Anomalous is Ubiquitous

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A panoramic tour of the scenic mountainous terrain of the Diffusion Kingdom

Joint research with *Joseph Klafter* TAU

Brownian Diffusion

Diffusion

- The most elemental random transport processes in Science and Engineering are **diffusions**
- The archetypal model of diffusion processes is **Brownian motion** (BM)
- **BM** is applied in a host of scientific fields – from Physics and Chemistry to Biology and Finance

Van Kampen (2007), Schuss (2009), ···

Brownian Landmarks

- Discovery (Brown 1827)
- Financial modeling (Bachelier 1900)
- Diffusion modeling (Einstein-Smoluchowski 1905)
- Mathematical construction (Wiener 1923)
- Stochastic calculus (Ito 1940s)
- Geometric BM (Samuelson 1965)
- Option pricing (Merton-Black-Scholes 1973)

Brownian Motion

Brownian motion (BM) is a random process whose increments are

- independent, stationary, and Gaussian

Brownian noise (BN) – the 'discrete derivative' of BM – is a sequence of

- i.i.d. Gaussian random variables

Brownian Universality

Random Walks:

$$W(t) = Z_1 + \cdots + Z_{[t]} \quad (t \geq 0)$$

Zs i.i.d. with zero mean and finite variance

Scaling:

$$W(t) = (1/\sqrt{n})W(nt)$$

Functional CLT:

Brownian motion is the universal stochastic scaling limit of random walks (Donsker 1951)

Macroscopic BM statistics emerge invariantly with respect to the microscopic RW statistics

BM reigned supreme the Diffusion Kingdom for almost a century

Anomalous Diffusion

In recent decades non-Brownian upheavals have been quaking the Diffusion Kingdom with ever increasing intensity

Bouchaud-Georges 1990,
Metzler-Klafter 2000, Klafter-Sokolov 2005, …

Anomalous Diffusion

Agenda

- Mean Square Displacements
- Auto Correlations
- Power Spectra
- Marginal Distributions

Mean Square Displacements

Brownian motion MSD:

$$E[|B(t)|^2] = ct$$

Brownian noise MSD:

$$E[|\Delta B(t)|^2] = c$$

'Anomalous' MSD:

$$E[|\xi(t)|^2] = ct^{\varepsilon}$$

Mean Square Displacements

Sub diffusion:

$0<\varepsilon<1$

- charge transport in amorphous semiconductors
  (Scher-Montroll 1975)
- contaminants propagation in ground water
  (Kirchner-Feng-Neal 2006)
- protein motion in intracellular media
  (Golding-Cox 2006) …

Mean Square Displacements

Super diffusion:

$\varepsilon > 1$

- tracer transport in turbulent flows
  (Solomon-Weeks-Swinney 1993)
- search patterns of foraging animals
  (Sims et. al. 2008)
- particle trajectories in plasma
  (Liu-Goree 2008) …

Auto Correlations

Brownian noise AC:

$$\mathrm{Cov}[\Delta B(t),\Delta B(t+l)] = 0$$

'Anomalous' AC:

$$\mathrm{Cov}[\xi(t),\xi(t+l)] \approx c/l^{\varepsilon}$$

Long-range dependence:

$$0<\varepsilon<1$$

The "Joseph effect" (Mandelbrot-Wallis 1968)

Auto Correlations

Long-range dependence:

- human walk (Hausdorff et. al. 1995)
- sedimentation (Segre et. al. 1997)
- atmospheric temp. variations (Bunde et. al. 1998)
- price volatility (Gopikrishnan et. al. 1999)
- heartbeat dynamics (Ivanov et. al. 2001)
- seismic coda (Campillo-Paul 2003) ···

Power Spectra

Brownian motion PS:

$$S_B(f) = c/|f|^2$$

Brownian noise PS:

$$S_{\Delta B}(f) = c$$

'Anomalous' PS:

$$S_\xi(f) = c/|f|^\varepsilon$$

Power Spectra

- Power-law power spectra are termed:

“1/f noise”

- 1/f noise is ubiquitous – over 1400 citations in

http://www.nslij-genetics.org/wli/1fnoise

- Typical exponent range:

$0<\varepsilon<2$

BN ‘boundary’ $\varepsilon=0$, BM ‘boundary’ $\varepsilon=2$

Marginal Distributions

Brownian motion/noise MD:

Gaussian – characterized by Fourier transform

$$F_B(\theta) = \exp(-c|\theta|^2)$$

'Anomalous' MD:

Levy – characterized by Fourier transform

$$F_\xi(\theta) = \exp(-c|\theta|^\varepsilon)$$

Exponent range:

$$0<\varepsilon<2$$

Marginal Distributions

Gaussian tails – super-exponential decay:

$$\Pr(|B(t)|>l) \approx \exp(-cl^2)$$

Levy 'heavy tails' – power-law decay:

$$\Pr(|\xi(t)|>l) \approx c/l^{\varepsilon}$$

The "Noah effect" (Mandelbrot-Wallis 1968)

Marginal Distributions

Levy marginal distributions:

- anomalous transport (Shlesinger et. al. 1993)
- plasma dynamics (Chechkin et. al. 2002)
- solar wind/flares (Watkins/Weron et. al. 2005)
- bill trajectories (Brockmann et. al. 2006)
- search processes (Condamin et. al. 2007)
- light scattering (Barthelemy et. al. 2008) ⋯

‘Anomalous’ stochastic phenomena as well as statistical selfsimilarity are widely observed also in internet-age data traffic

Leland et. al. 1994, Paxon-Floyd 1994,
Crovella-Bestavros 1996, Willinger et. al. 1997, …

Fractional Brownian Diffusion

Fractional Brownian Motion

Fractional Brownian motion (FBM) is a random process whose increments are

- dependent, stationary, and Gaussian

Fractional Brownian noise (FBN) – the 'discrete derivative' of FBM – is a sequence of

- stationary Gaussian random variables

(Mandelbrot-Van Ness 1968)

Fractional Brownian Motion

FBM is selfsimilar with Hurst exponent: $0<H<1$

- **$0<H<\frac{1}{2}$**

  negative correlation – anti-persistence

- **$H=\frac{1}{2}$**

  zero correlation – Brownian motion

- **$\frac{1}{2}<H<1$**

  positive correlation – persistence

Anomalous Behavior

Mean Square Displacement:

- anti-persistence => FBM *sub-diffusion*
- persistence => FBM *super-diffusion*

Correlations:

- persistence => FBN *long-range dependence*

Power Spectrum:

- anti-persistence/persistence => FBM *1/f noise*

Attaining FBM

FBM is attained as the stochastic scaling limit of superpositions of i.i.d. random processes:

- renewal processes (Mandelbrot 1969)
- correlated random walks (Davydov 1970)
- on-off processes (Taqqu et. al. 1997)
- OU processes (Leonenko-Taufer 2005) ···

In the superposition models yielding FBM a pre-set target output (FBM) is attained from specific classes of input processes

Signal Superposition Model

Signal Superposition Model

- A multitude of transmission sources
- Source k transmits the signal pattern

$$\mathbf{X}_k = (\mathbf{X}_k(t))_{t \geq 0}$$

- The transmission parameters of source k are:

  amplitude $\mathbf{a}_k$

  frequency $\boldsymbol{\omega}_k$

  initiation epoch $\boldsymbol{\tau}_k$

Signal Superposition Model

- The output process

$$Y = (Y(t))_{t\geq 0}$$

is the superposition of all source-transmissions:

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

Model Assumptions

- The signal patterns $\{X_k\}_k$ are i.i.d. copies of a general and arbitrary signal pattern

$$\mathbf{X} = (\mathbf{X}(\mathbf{t}))_{t\geq 0}$$

- The transmission parameters $\{(a_k,\omega_k,\tau_k)\}_k$ are a Poisson process with general intensity function

$$\lambda(a,\omega,\tau)$$

- The signal patterns and transmission parameters are mutually independent

Model Assumptions

Why Poisson?

- In large systems it is natural to assume that the transmission parameters $\{(a_k,\omega_k,\tau_k)\}_k$ are random
- The common modeling of random scattering of points in general domains is Poissonian
- Poisson processes have a wide spectrum of applications in Science and Engineering

(Kingman 1993, Wolff 1989, Embrechts et. al. 1997)

The signal superposition model is general and accommodates a span of stochastic models

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

Random Walks:

- Signal patterns: $X_k(t) = Z_k \cdot \chi(t>0)$

  $\{Z_k\}_k$ i.i.d. with zero mean and finite variance
- Amplitudes: $a_k = 1/\sqrt{n}$
- Frequencies: irrelevant
- Initiation epochs: Poisson process with rate **n**

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

Shot Noise processes:

- Signal patterns: $X_k(t) = \exp(-t)$
- Amplitudes: i.i.d. random variables
- Frequencies: $\omega_k = r$

  r relaxation constant
- Initiation epochs: Poisson process with unit rate

(Rice 1944-45, Lowen-Teich 1989-90, Eliazar-Klafter 2007)

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

M/G/∞ processes:

- Signal patterns: $X_k(t) = \chi(0<t<1)$
- Amplitudes: $a_k = 1$
- Frequencies: $\omega_k = 1/T_k$

  $\{T_k\}_k$ i.i.d. sojourn times
- Initiation epochs: Poisson process with unit rate

(Takacs 1962, Cox-Miller 1965, Cox-Isham 1980)

The signal superposition model

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

is a generalized version of the Einstein-Smoluchowski diffusion model

Universality

Universality

Considering a specific statistic **Φ(Y)** of the output process **Y** of the signal superposition model:

- Are there Poissonian parameter-randomizations that render the output-statistic **Φ(Y)** invariant with respect to the input signals **$\{X_k\}_k$** ?
- If yes, then what is the form of the resulting input-invariant output-statistic **Φ(Y)** ?

Universality

Agenda

- Mean Square Displacements
- Auto Correlations
- Power Spectra
- Marginal Distributions

Mean Square Displacements

$\Phi(\mathbf{Y})$ is the MSD of the output process

Theorem: Poissonian parameter-randomizations that render the output MSD invariant with respect to the input signal patterns exist and yield the power-law MSD

$$\mathbf{E[|Y(t)|^2] = c_X t^{\varepsilon}}$$

with exponent $\varepsilon>0$

Auto Correlations

$\Phi(\mathbf{Y})$ is the AC of stationary output processes

Theorem: Poissonian parameter-randomizations that render the output AC invariant with respect to the input signal patterns exist and yield the power-law AC

$$\mathrm{Cov}[Y(t),Y(t+l)] = c_X/l^{\varepsilon}$$

with exponent $0<\varepsilon<1$

Power Spectra

$\Phi(\mathbf{Y})$ is the PS of the output process

Theorem: Poissonian parameter-randomizations that render the output PS invariant with respect to the input signal patterns exist and yield the power-law PS

$$S_Y(f) = c_X/|f|^{\varepsilon}$$

with exponent $0<\varepsilon<2$

Marginal Distributions

$\Phi(\mathbf{Y})$ are the MDs of the output process

Theorem: Poissonian parameter-randomizations that render the output MDs invariant with respect to the input signal patterns exist and yield the Fourier transform

$$\mathbf{E[\exp(i\theta Y(t))] = \exp(-c_X|\theta|^{\varepsilon})}$$

with exponent $0<\varepsilon<2$ (here $\mathbf{c_X}$ depend also on $\mathbf{t}$)

MAGIC

Statistic considered:

- Mean Square Displacements
- Auto Correlations
- Power Spectra
- Marginal Distributions

Invariance result:

- Sub diffusion and super diffusion
- Long-range dependence
- 1/f noise
- Levy laws

M A G I C

- **Perfect coincidence between 'anomalous' empirical observations and theory**
- **Universal generation of 'anomalous' behavior via a single model and one approach**
- **'Anomalous' behavior emergent rather than pre-set as desired goal**
- **Robustness and resilience with respect to 'environmental changes'**

Emergence of Power Laws

‘Meta conclusion’:

Input-invariance universally yields Power Laws:

‘ $\Phi(Y) = c_X \cdot$ Power-Law ’

Einstein-Smoluchowski perspective:

The only macroscopic statistics which are invariant with respect to microscopic statistics are ‘anomalous’!

Katja Lindenberg **(UCSD):**

"Anomalous is Normal!"

(Phys. Rev. Lett. 2004, 2005, 2011)

Sources

Iddo Eliazar & Jospeh Klafter:

- PNAS 106 (2009) 12251
- J. Phys. A 42 (2009) 472003
- J. Phys. A 43 (2010) 132002
- Phys. Rev. E 82 (2010) 021109

Ultra Diffusion

Ultra Diffusion

BM characterization:

- Variance characterization of BM:

$$\mathrm{Var}[B(t)] = c \cdot t$$

- Fourier characterization of BM:

$$E[\exp(i\theta B(t))] = \exp(-c \cdot t \cdot |\theta|^2/2)$$

- Fourier characterization does *not* require a finite-variance condition to hold

Ultra Diffusion

Definition:

A stochastic process $(\xi(t))_{t\geq 0}$ is said to be an ultra diffusion if

$$\mathbf{E[\exp(i\theta\xi(t))] = \exp(-c\cdot\psi(t)\cdot\phi(\theta))}$$

In the finite-variance case

$$\mathbf{Var[\xi(t)] = (c\phi''(0))\cdot\psi(t)}$$

Ultra Diffusion

Examples:

- Poisson and compound Poisson processes
- Brownian motion and FBM
- Stable Levy motions and FSLM
- General Levy processes
- OU processes driven by stable Levy noises
- $M/G/\infty$ processes

Ultra Diffusion

Theorem: Poissonian parameter-randomizations that render the output process **Y** an ultra diffusion, invariantly with respect to the input signal patterns $\{X_k\}_k$, exist and yield the Fourier transform

$$E[\exp(i\theta Y(t))] = \exp(-c_X \cdot t^{\alpha} \cdot |\theta|^{\beta})$$

with exponents $\alpha>0$ and $0<\beta<2$

Eliazar & Klafter, J. Phys. A 43 (2010) 132002

Ultra Diffusion

Conclusions:

- Joint universal emergence of ‘sub/super diffusion’ and Levy laws:

$$\psi(t) = t^{\alpha} \quad \& \quad \phi(\theta) = |\theta|^{\beta}$$

- Intrinsic scaling (equality in law):

$$Y(t) = t^{\alpha/\beta} \cdot Y(1)$$

- Probability density at the origin:

$$f_{Y(t)}(0) = f_{Y(1)}(0) / t^{\alpha/\beta}$$

Self Similarity

Self Similarity

Definition:

A stochastic process $(\xi(t))_{t\geq 0}$ is said to be selfsimilar with Hurst exponent H if the scaled processes

$$(Y(st))_{t\geq 0} \quad \text{and} \quad (s^{H}Y(t))_{t\geq 0}$$

are equal in law (for any positive scale s)

Self Similarity

Examples:

- Brownian motion
- Fractional Brownian motion
- Stable Levy motions
- Fractional Stable Levy motions

Self Similarity

Theorem:

The output process **Y** is selfsimilar, with Hurst exponent *H*, invariantly of the input signal patterns $\{X_k\}_k$, if and only if the Poissonian intensity satisfies the scaling relation:

$$\lambda(s^H a,\omega/s,s\tau) = \lambda(a,\omega,\tau)/s^H$$

Eliazar & Klafter, Phys. Rev. Lett. 103 (2009) 040602

Finite-Variance Corollaries

- Selfsimilarity => sub/super diffusion:

$$E[|Y(t)|^2] = c_X t^{2H}$$

- Selfsimilarity => 1/f noise:

$$S_Y(f) = c_X/|f|^{1+2H}$$

- Selfsimilarity, stationary increments, $½<H<1$
  => long-range dependence of the 'derivative':

$$Cov[\Delta Y(t),\Delta Y(t+l)] \approx c_X/l^{2-2H}$$

Beyond Diffusion

The concept of input-invariant Poissonian randomizations is extremely powerful and is applicable well beyond the Diffusion Kingdom

Additional Applications

Propagating Populations:

- Eliazar & Klafter, Phys. Rev. E 82 (2010) 011112

Search Swarms:

- Eliazar & Klafter, J. Phys. A 44 (2011) 222001

Randomized CLTs:

- Eliazar & Klafter, Chem. Phys. 370 (2010) 290
- Eliazar & Klafter, Phys. Rev. E 82 (2010) 021122

Additional Applications

Ultraslow diffusions:

➢ Eliazar & Klafter, J. Phys. A 44 (2011) 405006

Log-Levy distributions and extreme randomness:

➢ Eliazar & Klafter, J. Phys. A 44 (2011) 415003

Stochastic flow cascades:

➢ Eliazar & Shlesinger, J. Stat. Phys. 146 (2012) 1

Spatially inhomogeneous relaxation processes:

➢ Eliazar & Benichou, J. Phys. A 45 (2012) 015003

Propagating Populations

Populations of propagating particles in the **d**-dimensional Euclidean space:

$$Y_k(t) = a_k X_k(\omega_k(t-\tau_k)) \quad (t \geq \tau_k)$$

- Invariance of the particles' displacements with respect to the spatial trajectories $\{X_k\}_k$
- Invariance of the particles' first passage times with respect to the spatial trajectories $\{X_k\}_k$

Search Swarms

Swarms of agents in a general topological space:

$$Y_k(t) = X_k(\omega_k(t-\tau_k)) \quad (t \geq \tau_k)$$

- The agents are observed only when in a target zone – a general subset of space
- Invariance of the agents' 'target-zone statistics' with respect to the spatial trajectories $\{X_k\}_k$

Randomized CLTs

The **stable Levy laws** and the **Extreme Value laws** are the universal stochastic limit laws of sums and extrema of i.i.d. random variables

CLT approach:

- deterministic and uniform scaling
- narrow domains of attraction

RCLT approach:

- Poissonian random and non-uniform scaling
- wide domains of attraction

The Diffusion Kingdom is vast, and we just spent a one-hour tour of the ‘anomalous’ peaks surrounding its Brownian center

Reading

Iddo Eliazar & Joseph Klafter

Anomalous is ubiquitous

Annals of Physics 326 (2011) 2517-2531

Iddo Eliazar & Joseph Klafter

A probabilistic walk up power laws

Physics Reports 511 (2012) 143-175

The End

Thank you