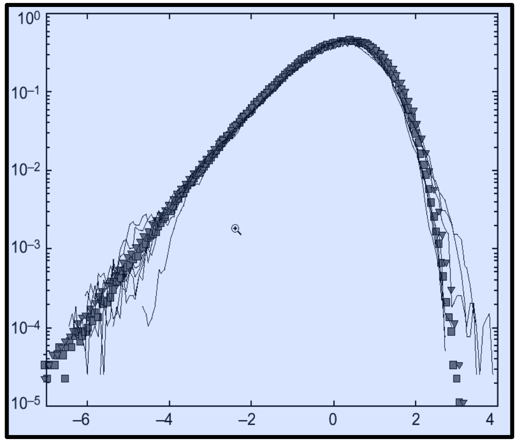
The Distribution of Spatially Averaged Critical Properties



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Aggregation, Inference and Rare Events in the Natural and Socio-economic Sciences
Warwick, 17th-18th May 2012

Bramwell, Holdsworth & Pinton, Nature, 396 552, 1998

S. T. Bramwell, K. Christensen, J.-Y. Fortin, P. C. W. Holdsworth, H. J. Jensen, S. Lise, J. M. Lopez, M. Nicodemi, J.-F. Pinton, M. Sellitto, PRL 84, 3744 - 3747 (2000).

S. T. Bramwell, J.-Y. Fortin, P. C. W. Holdsworth, S. Peysson, J.-F Pinton, S. Portelli, M. Sellitto, PRE 2001

nature physics

ARTICLES

PUBLISHED ONLINE: 3 MAY 2009 | DOI:10.1038/NPHYS1268

The distribution of spatially averaged critical properties

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THANKS TO

- (ENS Lyon:) Peter Holdsworth, Jean-Francois Pinton, Jean Yves Fortin, Mauro Sellito, Babtiste Portelli, Stephan Peysson, Pascale Archambault, Maxime Clusel,
- (UCL:) Tom Fennell, Simon Banks, Andrea Taroni
- Imperial College: Henrik Jensen, Kim Christensen
- Discussions: Zoltan Racz, Eric Bertin

Three Incongruous Facts

- Many phenomena show power law distributions, suggestive of "scale invariance"

- Scale-invariant spatially averaged physical properties show a particular non-power law distribution that is apparently related to an extreme value distribution

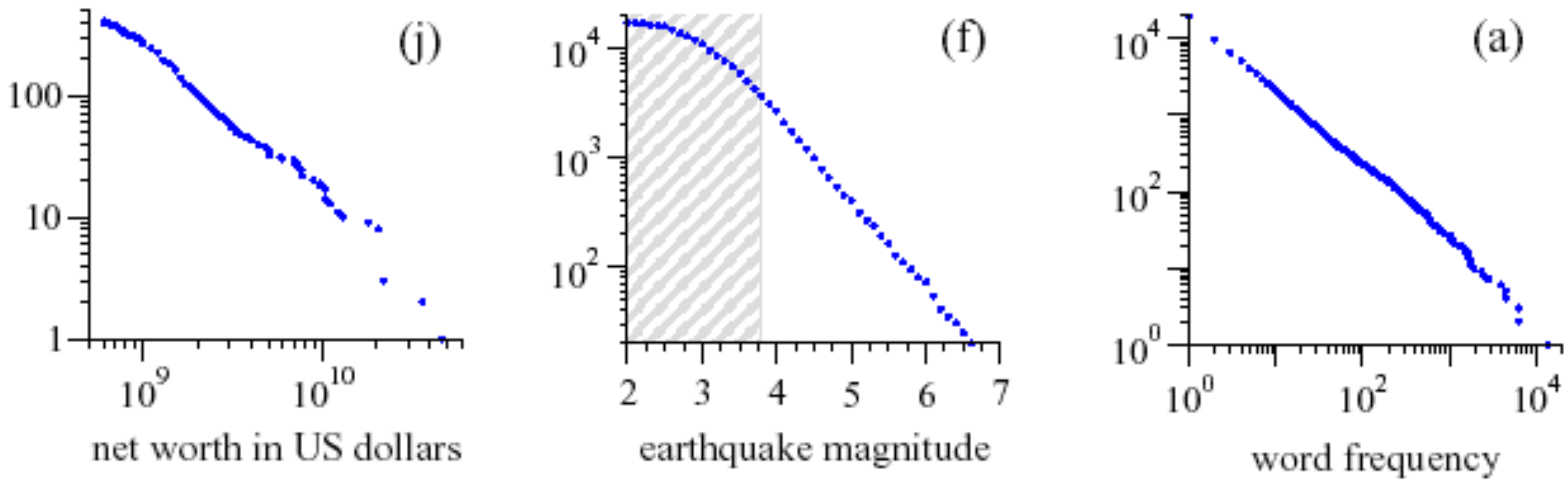
- To complicate matters the (rigorous) prediction for critical order parameter distributions is neither power law nor quasi extreme value, nor is it often observed

PLAN

1. Background to the problem
2. A renormalisation approach
3. Conclusion: a simple predictive rule

Power Law Distributions

Wealth, word frequency and earthquakes



From Newman, Contemporary Physics, 2005

$$P_X(x) \sim x^{-\alpha}$$

In public conscience connected with “scale invariance” and “criticality”

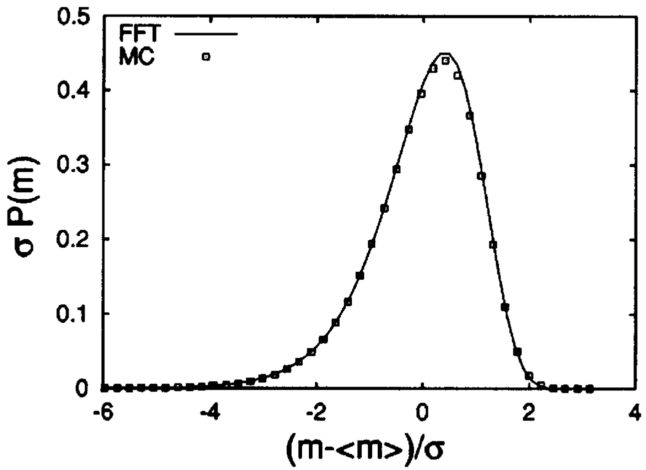
Equilibrium critical phenomena should be the best understood critical systems

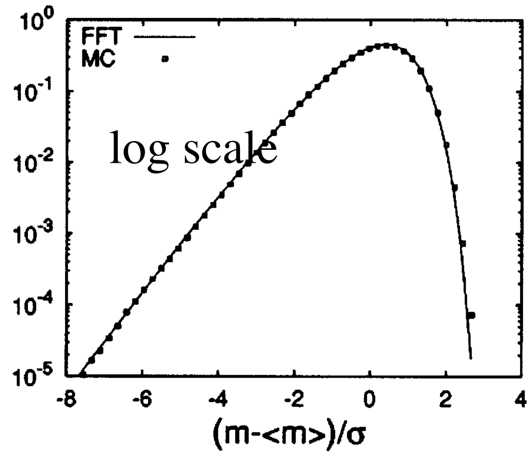
But order parameter distributions are NOT power law

Theory *(Bruce 1995)* (δ = 15)

$$p_Z(z) \sim z^{(\delta-1)/2} e^{-cz^{\delta+1}}$$

But when we calculated the order parameter distribution of the 2D-XY model in harmonic limit (a tractable model of critical phenomena) we found yet another distribution, with exponential tail



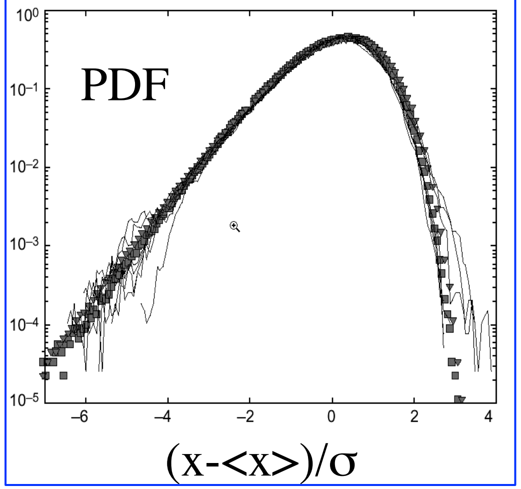


We noticed the same distribution was exhibited in a turbulence experiment…

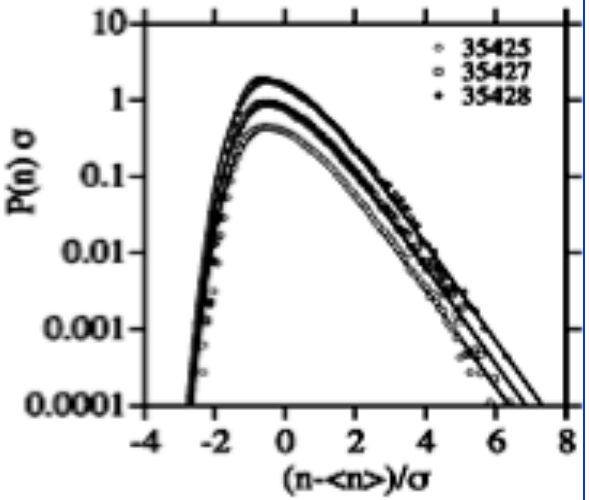
… and argued that it should be widely observed for spatially averaged properties in equilibrium critical systems and driven systems at steady state

Many "BHP" distributions found in experiment and simulation

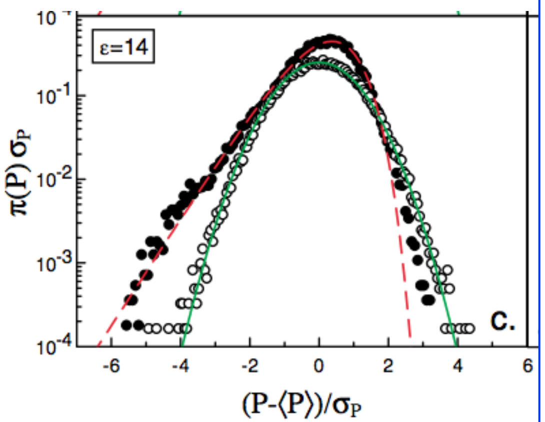
Bramwell, Holdsworth & Pinton, Nature, 1998 power consumption turbulent flow and magnetic critical model



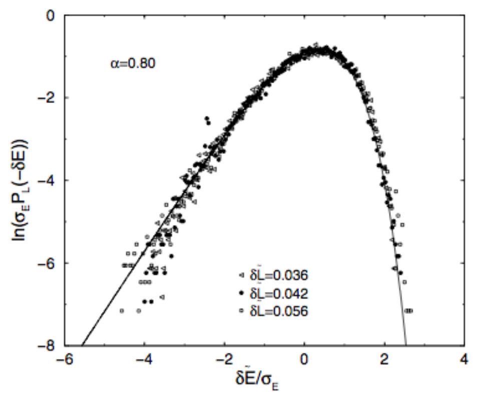
Van Milligan et al., Phys. Plamas 2005 plasma density, tokamak



Toth Katona et al. PRL 2003 injected power electroconvection

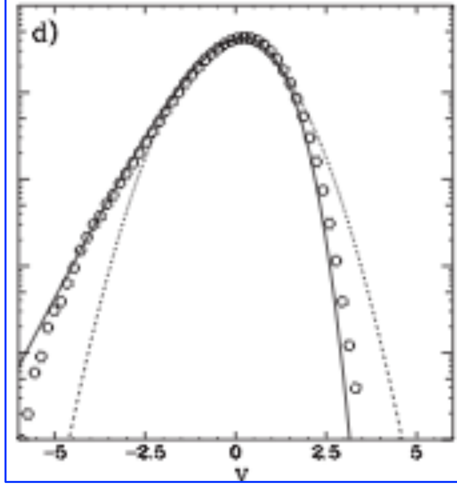


Javier Brey et al. PRL 2005 Simulated energy at instability in granular gas



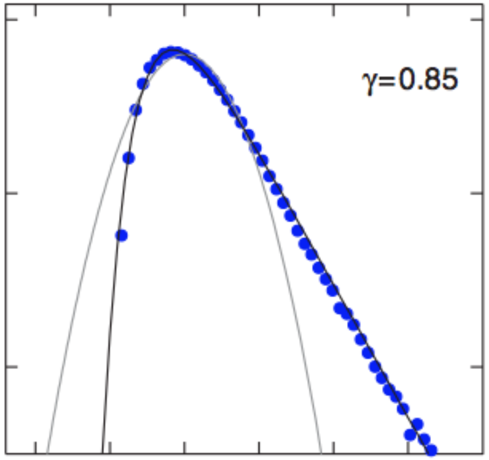
Joubaud et al. PRL 2008

Fré edericksz transition



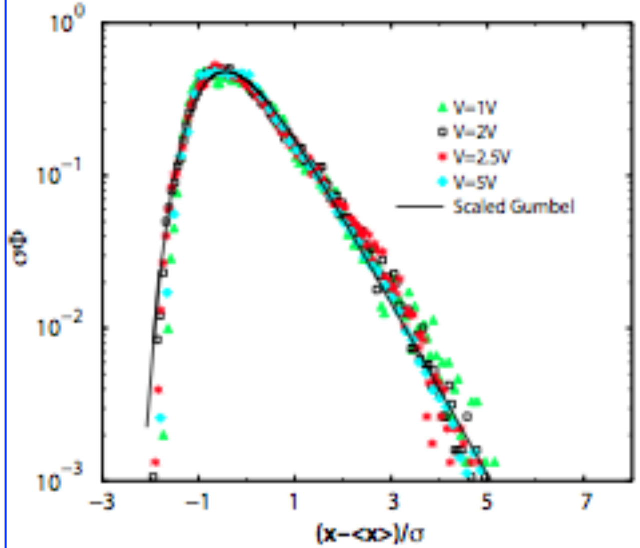
Planet et al. PRL 2009

velocity fluctuations of imbibation front



Millitheler et al. 2011

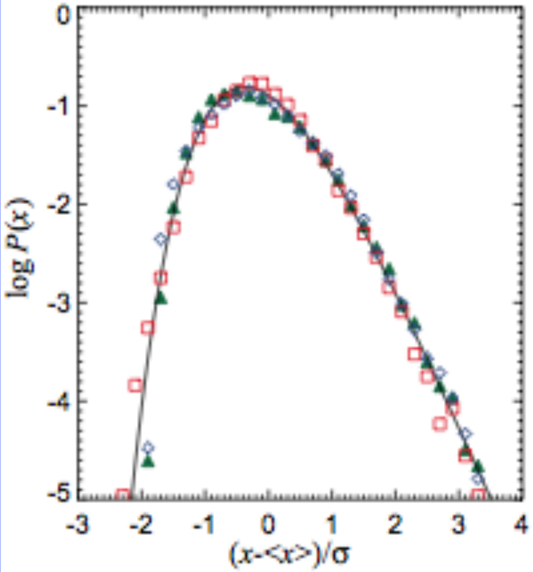
bacteriorhodopsin protein current fluctuations



Rypdal et al. NJP 2008

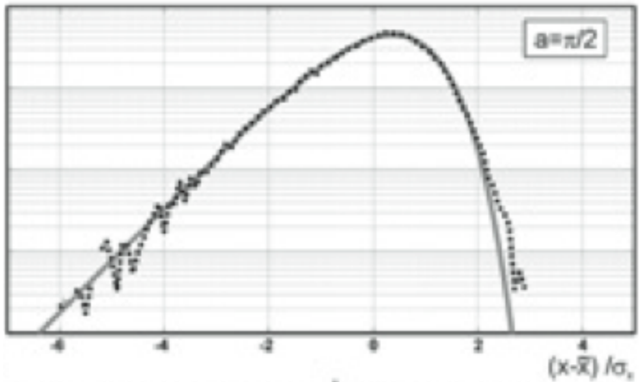
dust kinetic energy (exp.)

sandpile & turbulence models



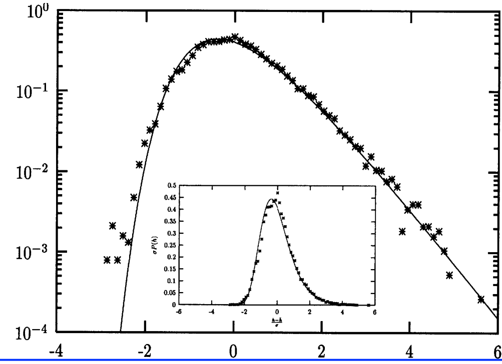
Barucci et al. EPL 2010

Radar amplitude



Bramwell et al. EPL 2002

Danube water level



Generalised Gumbel Distribution

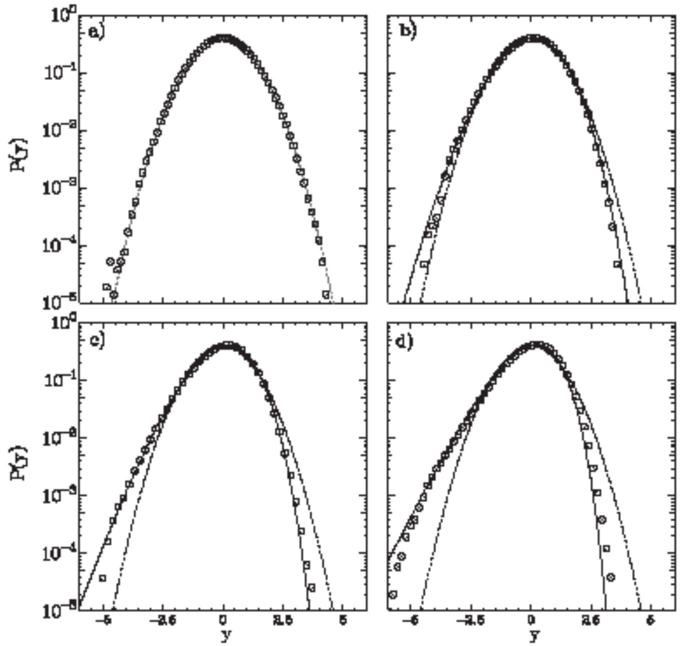
[Bramwell et al., Racz et al., Watkins, Chapman et al., Clusel & Bertin]

$$p(z) \sim [e^{-z-e^{-z}}]^{a} = G(z)^{a}$$

G = Gumbel distribution of extreme value statistics

- $a \sim \pi/2$ *(Bramwell et al., PRL 2000)*
- $a = 1$ 1/f noise *(Antal et al., PRL 2001)*
- $a \sim a_0 + (L/\xi)^D$ *(Portelli, et al., PRE 2001)*
- *Formal relation with EVS (Bertin & Clusel)*

- *The exponent a is argued to control the effective number of degrees of freedom in the critical system and large a tunes towards central limit theorem*



Fredeericksz Transition in Liquid Crystals: PDF measured and L/ξ tuned experimentally. Joubaud et al. PRL 2008

Can we reconcile the "three incongruous facts"? i.e. *(1) many phenomena show power law distributions, suggestive of "scale invariance"; (2) scale-invariant spatially averaged physical properties show BHP or generalised Gumbel (GG) distribution; (3) "critical" order parameter distribution is neither power law nor GG, and is very rare.*

Aims

1. Define scale invariance
2. Explain why spatially averaged properties are not power law distributed
3. Understand relation of generalised Gumbel (GG) and "critical" distribution
4. Explain why GG common and "critical" rare

Another aim: can we find a rule to predict the form of a distribution from the scaling of its moments, that might be practically useful?

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Notation

The symbol N represents a dimensionless system size

The symbol P represents CDF or CCDF depending on context

The dimensionless spatially averaged property X takes real positive values x

X is intensively defined, so *intuitively* we might guess $P_N(x) \sim P_{2N}(x)$

Subtleties concerning the fact that x is positive only may be glossed over for clarity (but can be treated with precision if required).

The sign of the skewness may depend on precise definition of X

Definition of scale invariance and its consequences

Basic assumption is that the distribution P is of the form

$$P_N(x) = F(Ng(x))$$ P is a CDF or CCDF

That is, N is a system size, so 1/g is a characteristic scale and we would expect functional behaviour of P to be dominated by singularities in 1/g.

Define "scale invariance" by $g(x)/\lambda = g(R_\lambda(x))$ $\qquad P_N(x) = P_{N/\lambda}(R_\lambda(x))$

Linearise transformation R around fixed point at g=0; case of x =∞

$$(1/x)' = (1/x)\lambda^{\phi} \qquad (1/x)' = (1/x) + x_0(1/x)^2 \ln(\lambda)$$

x classified as a *relevant*, *irrelevant* or *marginal* variable (critical exponent ϕ).

rel/irrel $P = F\left(Ng_0 x^{\pm 1/|\phi|}\right)$ $\qquad P = F(Ng_0 e^{\pm x/c})$ marginally rel/irrel

Limit distributions P(z) with z defined as follows

$$P(x/\sigma) = F((x/\sigma)^{\pm 1/|\phi|}) \qquad \mu, \sigma \sim N^{\pm|\phi|}$$

$$P((x-\mu)/\sigma) \sim F(e^{\pm(x-\mu)/c}) \qquad \mu \sim \pm c \ln N, \sigma \sim c$$

Choice of the function F

(1) **CLT is satisfied** if we chose $F(g) = \Phi \circ \sqrt{g}$ with g=z^2

<u>Heuristic argument for critical order parameters</u>. Interactions are slowly turned on. Retain same for for F but allow g to evolve towards the singularity. Critical point theory gives $|\phi = 1/(\delta+1)|$, hence we predict

$$p_Z(z) \sim z^{(\delta-1)/2} e^{-cz^{\delta+1}}$$

...as expected!

Critical point theory shows F always exists but may not be universal (hyperscaling)

(2) Large deviations theory suggests $F(g) = \exp(-g)$

We recover the "three types" of extreme value theory (Fisher-Tippet-Gendenko)!

Weibull	Gumbel	Fréchet
$P(z) \sim 1 - e^{-z^{1/\lvert\phi\rvert}}$	$P(z) \sim 1 - e^{-e^{z}}$	$P(z) \sim e^{-1/z^{\phi}}$

Rule for the Limit Distributions

In terms of reduced variable Z, the limit distributions take the form $P(z) = F(g(z))$

If we know how mean varies with system size, then we know g

Qualitatively, the results are similar for both choices of F given above

Mean varies as		**PDF is**
$N^{-\lvert\phi\rvert} \rightarrow g(z) = z^{1/\lvert\phi\rvert}$	(relevant)	critical type or Weibull
$-\ln N \rightarrow g(z) = e^{z}$	(marginally relevant)	(generalised) Gumbel
$\ln N \rightarrow g(z) = e^{-z}$	(marginally irrelevant)	(generalised) Gumble
$N^{\lvert\phi\rvert} \rightarrow g(z) = 1/z^{1/\lvert\phi\rvert}$	(irrelevant)	power law or Fréchet

However these two can be ruled out for most *physical* properties of interest as once intensively defined, such properties are very unlikely to increase with size

This leaves the "Gumbel" type and the the "critical" type. A detailed study of the XY model reveals a competition between the two (Bramwell, see also work by Botet et al.)

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Summary and Conclusions

$$P_N(x) = F(Ng(x))$$

• Proposed a general definition of scale invariance, classified by singularities in 1/g(x).
• CLT, critical distributions, extreme value distributions and (we expect) all spatially averaged physical properties should share this scale invariance.

• Have explained why scale invariant physical properties are not power law distributed.
• Generalised Gumbel (GG) and “critical” distributions no longer appear incongruous, but GG is very common and “critical” distribution is very rare (–unstable behaviour?).

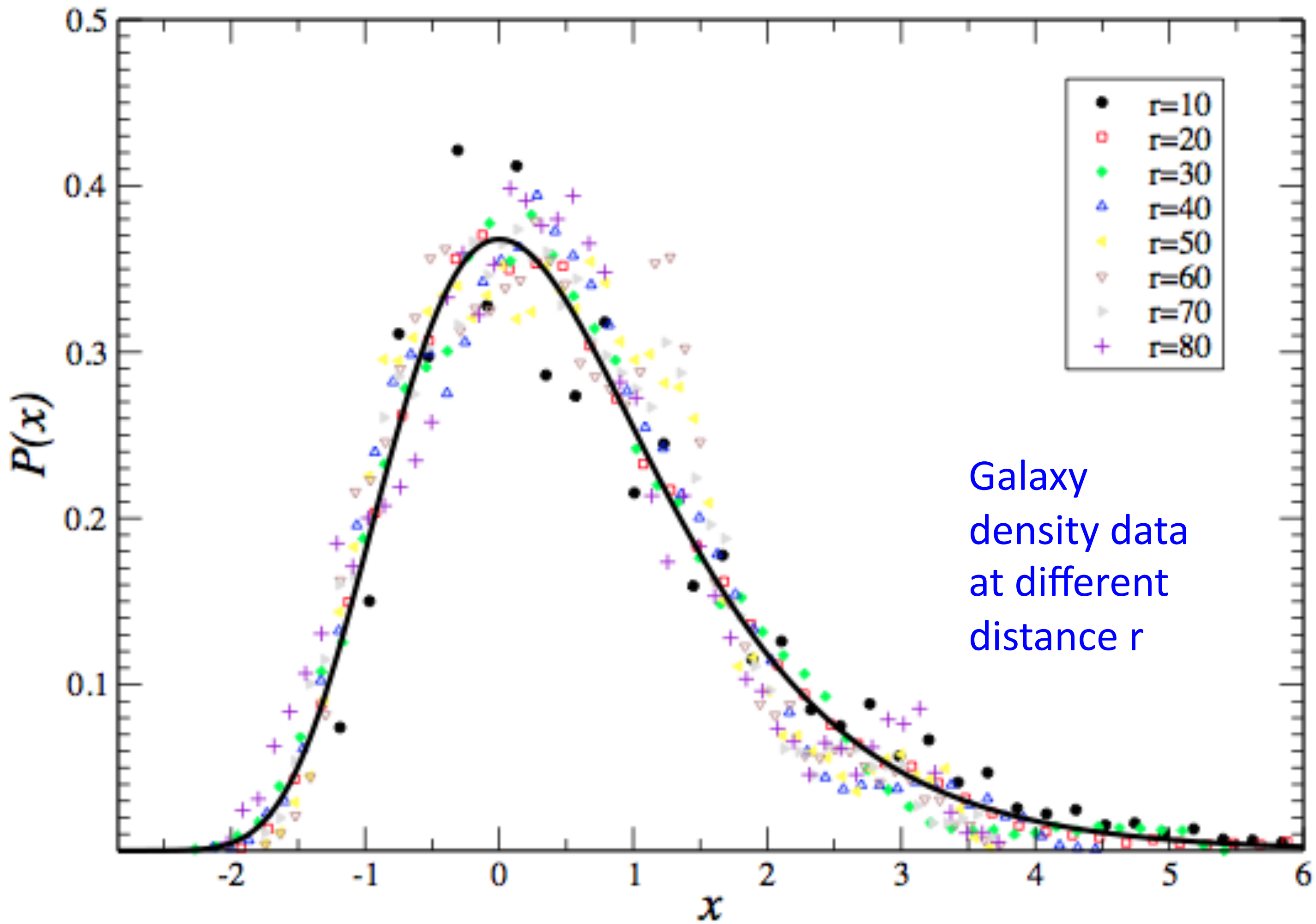
• F is generally not universal: but results insensitive to this (- domain of attraction to exponential?): perhaps some hidden universality in F (to give e.g. BHP distribution)?

For size dependence of mean of X, I conjecture a simple and practical rule:

- increases rapidly with size -> irrelevant -> try power law
- evolves slowly with size -> marginal -> try GG
- decreases rapidly with size -> relevant -> try critical or Weibull

Labini and Pietronero "the Complex Universe…" (J. Stat. Mech. 2010)

According to the conjecture of Bramwell for critical systems [60], if a spatially averaged quantity depends only weakly (say logarithmically) on the system size, the distribution of this quantity follows the Gumbel distribution. This is indeed what we see in the galaxy data.



Galaxy density data at different distance r if there were no phenomena which are independent of all but a manageably small set of conditions, physics would be impossible.

Eugene Wigner