Chapter 4

Problems

1. Let us assume that in an epidemic, infected individuals can infect healthy ones at rate α, and they can also recover at rate β. As we saw in the previous section, the epidemic dies out for $\alpha \leq \beta$, but goes on forever for $\alpha > \beta$.

   (a) In the critical case, $\alpha = \beta$, an infected person infects someone or recovers with the same probability 1/2, and the outbreak ends with probability one. How big is the size of the outbreak? In other words, if N is the total number of infected people during an outbreak, what is its distribution $A_n = P(N = n)$?

   (b) What is the average outbreak size?

   (c) Obtain the outbreak size distribution for general rates α, β. What is $F(1)$ in this case? What is the average outbreak size?

2. Let us model virus spreading within a body. We call virus particles (DNA in a capsid) A, and infected cells B. After an exponential waiting time with parameter c, a virus particle either infects a cell (say w.p. p) or it doesn't (w.p. $1 - p$). An infected cell upon death produces N new viruses. Hence a simple model of this process is

$$A \xrightarrow{c} \begin{cases} B & \text{w.p. } p \\ \emptyset & \text{w.p. } 1-p \end{cases} \qquad B \xrightarrow{1} \overbrace{AAA\ldots A}^{N}$$

   (a) Write down the backward Kolmogorov equations for the generating functions $F_A(x, y, t)$ and $F_B(x, y, t)$.

   (b) Write down equations for the eventual extinction probabilities $q_A = F_A(0, 0, \infty)$ and $q_B = F_B(0, 0, \infty)$.

   (c) Derive an equation for q_A and look for the smallest non-negative solution. Show that $q_A < 1$ exists only for $R = pN > 1$. [1]

   (d) Try to sketch $q_A(p)$: the probability that a single virus particle cannot infect the body, in terms of the cell infection probability (virulence).

[1] The parameter R is called the *basic reproductive ratio*, which tells us the average number of virus particles an initial virus becomes during a single "cycle".