Multifractal inference

Ola Løvsletten

joint work with Martin Rypdal

Spring, 2012

UNIVERSITY OF TROMSØ

About our project

- Phd student at UiT with M. Rypdal as supervisor
- Employed at a local bank, a co-operation project supported by the national research council.
- In the bank I work in the risk management group which counts $5 + \frac{1}{4}$ positions. (I am the $\frac{1}{4}$).
- Specfically I work with market risk (losses due to changes in observerable variables).
- Topic of research: Modern statistical methods in finance, which translates into multifractals.

Overview and outline

- 'Approximated maximum likelihood estimation in multifractal random walks' , Løvsletten and Rypdal 2012, Physical Review E.
- Start with recalling what MLE is, and some simpler examples, fractional Brownian motions:
  - Computation of the likelihood (pdf)
  - Some practical information
  - A small Monte Carlo study, MLE vs. moment estimator
- Multifractals
  - Motivation in finance
  - Properties
  - Different models and inference.
  - Monte Carlo and practical info
- Current research

Overview and outline

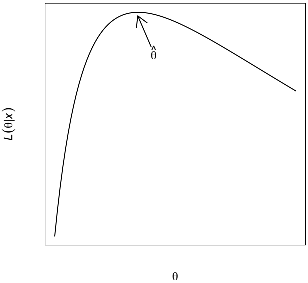
- 'Approximated maximum likelihood estimation in multifractal random walks' , Løvsletten and Rypdal 2012, Physical Review E.
- Start with recalling what MLE is, and some simpler examples, fractional Brownian motions:
  - Computation of the likelihood (pdf)
  - Some practical information
  - A small Monte Carlo study, MLE vs. moment estimator
- Multifractals
  - Motivation in finance
  - Properties
  - Different models and inference.
  - Monte Carlo and practical info
- Current research

Preliminary

Situation: We have some data $\mathbf{x} = (x_1, \ldots, x_n)$, a model density $p(\mathbf{x}|\theta) : \mathbb{R}^n \to \mathbb{R}_+$ and wish to estimate the parameter vector θ. One popular method is maximum likelihood estimatation (MLE) :

$$\hat{\theta} = \arg\max_{\theta} L(\theta|\mathbf{x}),$$

where $L(\theta|\mathbf{x}) = p(\mathbf{x}|\theta)$ with $\mathbf{x}$ fixed (the data plugged into the pdf).



Example 1: Brownian motion with drift

- Model:

$$dX(t) = \mu dt + \sigma dB(t)$$

- Define $x_t = X(t) - X(t-1)$
- The random variables $x_1, \dots, x_n$ are independent $\mathcal{N}(\mu, \sigma^2)$.
- Analytical expression for the likelihood from which we can find the maximum by differentiation.
- ML estimates: $\hat{\mu} = n^{-1} \sum x_i \quad, \hat{\sigma}^2 = n^{-1} \sum (x_i - \bar{x})^2$
- Sidenote: Louis Bachelier, in his study of financial prices, suggested this model as early as $\sim$ 1900. It was later realized that it was the logarithmic prices, rather than the prices themselves, that should be modeled as a Brownian motion.

Example 1: Brownian motion with drift

- Model:

$$dX(t) = \mu dt + \sigma dB(t)$$

- Define $x_t = X(t) - X(t-1)$
- The random variables $x_1, \ldots, x_n$ are independent $\mathcal{N}(\mu, \sigma^2)$.
- Analytical expression for the likelihood from which we can find the maximum by differentiation.
- ML estimates: $\hat{\mu} = n^{-1} \sum x_i \quad , \hat{\sigma}^2 = n^{-1} \sum (x_i - \bar{x})^2$
- Sidenote: Louis Bachelier, in his study of financial prices, suggested this model as early as $\sim$ 1900. It was later realized that it was the logarithmic prices, rather than the prices themselves, that should be modeled as a Brownian motion.

Example 2: fractional Brownian motion

- Model:

$$dX(t) = \mu dt + \sigma dB_H(t)$$

- $B_H(t)$ a fractional Brownian motion:
  - a Gaussian process with longe-range dependent increments
- $x_t = X(t) - X(t-1)$ a fractional Gaussian noise.
- The Hurst-exponent H is a measure of the dependence between x_t and x_s:
  - $H < 0.5$, anti-correlated
  - $H = 0.5$, uncorrelated (and independent-why?)
  - $H > 0.5$, positive correlation
- Let Σ_n to be the covariance-matrix of $(x_1, \dots, x_n)$.
- The pdf:

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma_n^{-1}(\mathbf{x}-\mu)\right)$$

Example 2: fractional Brownian motion

- Model:

$$dX(t) = \mu dt + \sigma dB_H(t)$$

- $B_H(t)$ a fractional Brownian motion:
  - a Gaussian process with longe-range dependent increments
- $x_t = X(t) - X(t-1)$ a fractional Gaussian noise.
- The Hurst-exponent H is a measure of the dependence between x_t and x_s:
  - $H < 0.5$, anti-correlated
  - $H = 0.5$, uncorrelated (and independent-why?)
  - $H > 0.5$, positive correlation
- Let Σ_n to be the covariance-matrix of $(x_1, \ldots, x_n)$.
- The pdf:

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma_n^{-1}(\mathbf{x}-\mu)\right)$$

Example 2: fractional Gaussian noise

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma_n^{-1}(\mathbf{x}-\mu)\right)$$

- Problem: Inversion of the covariance matrix Σ_n.
- Consider the structure of the covariance matrix:

$$\Sigma_n(i,j) = \mathbb{E}(x_i x_j) - \mu^2 = \gamma(|i-j|), \quad (1)$$

  so all the entries at a specific diagonal are equal.
- Reduces the computational cost of evaluating the likelihood to $\mathcal{O}(n^2)$ by the Durbin Levinson algorithm.
- The DL algorithm can also be used to simulate stationary Gaussian processes.

Example 2: fractional Gaussian noise

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma_n^{-1}(\mathbf{x}-\mu)\right)$$

- Problem: Inversion of the covariance matrix Σ_n.
- Consider the structure of the covariance matrix:

$$\Sigma_n(i,j) = \mathbb{E}(x_i x_j) - \mu^2 = \gamma(|i-j|), \tag{1}$$

  so all the entries at a specific diagonal are equal.
- Reduces the computational cost of evaluating the likelihood to $\mathcal{O}(n^2)$ by the Durbin Levinson algorithm.
- The DL algorithm can also be used to simulate stationary Gaussian processes.

Example 2: fractional Gaussian noise

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma_n^{-1}(\mathbf{x}-\mu)\right)$$

- Problem: Inversion of the covariance matrix Σ_n.
- Consider the structure of the covariance matrix:

$$\Sigma_n(i,j) = \mathbb{E}(x_i x_j) - \mu^2 = \gamma(|i-j|), \tag{1}$$

  so all the entries at a specific diagonal are equal.
- Reduces the computational cost of evaluating the likelihood to $\mathcal{O}(n^2)$ by the Durbin Levinson algorithm.
- The DL algorithm can also be used to simulate stationary Gaussian processes.

Example 2: fractional Gaussian noise

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma_n^{-1}(\mathbf{x}-\mu)\right)$$

- Problem: Inversion of the covariance matrix Σ_n.
- Consider the structure of the covariance matrix:

$$\Sigma_n(i,j) = \mathbb{E}(x_i x_j) - \mu^2 = \gamma(|i-j|), \tag{1}$$

  so all the entries at a specific diagonal are equal.
- Reduces the computational cost of evaluating the likelihood to $\mathcal{O}(n^2)$ by the Durbin Levinson algorithm.
- The DL algorithm can also be used to simulate stationary Gaussian processes.

Some practical information

- I use the freely available software R.
- There is written a package (add-on) called LTSA, Linear Time Series Analysis, which contains the aforementioned Durbin-Levinson algorithm.
- Another interesting package is FGN (which uses LTSA).
- Simulating fractional Gaussian noise and using MLE is two lines of code:

  ```
  x<-SimulateFGN(n=10^4,H=0.9)
  FitFGN(x)
  ```
- McLeod et.al (2007) Algorithms for Linear Time Series Analysis: With R Package, Journal of statistical software

A small Monte Carlo study

fractional Brownian motion (fBm)

- For a fBm $B_H(t)$ we have that

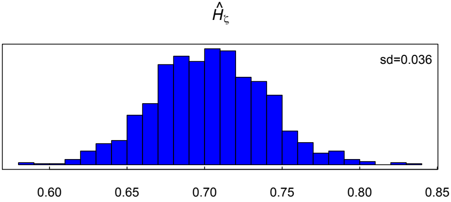
$$\mathbb{E}|B_H(t+\delta t) - B_H(t)|^q \propto \delta t^{\zeta(q)}$$

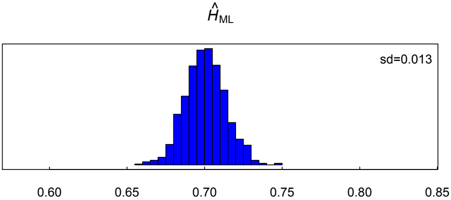
  with $\zeta(q) = Hq$.
- As an alternative to MLE we consider the moment-estimator $\hat{H}_\zeta = \hat{\zeta}(2)/2$
- $\hat{\zeta}(2)$ found by a least square fit to the second order sample moments of $B_H(t+\delta t)$.

A small Monte Carlo study

fractional Brownian motion (fBm)

Simulated $n_{\mathrm{MC}} = 500$ sample paths, each with $n = 2500$ and $H = 0.7$. For each realization the Hurst exponent is estimated by the two estimators.



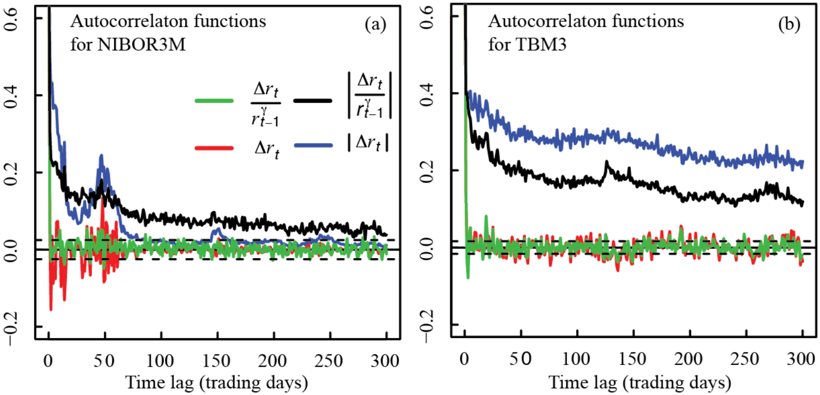


Overview and outline

- 'Approximated maximum likelihood estimation in multifractal random walks' , Løvsletten and Rypdal 2012, Physical Review E.
- Start with recalling what MLE is, and some simpler examples, fractional Brownian motions:
  - Computation of the likelihood (pdf)
  - Some practical information
  - A small Monte Carlo study, MLE vs. moment estimator
- Multifractals
  - Motivation in finance
  - Properties
  - Different models and inference.
  - Monte Carlo and practical info
- Current research

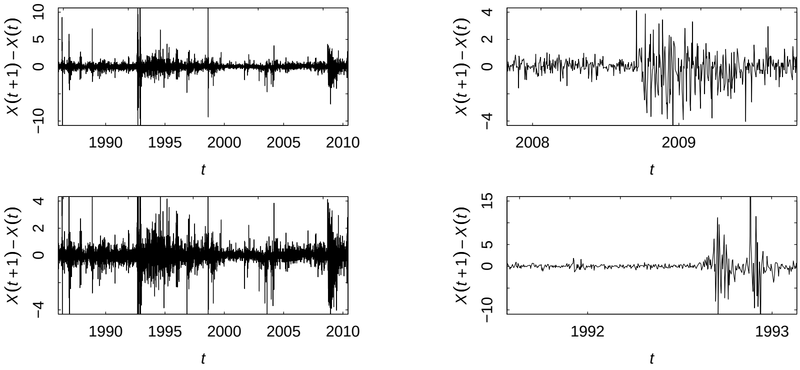
Time-varying volatility

An empirical look



Time-varying volatility

An empirical look



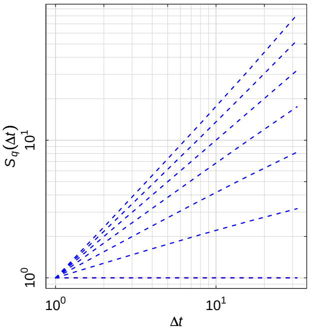
Multifractal processes

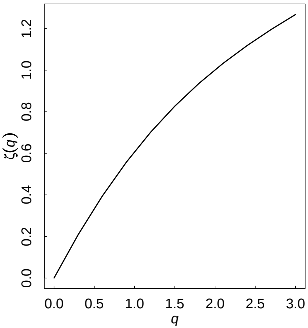
Definition

A stochastic prosess $\{X(t), t \in [0, T]\}$ *is multifractal if*

$$\mathbb{E}|X(t)|^q \sim t^{\zeta(q)},$$

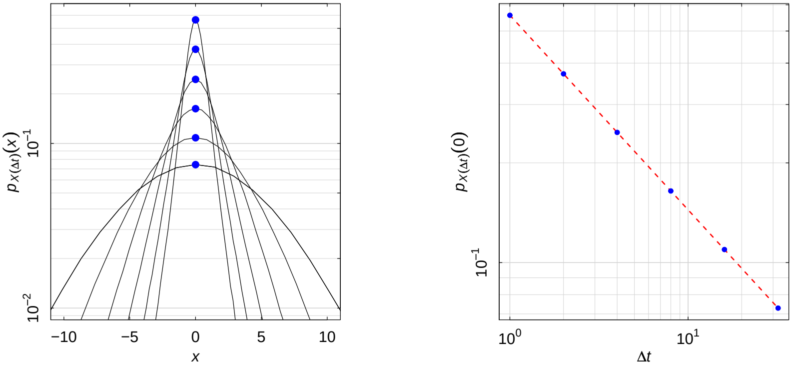
and $\zeta(q)$ *is strictly concave.*





Multifractal processes

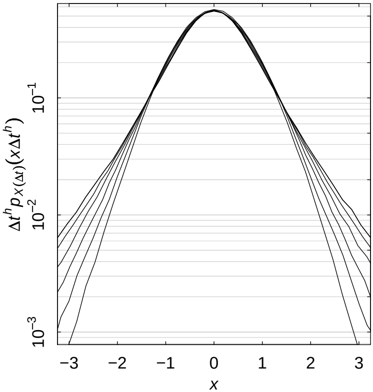
Properties



$$p_{X(\Delta t)}(0) \sim \Delta t^{-\nu}$$

Multifractal processes

Properties



$$\Delta t^{\nu} p_{X(\Delta t)}(x\Delta t^{\nu})$$

Multifractal models

Definition

A stochastic prosess $\{X(t), t \in [0, T]\}$ *is multifractal if*

$$\mathbb{E}|X(t)|^q \sim t^{\zeta(q)},$$

and $\zeta(q)$ *is strictly concave.*

- Multifractal model of asset return, Mandelbrot, Calvet and Fisher, 1997
- Markov switching multifractal, Calvet and Fisher 2001
- Multifractal random walk, Bacry et al. 2001

All of this models can be written as

$$X(t) = B(m([0, t]))$$

where $B(t)$ is a Brownian motion and m is a multifractal random measure. Note: $\mathbb{E}|X(t)|^q = c_q\, \mathbb{E}|m([0, t])|^{q/2}$.

Multifractal models

Definition

A stochastic prosess $\{X(t), t \in [0, T]\}$ *is multifractal if*

$$\mathbb{E}|X(t)|^q \sim t^{\zeta(q)},$$

and $\zeta(q)$ *is strictly concave.*

- Multifractal model of asset return, Mandelbrot, Calvet and Fisher, 1997
- Markov switching multifractal, Calvet and Fisher 2001
- Multifractal random walk, Bacry et al. 2001

All of this models can be written as

$$X(t) = B(m([0, t]))$$

where $B(t)$ is a Brownian motion and m is a multifractal random measure. Note: $\mathbb{E}|X(t)|^q = c_q \, \mathbb{E}|m([0, t])|^{q/2}$.

Multifractal models

Definition

The (log-normal) multifractal measure m defining the multifractal random walk is given by

$$m([0,t]) = \exp(h(t))$$

where $h(t)$ is a 'Gaussian' centered process with covariances

$$Cov(h(t),h(s)) = \lambda^2 \log^+ \frac{R}{|t-s|}$$

To verify the multifractality one can show the realation

$$h(at) \stackrel{\mathrm{d}}{=} h(t) + Y(a)$$

where $Y(a) \stackrel{\mathrm{d}}{\sim} N(0, -\lambda^2 \log a)$ independent of $h(t)$.

Multifractal random walk

For the discrete time MRW model the observation equations are given by

$$x_t = \sigma \exp(h_t/2)\varepsilon_t \qquad (2)$$

where h_t is a Gaussian process with covariances

$$Cov(h_t, h_s) = \lambda^2 \log^+ \frac{R}{|t-s|+1}.$$

In the basic stochastic volatility model (Taylor, 1986) the observation equations are the same, but h_t follows an AR(1) process:

$$h_t = \phi h_{t-1} + \sigma_u u_t.$$

Multifractal random walk

For the discrete time MRW model the observation equations are given by

$$x_t = \sigma \exp(h_t/2)\varepsilon_t \tag{2}$$

where h_t is a Gaussian process with covariances

$$Cov(h_t, h_s) = \lambda^2 \log^+ \frac{R}{|t-s|+1}.$$

In the basic stochastic volatility model (Taylor, 1986) the observation equations are the same, but h_t follows an AR(1) process:

$$h_t = \phi h_{t-1} + \sigma_u u_t.$$

The likelihood

The observation equations:

$$x_t = \sigma \exp(h_t/2)\varepsilon_t.$$

The pdf of $\mathbf{x} = (x_1, \ldots, x_n)$ is given by the integral

$$p(\mathbf{x}) = \int_{\mathbb{R}^n} p(\mathbf{h}, \mathbf{x}) d\mathbf{h} = \int_{\mathbb{R}^n} p(\mathbf{h}) p(\mathbf{x}|\mathbf{h}) d\mathbf{h}.$$

- No closed-form solution for the integral
- Laplace approximation:

$$p(\mathbf{x}) \approx b_n \,|\det \Omega|^{-1/2} p(\mathbf{h}^*) p(\mathbf{x}|\mathbf{h}^*), \tag{3}$$

with $\mathbf{h}^*$ the maximum of $\mathbf{h} \mapsto \log p(\mathbf{x}, \mathbf{h})$,
and Ω the corresponding Hessian matrix.

The likelihood

- The Laplace approximation:

$$p(\mathbf{x}) \approx b_n \, |\det \Omega|^{-1/2} p(\mathbf{h}^*) p(\mathbf{x}|\mathbf{h}^*),$$

  with $\mathbf{h}^*$ the maximum of $\mathbf{h} \mapsto \log p(\mathbf{x}, \mathbf{h})$,
  and Ω the corresponding Hessian matrix.

- The form of the first order derivatives:

$$\frac{\partial \log p(x, h)}{\partial h_i} = b_i + \sum_j A_{ij} \, h_j + g_i(x_i, h_i) \, ,$$

- For the basic SV model the Markov property of h_t implies that the matrices A and Ω are tri-diagonal $\rightsquigarrow$ computational ok.

The likelihood

- The Laplace approximation:

$$p(\mathbf{x}) \approx b_n \, |\det \Omega|^{-1/2} p(\mathbf{h}^*) p(\mathbf{x}|\mathbf{h}^*),$$

  with $\mathbf{h}^*$ the maximum of $\mathbf{h} \mapsto \log p(\mathbf{x}, \mathbf{h})$,
  and Ω the corresponding Hessian matrix.
- The form of the first order derivatives:

$$\frac{\partial \log p(\mathbf{x}, h)}{\partial h_i} = b_i + \sum_j A_{ij} \, h_j + g_i(x_i, h_i) \,,$$

- For the mrw we suggest a second approximation

$$p(h_t|\mathbf{h}_{t-1}) \approx p(h_t|h_{t-1}, \ldots, h_{t-\tau}), \tag{4}$$

  which makes the matrices A and Ω band-diagonal with bandwidth τ.

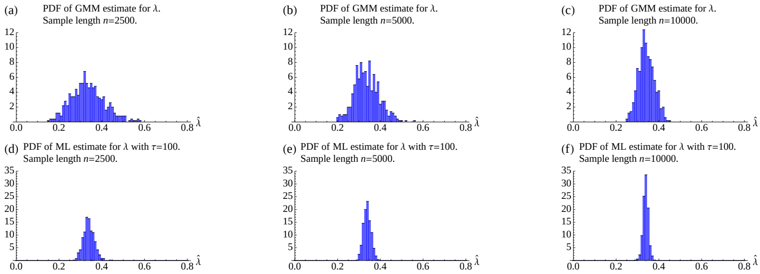
Some practical information

Multifractal random walk

- We have implemented the MLE procedure for the mrw model in a R package.
- Version 0.2 of this package is hopefully soon to come, with smoothing, filtering and volatility forecasting, together with density forecast.
- We have also implemented utility functions for the MSM (part of my master) and the basic SV model.

A small Monte Carlo study

Multifractal random walk



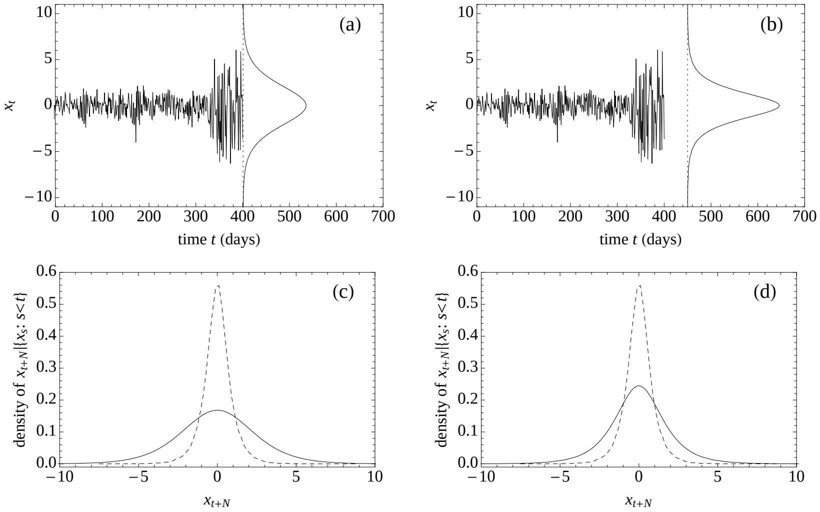
n	sd($\hat{\lambda}_{ML}$)	sd($\hat{\lambda}_{GMM}$)
2500	0.03	0.08
5000	0.02	0.05
10000	0.01	0.04

Concluding remarks

- We are not saying 'always use ML'.
- For the use of the approximated likelihood method for other inference problems:
  - A multifractal approach towards inference in finance, Løvsletten O. and Rypdal M. 2012, arxiv
- A generalization of the (discrete-time) mrw to fractional MRW and multifractal Ornstein-Uhlenbeck processes:
  - Modeling electricity spot prices using mean-reverting multifractal processes, Rypdal M. and Løvsletten O. 2012, submitted
  - These models are also relevant in e.g. turbulence and magnetospheric physics.
- We are very much open for collaboration!

Current research

Forecasts based on the MRW model.



Bacry E., Muzy J.F. and Delour J. (2012)
Multifractal random walk
Physical Review E

Calvet L. and Fisher, A. (2001)
Forecasting multifractal volatility.
Journal of Econometrics.

Calvet, L. and Fisher, A. (2003)
Regime-switching and the estimation of multifractal processes.
NBER working paper series.

Løvsletten O. and Rypdal M. (2012)
Approximated maximum likelihood estimation in multifractal random walks
Physical Review E

Mandelbrot B., Fisher A. and Calvet L (1997).
A Multifractal Model of Asset Returns.
Cowles Foundation for Research in Economics.

McLeod et.al (2007)

Algorithms for Linear Time Series Analysis: With R Package
Journal of statistical software.