1) has been proved by

1. Eyink (1994) ;
2. Constantin, E and Titi (1994) : the proof uses the Besov class $B_3^{\alpha,\infty}$.

In other words, a solution needs to be "a little" smoother than just 1/3.

# "Wild solutions" of 3D Euler

For (2), that is : *"∃ solutions with $C^{0,\alpha}$ regularity for any $\alpha \leq \frac{1}{3}$ which have dissipating energy"* is generally an open problem but some solutions have been shown to exist :

1. Scheffer (1993) ;
2. Shnirelman (1997, 2000) ;
3. **De Lellis & Szekelyhdi** (2009,2010,2013), followed by Wiedemann (2011), have shown that ∃ an infinite set of energy dissipating solutions in both $\mathbb{R}^2$ and $\mathbb{R}^3$.
4. Such solutions, for which De Lellis & Szekelyhdi coined the name "wild", are not physical. Such a flow, starting from rest without external forcing, begins to move & then goes back to rest.
5. See Bardos & Titi (2010,2012) and connections with work on vortex sheet solutions (DiPerna & Majda 1987) & their weak limit.

## Theorem (Ertel's Theorem (1942))

*If $\omega(\boldsymbol{x},\, t)$ satisfies the 3D Euler equations then any arbitrary differentiable $\mu(\boldsymbol{x},\, t)$ satisfies*

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu) = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right).$$

**Proof :** Ertel (1942) ; Truesdell & Toupin (1960) ; Hoskins *et al* (1985) ; Ohkitani (1993) ; Kuznetsov & Zakharov (1997) ; Viudez (2001).

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu) = \left(\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\right)\cdot\nabla\mu + \boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right)$$

**Ohkitani (1993)** took $\mu \equiv u_i$ where $Du_i/Dt = -p_{,i}$ ; noting that $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}$

$$\frac{D(S\boldsymbol{\omega})}{Dt} = -P\boldsymbol{\omega}$$

**where the matrix $P = \{p_{,ij}\}$ is the pressure Hessian $p$.** Thus we have

$$\frac{\mathbf{D}\boldsymbol{\omega}}{\mathbf{Dt}} = \mathbf{S}\boldsymbol{\omega} \qquad \text{and} \qquad \frac{\mathbf{D}(\mathbf{S}\boldsymbol{\omega})}{\mathbf{Dt}} = -\mathbf{P}\boldsymbol{\omega}\,.$$

The gradient matrix $M_{ij} = \partial u_j / \partial x^i$ satisfies ($\mathrm{tr}\, P = -\mathrm{tr}\,(M^2)$)

$$\frac{DM}{Dt} + M^2 + P = 0\,, \qquad\qquad \mathrm{tr}\, M = 0\,.$$

Attempts have been made to model the pressure Hessian by **a constitutive closure.** The idea goes back to :

Léorat (1975) ; Vieillefosse (1984) ; Cantwell (1992)

who assumed that the Eulerian pressure Hessian $P = \{p_{,ij}\}$ is isotropic. This results in the *restricted Euler equations* with

$$P = -\tfrac{1}{3} I\, \mathrm{tr}\,(M^2)\,.$$

**Constantin's distorted Euler equations (1986) :** With the Lagrangian path map $\boldsymbol{a} \mapsto \boldsymbol{X}(\boldsymbol{a}, t)$, $N = M \circ \boldsymbol{X}$ Euler equns are re-written as

$$\frac{\partial N}{\partial t} + N^2 + Q(\boldsymbol{x}, t)\, Tr(N^2) = 0\,, \qquad Q_{ij} = R_i R_j \qquad R_i = (-\Delta)^{-1/2} \partial_i$$

'Distorted Euler equns' : replace $Q(t)$ by $Q(0) \Rightarrow$ rigorous blow up.

# Other models? Connection with the 2D surface QG equations?

In $\mathbb{R}^2$ if $\boldsymbol{u}$ is chosen as $\boldsymbol{u} = \nabla^\perp \psi$ with $\theta = -(-\Delta)^{1/2}\psi$, and $\boldsymbol{\mathcal{B}} = -\nabla^\perp \theta$ then

$$\frac{D\boldsymbol{\mathcal{B}}}{Dt} = \boldsymbol{\mathcal{B}} \cdot \nabla \boldsymbol{u} \qquad \text{vortex stretching term as in 3D Euler}$$

are the 2D surface QG equations of GFD discussed by Constantin, Majda & Tabak (1994) who conjectured that fronts might be finite time singularities.

1. Ohkitani & Yamada (1997) found numerically growth that was double exponential in time ;
2. Confirmed by Constantin, Nie & Schorghofer (1998) ;
3. Cordoba (1998) then showed the absence of a singularity for a *simple* hyperbolic saddle ;
4. See Cordoba, Fefferman & Rodrigo (2004) & Rodrigo (2004).

They suggested a principle : *If the level set topology in the temperature field for the 2D-QG active scalar in the region of strong scalar gradients does not contain a hyperbolic saddle, then no finite time singularity is possible".*

## The 3D incompressible Navier-Stokes equations

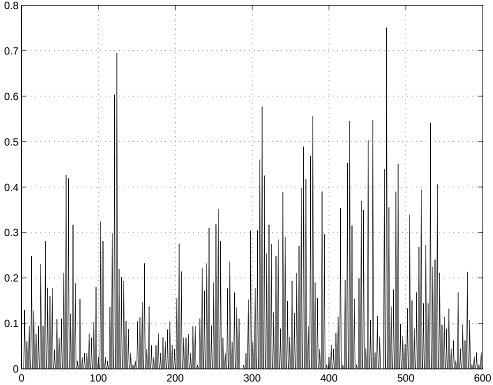
Consider the **3D Navier-Stokes equations** in the domain $[0, L]^3_{per}$

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x})\,; \qquad \text{div}\,\boldsymbol{u} = 0\,; \qquad (\text{div}\,\boldsymbol{f} = 0)$$

## Characteristics of turbulent NS-flows

1. Intermittent events are manifest as violent spiky surges away from space-time averages in both vorticity $\omega$ & strain $S_{i,j}$ – clustering.
2. Spectra have non-Gaussian characteristics (spatio-temporal clustering) :
   - The flatness in a Gaussian flow is 3 ;
   - Batchelor & Townsend 1949's wind tunnel measurements were $\leq 5.6$ ; Kuo & Corrsin 1971; Sreenivasan & Meneveau 1988/1991
   - The Forex market is $\approx 12+$
   - The atmospheric boundary layer is $\approx 40+$.
3. **NSE computations :** Orszag & Patterson 1972; Kerr 1985; Eswaran & Pope 1988, Jimenez *et al* 1993, Mohin & Mahesh 1998; Kurien & Taylor 2005; Ishihara *et al* 2009, Donzis *et al* 2008/10/12; Kerr 2013.

# Intermittency from wind tunnel data : (Hurst & Vassilicos)




Dissipation-range intermittency from wind tunnel turbulence where hot wire anemometry has been used to measure the longitudinal velocity derivative at a single point (D. Hurst & J. C. Vassilicos). The horizontal axis spans 8 integral time scales with $Re_\lambda \sim 200$.

# Visualization from the TAMU $4096^3$ data-base : Courtesy of Diego Donzis

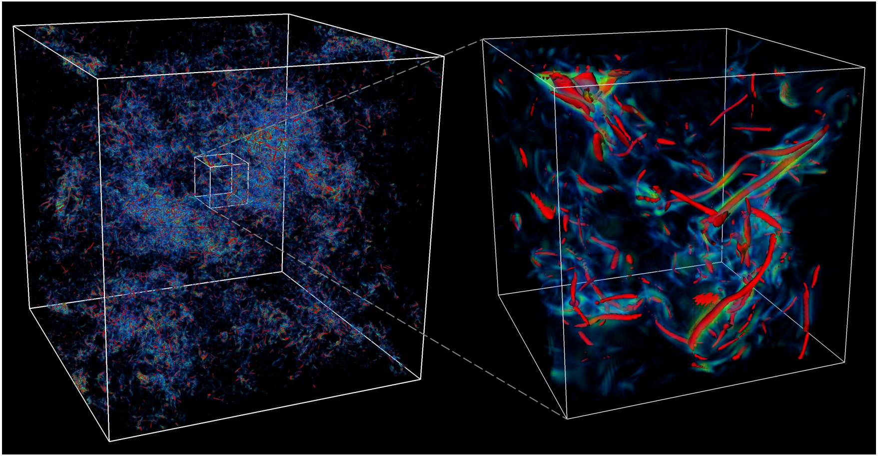


Figure: The enstrophy $|\boldsymbol{\omega}_i|^2$ (red) & the dissipation $2\nu S_{i,j}S_{j,i}$ (blue-green) at $Re_\lambda = 1000$.

# Vorticity iso-surfaces

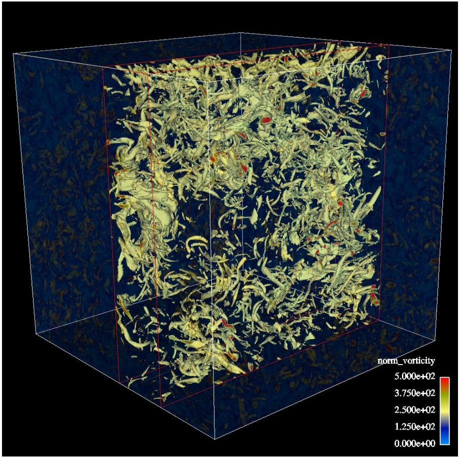



Figure: Vorticity iso-surfaces in a $512^3$ sub-domain of the LANL decaying $2048^3$ NS-simulation at $Re_\lambda \sim 200$ : **courtesy of Mark Taylor & Susan Kurien 2005**.

## Kolmogorov's arguments : (Frisch 1995)

From Leray's energy inequality the time-averaged energy dissipation rate $\varepsilon$ is

$$\varepsilon = \nu\, L^{-3} \left\langle \int_{\mathcal{V}} |\omega|^2 dV \right\rangle_T \qquad \text{where} \qquad \langle\cdot\rangle_T = \frac{1}{T}\int_0^T \cdot\, d\tau$$

& defines the inverse Kolmogorov length

$$L\lambda_k^{-1} = \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \sim Re^{3/4}\,.$$

It is based on the Foias-Doering (2002) result : for NS-solns $Gr \leq c\, Re^2$.

## Velocity structure functions

**Velocity structure functions** are used to study intermittency

$$\left\langle\, |u(\boldsymbol{x}+\boldsymbol{r}) - u(\boldsymbol{x})|^p \,\right\rangle_{ens.\,av.} \sim r^{\,\zeta_p}$$

$\zeta_p$ is a concave curve below linear ($p > 3$) : anomalous scaling.

1. In 1934 Leray introduced the idea of weak solutions of the 3D NSE, in the sense of distributions. Weak solutions are not ordinary functions but only exist in the sense that they can be integrated against a test function – one has to go through a process of a Galerkin expansion followed by a weak convergence proof.
2. **Weak solutions of 3D-NSE exist but there is no proof uniqueness :** if this is all there is then it produces a serious computational problem.
3. **Books :**
   - Constantin, Foias, *Navier-Stokes equns*, Chicago Univ Press 1988 ;
   - JDG/Doering, *Applied analysis of the Navier-Stokes equations*, CUP (1995) ;
   - Foias, Manley, Rosa & Temam, *Navier-Stokes equations & Turbulence*, CUP (2001) ;
   - Temam, *Infinite Dimensional Dynamical Systems in Mechanics & Physics*, vol 68, Springer-Verlag, (1988).
   - Robinson, *Infinite-dimensional dynamical systems*, CUP (2001).
   - Robinson, Rodrigo & Sadowski – to appear ... sometime

1. **History :** Leray (1934) ; Prodi (1962), Serrin (1963) & Ladyzhenskaya (1964) : every Leray-Hopf solution $\boldsymbol{u}$ of the 3D-NSE with

$$\boldsymbol{u} \in L^r\left((0,\ T)\,;L^s\right) \text{ is regular on } (0,\ T] \text{ provided } 2/r + 3/s = 1$$

with $s \in (3,\ \infty]$ or if $\boldsymbol{u} \in L^\infty\left((0,\ T)\,;L^p\right)$ with $p > 3$. **For weak solns : existence is known but uniqueness is open.**

2. When $s = 3$ : **Escauriaza, Seregin & Sverák (2003)** have settled the case of $L^\infty\left((0\,;T);L^3\right)$ using scaling & backward uniqueness.

3. **Modern developments :**
   (i) Assumptions on the pressure or 1 velocity derivative : Kukavica & Ziane (2006, 2007), Zhou (2002,2005), Cao & Titi (2008, 2010), Cao (2010), Cao, Qin, Titi (2008) ; Galdi (2002).
   (ii) Direction of vorticity : Constantin & Fefferman (1993) ; Vasseur (2008) ; da Veiga & Berselli (2002), da Veiga (2013).
   (iii) Seregin & Sverak (2002) : smooth solutions of the 3D-NSE remain smooth for as long as either the pressure is bounded from below, or the Bernoulli pressure $\frac{1}{2}|\boldsymbol{u}|^2 + p$ is bounded from above.

# NS regularity in contrast : what results do we have & what do we need?

1. We have regularity for short times or small i.d. (Ladyzhenskaya 1964) but we need to extend this to arbitrarily long times & large i.d.
2. "Leray-type methods" (CF88, FMRT01) give Leray's energy inequality

$$\tfrac{1}{2}\frac{d}{dt}\int_{\mathcal{V}} |\boldsymbol{u}|^2 dV \leq -\nu \int_{\mathcal{V}} |\nabla \boldsymbol{u}|^2 dV + \|\boldsymbol{u}\|_2 \|\boldsymbol{f}\|_2$$

   from which we conclude

$$\langle H_1 \rangle_T < \infty \qquad\qquad H_n = \int_{\mathcal{V}} |\nabla^n \boldsymbol{u}|^2 dV = \int_{\mathcal{V}} |\nabla^{n-1} \boldsymbol{\omega}|^2 dV$$

   **but we need $H_1 < \infty$ pointwise in time. Contrast with the BKM condition $\int_0^T \|\boldsymbol{\omega}\|_{L^\infty}\, d\tau < \infty$ for Euler.**
3. We have $\left\langle H_n^{\frac{1}{2n-1}} \right\rangle_T < \infty$ (FGT 1981) but we need $\left\langle H_n^{\frac{2}{2n-1}} \right\rangle_T < \infty$ .
4. We have $\langle \|\boldsymbol{u}\|_\infty \rangle_T < \infty$ but we need $\langle \|\boldsymbol{u}\|_\infty^2 \rangle_T < \infty$ .
5. We have $\left\langle \|\boldsymbol{\omega}\|_\infty^{1/2} \right\rangle_T < \infty$ but we need $\langle \|\boldsymbol{\omega}\|_\infty \rangle_T < \infty$ (BKM) .
6. We have pointwise control over $\|\boldsymbol{u}\|_2$ but we need control over $\|\boldsymbol{u}\|_3$ (Escauriaza, Seregin & Sverák 2003).

- Scheffer (*Pacific J. Math.* 66, 535, 1976) – see also Leray (1934) – showed that the singular set in time has zero $\frac{1}{2}$-dimensional Hausdorff measure: see chapter 10 of Constantin & Foias (1988): pgs 93-95.

- **Caffarelli, Kohn & Nirenberg** (*CPAM* 35, 771–831, 1982), based on the partial regularity concept of 'suitable weak solns' (following Scheffer's (1980) estimate of $5/3$), showed that the singular set in **space-time has zero 1-dim Hausdorff measure**.

- For further papers on this topic see also Ladyzhenskaya-Seregin (1999); Seregin (2001-05); Lin (1998); Choe-Lewis (2000); Vasseur (2007); Kukuvica (2008); Robinson & Rodrigo (2009).

- **Conclusion :** Singularities are rare events, if they occur at all.

**Physics literature :** Mandelbrot (1976); Halsey (1986); Frisch-Parisi (88) & Sreenivasan-Meneveau (91) suggest dissipation concentrates on fractal sets.

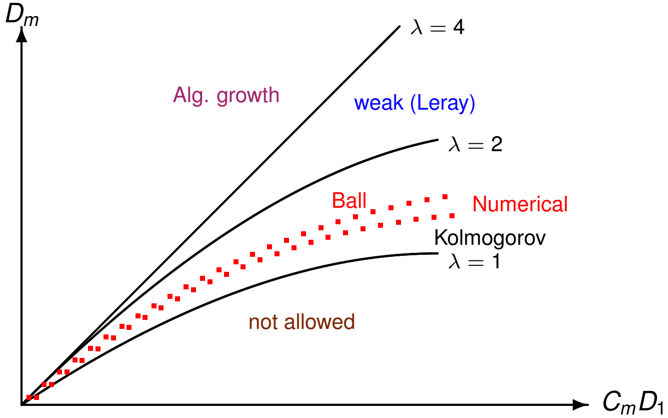
## Lastly : some of my own recent collaborative work

High moments of vorticity are of interest : let $\varpi_0 = \nu L^{-2}$

$$\Omega_m = \left(L^{-3}\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m} dV\right)^{1/2m} \qquad D_m = \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m} \qquad \alpha_m = \frac{2m}{4m-3}\,.$$

The $D_m$ have the following properties :

1. Analytically : $\langle D_m\rangle \leq c\,Re^3$ (JDG 2011) ;
2. Numerically, in 3 independent experiments ($1024^2 \times 2048$ ; $512^3$; $4096^3$), they sit on a descending scale $D_{m+1} < D_m$ such that
$$D_m \leq D_1^{a_m}\,.$$
(Donzis, Gupta, JDG, Kerr, Pandit & Vincenzi JFM2013, Arxiv :1402.1080v2 (2014)).
3. Moreover, Hölder's inequalities suggests a *scaling ansatz* :
$$a_m = \frac{m\lambda + 1 - \lambda}{4m-3} \qquad 1 \leq \lambda \leq 4$$
Note that $\lambda = 1$ is the lower bound $\Omega_1 \leq \Omega_m$.




$$\dot{D}_1 \leq \varpi_0 \left( -\frac{D_1^2}{E} + D_1^{1+\frac{1}{2}\lambda} \right) \qquad E = L^{-1}\nu^{-2} \int_{\mathcal{V}} |\boldsymbol{u}|^2 dV \leq Gr^2$$

| | $\mathcal{E} \sim k^{-q}$ | $L \langle \kappa \rangle$ |
|---|---|---|
| | $q = 3 - 4/3\lambda$ | $Re^{3\lambda/4}$ |
| $\lambda = 1$ | $5/3$ | $Re^{3/4}$ |
| $\lambda = 1.5$ | $19/9$ | $Re^{9/8}$ |
| $\lambda = 4$ | $8/3$ | $Re^{3}$ |

# Final conclusions

## Euler equations

1. A direct analytical proof of $3D$ Euler blow-up/non-blow-up is required.
2. Do boundaries make a difference?

## NS equations

1. By scaling arguments, Leray-Hopf methods seem to reach the $\|\boldsymbol{u}\|_3$-result but no lower (Escauriaza, Seregin & Sverák 2003).
2. Other than the "direction of vorticity" arguments of Constantin & Fefferman (1993) little is known about local behaviour of vorticity :
   - Numerics suggest solutions are regular & act on low-dim sets.
   - Why does vorticity & strain accumulate into tubes?
   - Why does the sign of $\Delta p$ change from +ve (inside a tube) to -ve (outside a tube)?
   - What is significant about this topology?

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