Information theory for complex systems

On the Equivalence between Stochastic Baker's Maps and Two-Dimensional Spin Systems

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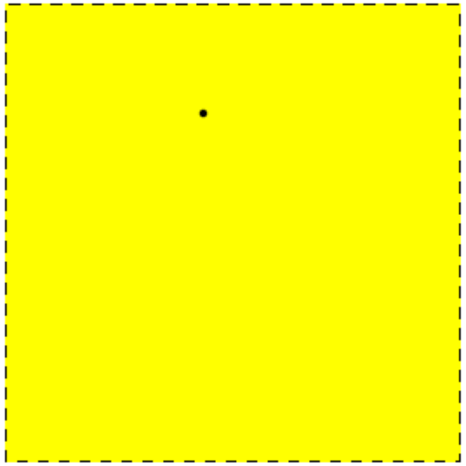
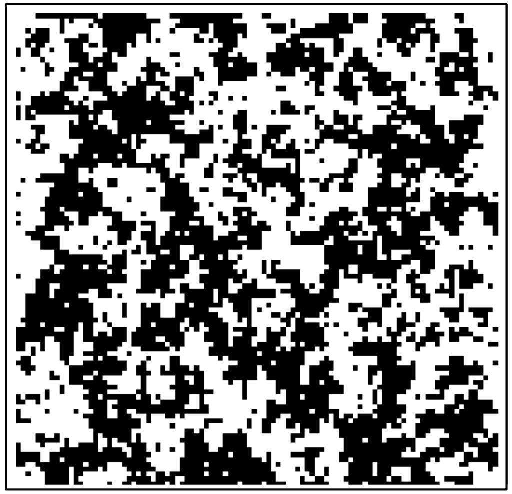
Background



Baker's transformation



Ising spin model

What is the mathematical connection between these two systems?
Can they be described and analyzed in similar ways?

Dyadic representation

The baker's map can be written

$$(x',y') = f(x,y) = \left(2x - \lfloor 2x \rfloor, \frac{y}{2} + \frac{\lfloor 2x \rfloor}{2}\right)$$

where $\lfloor 2x \rfloor$ is the integer part of $2x$, and $(x,\ y) \in [0,\ 1] \times [0,\ 1]$.

Positions in binary form:

$$x = 0.x_0x_1...x_n... \qquad y = 0.y_0y_1...y_n...$$

with x_i and $y_i \in \{0,\ 1\}$.

Composed into a bi-infinite sequence:

$$(x,\ y) = (...y_n...y_1y_0\ .\ x_0x_1...x_n...)$$

Then the original baker's map is just a shift operation:

$$f(x,\ y) = (...y_n...y_1y_0x_0\ .\ x_1...x_n...)$$

The stochastic baker's map

Consider instead the following map

$$(x',y') = g(x,y) = \left(2x - \lfloor 2x \rfloor, y/2 + \xi(x,y)/2\right)$$

where we have replaced the $\lfloor 2x \rfloor/2$ term in y' with a stochastic variable $\xi(x, y) \in \{0, 1\}$ that may depend on position in state space (x, y).

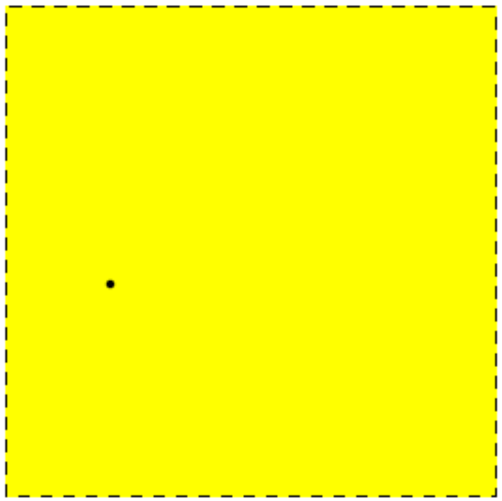
In order to get symmetry between x and y we also randomly switch between x and y, i.e., random choice of compressing in horizontal or vertical direction, to form the stochastic baker's map φ :

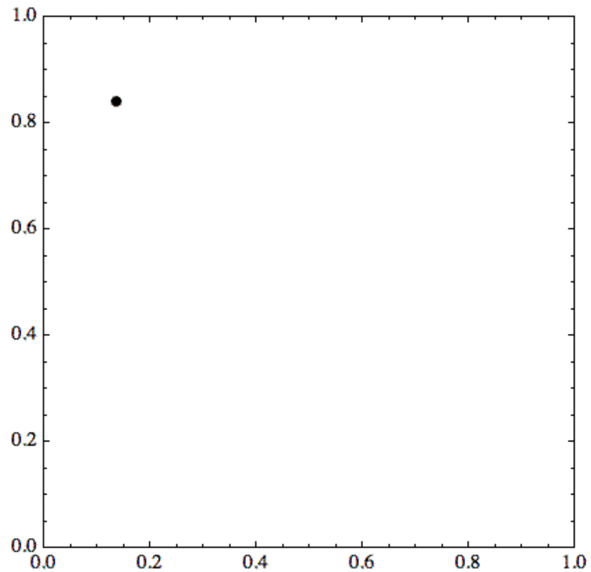
$$(x',y') = \varphi(x,y) = \begin{cases} \left(2x - \lfloor 2x \rfloor, y/2 + \xi(x,y)/2\right) & \text{with probability } 1/2 \\ \left(x/2 + \xi(y,x)/2, 2y - \lfloor 2y \rfloor\right) & \text{with probability } 1/2 \end{cases}$$

Shuffling and twisting baker's map

$$(x',y') = \varphi(x,y) = \begin{cases} \left(2x - \lfloor 2x \rfloor, y/2 + \xi(x,y)/2\right) & \text{with probability } 1/2 \\ \left(x/2 + \xi(y,x)/2, 2y - \lfloor 2y \rfloor\right) & \text{with probability } 1/2 \end{cases}$$

ξ characterized by conditional probability $q_0(x,\ y) = \mathrm{P}(\xi{=}0 \mid x,\ y)$.





Dyadic representation

A time series of consecutive steps of the map φ can look like:

t	$\ldots$	y_n	y_{n-1}	$\ldots$	y_1	y_0	x_0	x_1	$\ldots$	x_{n-1}	x_n	$\ldots$
$t+1$	y_n	y_{n-1}	$\ldots$	y_1	y_0	ξ_1	x_1	$\ldots$	x_{n-1}	x_n	$\ldots$	$\ldots$
$t+2$	$\ldots$	y_n	y_{n-1}	$\ldots$	y_1	y_0	ξ_2	x_1	$\ldots$	x_{n-1}	x_n	$\ldots$
$t+3$	$\ldots$	$\ldots$	y_n	y_{n-1}	$\ldots$	y_1	ξ_3	ξ_2	x_1	$\ldots$	x_{n-1}	x_n
$t+4$	$\ldots$	y_n	y_{n-1}	$\ldots$	y_1	ξ_4	ξ_2	x_1	$\ldots$	x_{n-1}	x_n	$\ldots$

When symbols are shifted into the "wall" $|$ they disappear and reappear on the other side as $\xi(x, y)$, a stochastic variable that may depend on the full sequences for x and y.

Dynamics characterized by an invariant measure $\mu_b(x, y)$ over state space $[0, 1] \times [0, 1]$.

Entropy rate of φ

The entropy rate can be derived from the symbolic dynamics description, and a certain *resolution* or binary precision n.

…	y_n	y_{n-1}	…	y_1	y_0	x_0	x_1	…	x_{n-1}	x_n	…
y_n	y_{n-1}	…	y_1	y_0	ξ_1	x_1	…	x_{n-1}	x_n	…	…
…	y_n	y_{n-1}	…	y_1	y_0	ξ_2	x_1	…	x_{n-1}	x_n	…
…	…	y_n	y_{n-1}	…	y_1	ξ_3	ξ_2	x_1	…	x_{n-1}	x_n
…	y_n	y_{n-1}	…	y_1	ξ_4	ξ_2	x_1	…	x_{n-1}	x_n	…

resolution level

If we have an arbitrarily long history there is almost always

1. no uncertainty of what is shifted in from lower levels of resolution
2. 1 bit of uncertainty from the shift direction
3. uncertainty s_ξ due to ξ

Entropy rate of φ

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…	y_n	y_{n-1}	…	y_1	y_0	x_0	x_1	…	x_{n-1}	x_n	…
y_n	y_{n-1}	…	y_1	y_0	ξ_1	x_1	…	x_{n-1}	x_n	…	…
…	y_n	y_{n-1}	…	y_1	y_0	ξ_2	x_1	…	x_{n-1}	x_n	…
…	…	y_n	y_{n-1}	…	y_1	ξ_3	ξ_2	x_1	…	x_{n-1}	x_n
…	y_n	y_{n-1}	…	y_1	ξ_4	ξ_2	x_1	…	x_{n-1}	x_n	…

If we have an arbitrarily long history there is almost always

1. no uncertainty of what is shifted in from lower levels of resolution
2. 1 bit of uncertainty from the shift direction
3. uncertainty s_ξ due to ξ

The entropy rate from ξ is then

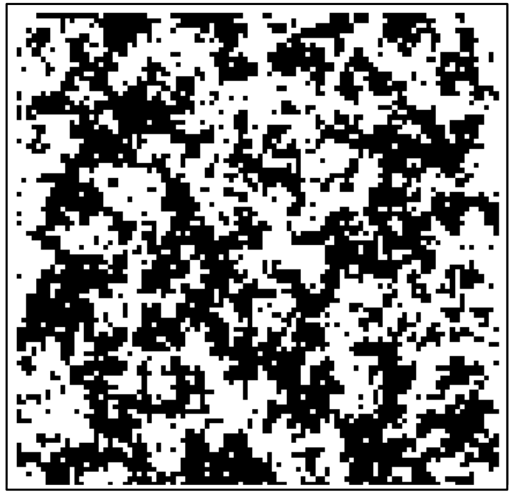
$$s_\xi = \sum_{x,y} \mu_b(x,y)\sigma(q_0(x,y))$$

where σ is the entropy function

$$\sigma(p) = p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$$

Information theory for equilibrium spin systems

- Consider interactions between nearest neighbours only, i.e., energy contributions only come from local spin interaction.
- Equilibrium is defined by a maximum in the entropy s, given a specified value on the energy u.
- Ising model in 1d and 2d...



Ising spin model

1D: entropy for symbol sequences

1	0	1	0	0	1	0	?

The entropy s, or the average uncertainty about the next symbol, can be derived from the conditional probability of the next symbol, with an increasing number of preceding symbols $(x_1, \ldots, x_{m-1})$:

$$s = \lim_{m \to \infty} \sum_{x_1,\ldots,x_{m-1}} p(x_1,\ldots,x_{m-1}) \sum_{x_m} p(x_m \mid x_1,\ldots,x_{m-1}) \log \frac{1}{p(x_m \mid x_1,\ldots,x_{m-1})}$$

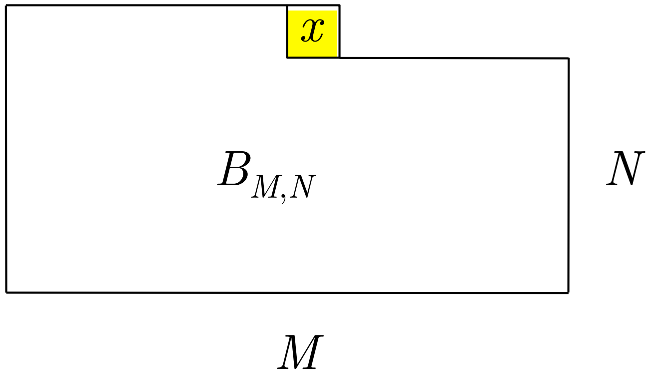
2D: one state per step

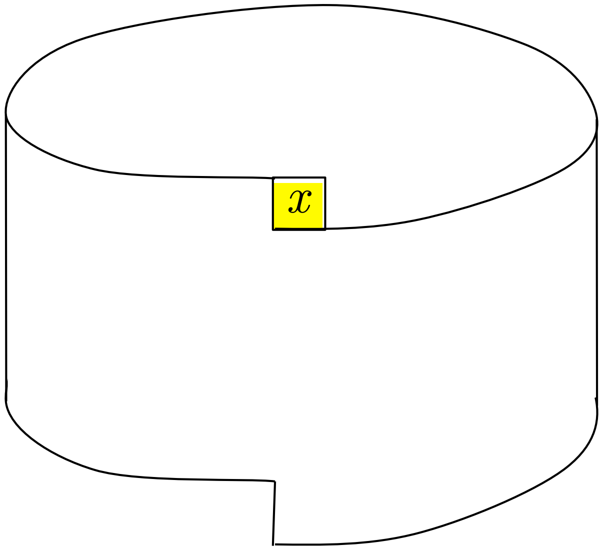
- Generalization of the 1D entropy expression adding one state at the time.

$$s = \lim_{M,N \to \infty} \sum_{B_{M,N}} p(B_{M,N}) S[p(x \mid B_{M,N})]$$

Conditional probability for x given B

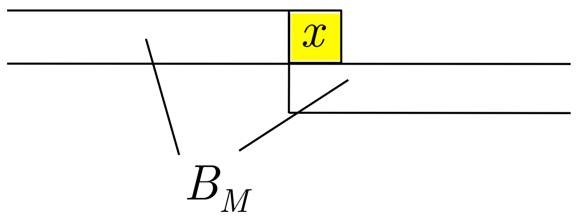
Invariant measure μ gives the probability for B





2D: one state per step

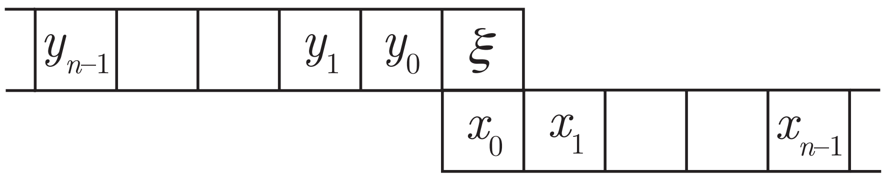
- If interactions are only between nearest neighbours, we don't need the infinity limit for N (but still for M),



Information theory for equilibrium 2d spin systems

- Equilibrium is defined by a maximum in the entropy s, given a specified value on the energy u.

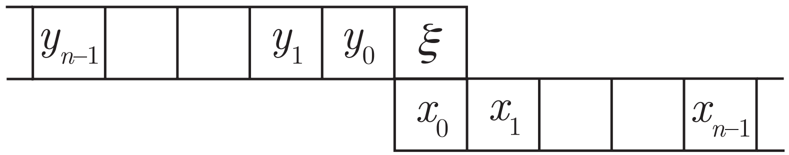
- This implies that the block shape required for calculating the entropy is simplified – the state ξ need not depend on anything below the rows of states to the left and right below:



- This means that the spin system is characterized by the measure μ over blocks of spins B containing $(y_{n\text{-}1}\ldots y_1 y_0\_\, x_0 x_1 \ldots x_{n\text{-}1})$ and the conditional probability $P(\xi='0') = p_0(y_{n\text{-}1}\ldots y_1 y_0\_\, x_0 x_1 \ldots x_{n\text{-}1})$ in the limit $n \to \infty$.

Equivalent representations

- Repeated applications of conditional probability for ξ given the configuration $B_\infty = (\ldots y_{n\text{-}1} \ldots y_1 y_0 \_ x_0 x_1 \ldots x_{n\text{-}1} \ldots)$ means that we move along the edge, building up a new row on top of x.



- One step in this process can then be represented by

$$(\ldots y_{n\text{-}1} \ldots y_1 y_0 \_ x_0 x_1 \ldots x_{n\text{-}1} \ldots) \rightarrow (\ldots y_{n\text{-}1} \ldots y_1 y_0 \xi \_ x_1 \ldots x_{n\text{-}1} \ldots)$$

- This is formally identical to the basic step in the stochastic baker's map:

$$g(\ldots y_{n\text{-}1} \ldots y_1 y_0 \,.\, x_0 x_1 \ldots x_{n\text{-}1} \ldots) = (\ldots y_{n\text{-}1} \ldots y_1 y_0 \xi \,.\, x_1 \ldots x_{n\text{-}1} \ldots)$$

Equivalent invariant measures

- Therefore, if we choose the conditional probability in the baker's map identical to the one for ξ in the spin system:

$$q_0(y_{n\text{-}1} \ldots y_1 y_0 \,.\, x_0 x_1 \ldots x_{n\text{-}1}) = p_0(y_{n\text{-}1} \ldots y_1 y_0 \_ x_0 x_1 \ldots x_{n\text{-}1})$$

then the spatial statistics of the spin system is identical to the statistics of the corresponding stochastic baker's map and they are both characterized by the same invariant measure

$$\boldsymbol{\mu}_b = \boldsymbol{\mu}$$

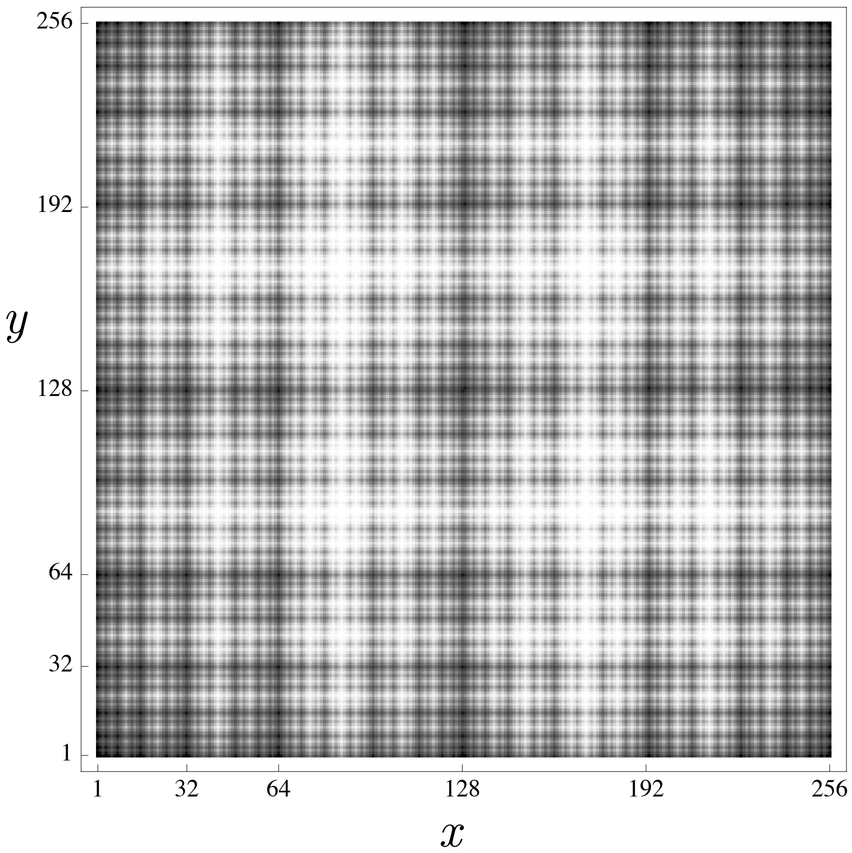
- **This means that for each equilibrium 2d spin system, with nearest neighbour interaction, there is a corresponding stochastic baker's map that contains all the properties of the spin system.**

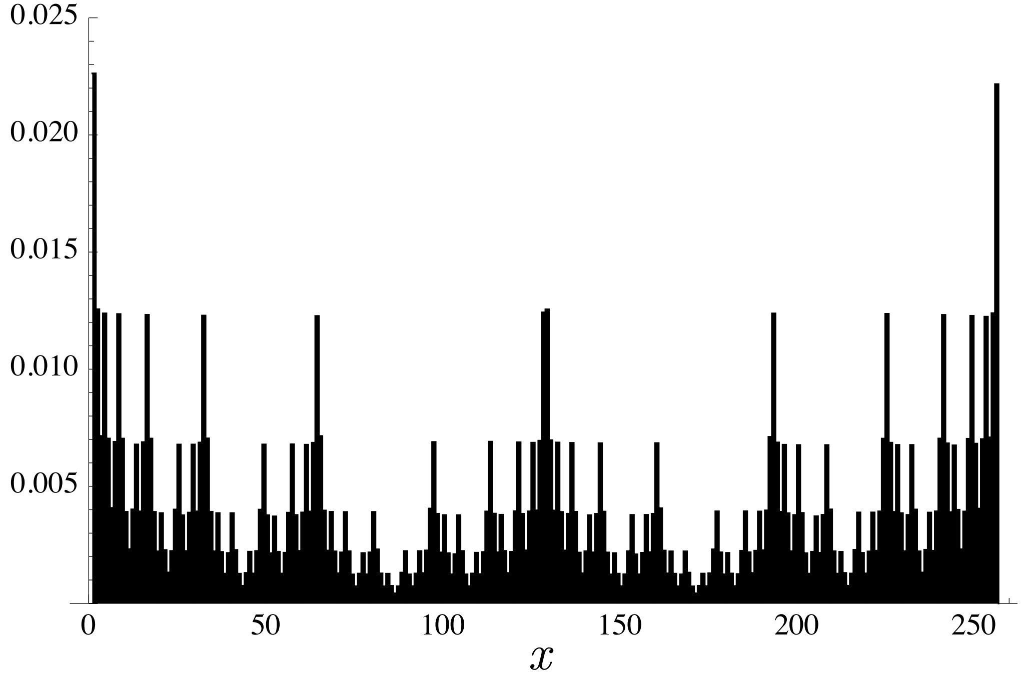
One example

- Take the 2d Ising model with spins 0 and 1, and interaction constant $J = 1$.
- Local energy contribution (in terms of k_B) is –1 and +1 from parallel and anti-parallel spins, respectively.
- We have run Monte Carlo simulations, at $T = 4$ (above T_c), collecting $3.1 \cdot 10^6$ spin configurations and from that derived the spin system conditional probabilities $p_0(y_2 y_1 y_0 \_\, x_0 x_1 x_2)$, based on block size $n = 3$.
- Then the same conditional probabilities q_0 defines a stochastic baker's map. We estimate the invariant measure μ_b at a binary resolution of $m = 8$, i.e., the unit square is divided into $2^8 \times 2^8$ cells.

Invariant measure

The invariant measure for the baker's map at binary resolution $m = 8$ and temperature $T = 4$:





projected measure

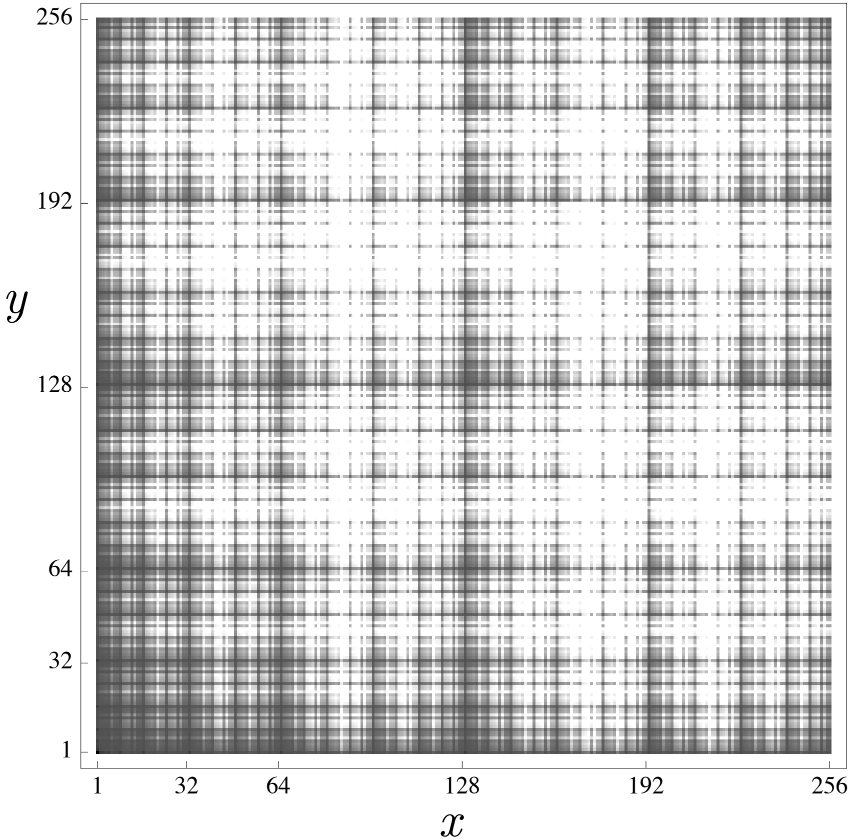
Free energy calculations

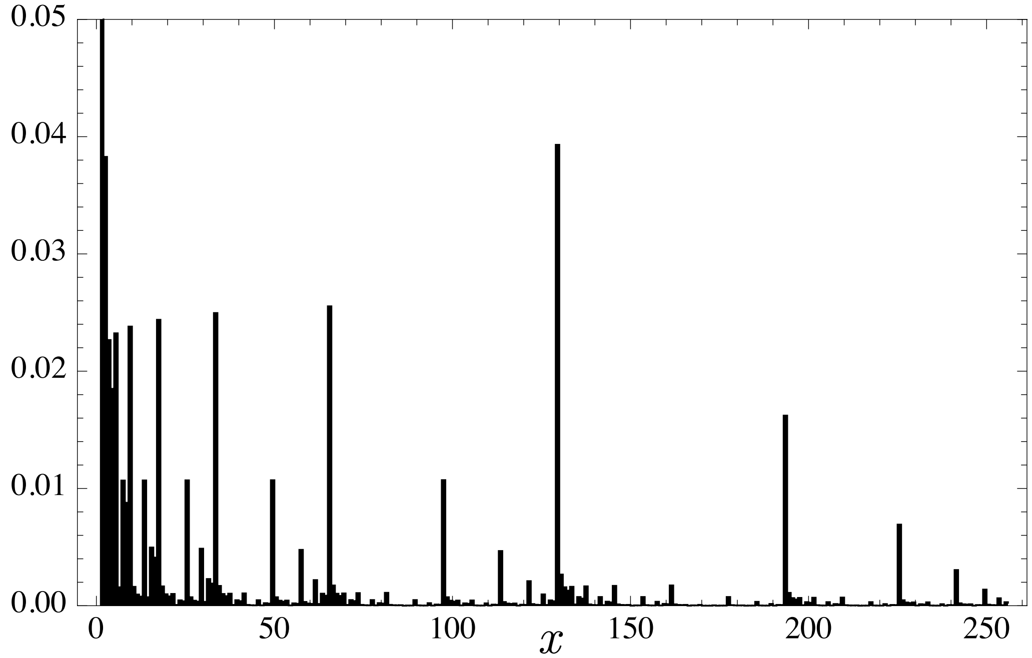
- Free energy $g = u - T\,s$
- Energy u is calculated from local averages (both in MC and in baker's map), while entropy s is estimated from blocks B_n with n=3.

- Calculation of the free energy for the this example results in:
  - Exact value $g_{\mathrm{ex}} \approx -3.02643$
  - Monte Carlo $g_{\mathrm{MC}} \approx -3.02643$
  - Baker's map $g_{\mathrm{b}} \approx -3.02634$ (so deviation $<10^{-4}$)

Invariant measure

The invariant measure for the baker's map at $T = 2.2 < T_c$:





projected measure

Invariant measure

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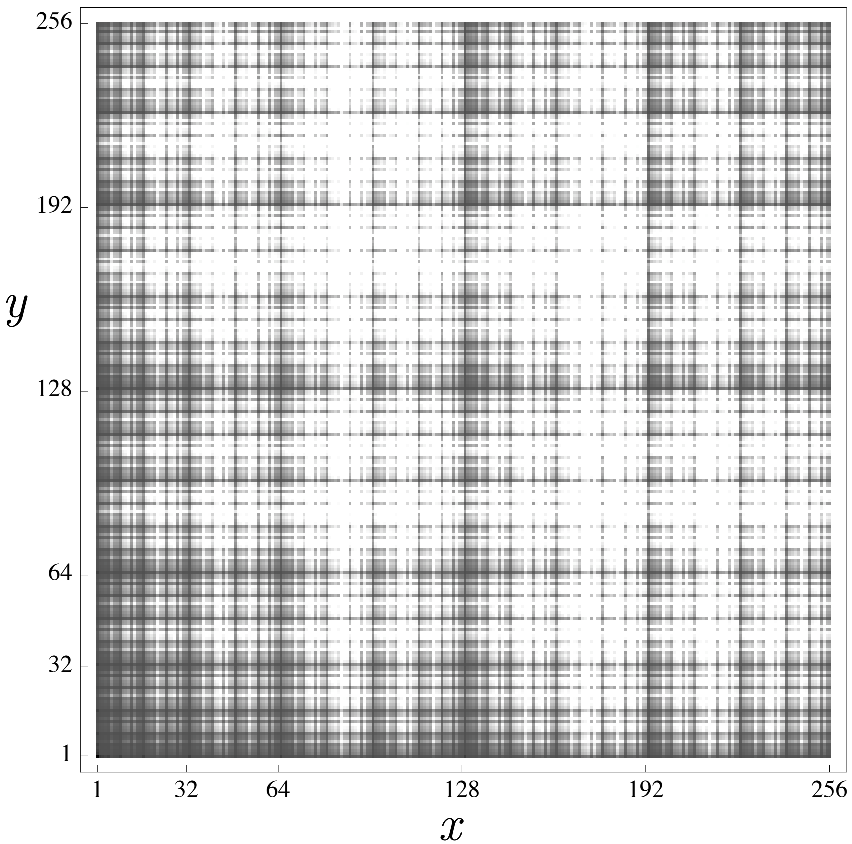

Free energy calculations

Table 1: Free energy for the stochastic baker's map at different levels of resolution of state space m and block size n for the conditional probabilities. The rightmost column shows the Monte Carlo estimates. The exact value of the free energy with this precision is −2.0907 [15].

n	$m=1$	$m=2$	$m=3$	$m=4$	$m=8$	MC
1	−2.0807	−2.0804	−2.0805	−2.0808	−2.0815	−2.1009
2	–	−2.0894	−2.0890	−2.0888	−2.0887	−2.0928
3	–	–	−2.0905	−2.0904	−2.0902	−2.0913
4	–	–	–	−2.0907	−2.0906	−2.0909
5	–	–	–	–	−2.0906	−2.0907

Summary

- We have demonstrated that there is a representation that makes a class of stochastic baker's maps formally equivalent to two-dimensional equilibrium spin systems with nearest-neighbour interaction.
- The generalization to longer (but still finite) interaction distance is (relatively) straightforward.
- The invariant measure of the stochastic baker's map is identical to the translation invariant measure of the spin system.
- Thus:
  For any two-dimensional equilibrium spin system there is a corresponding stochastic baker's map that contains all the characteristics of the spin system.

More information...

- Paper (free access), K. Lindgren, EPL **90**, 30011 (2010):

  http://iopscience.iop.org/0295-5075/90/3/30011/fulltext/

- Lecture notes (draft) available on course web site:

  http://studycas.com/node/114

Two envelopes puzzle

Let us say you are given two indistinguishable envelopes, each of which contains a positive sum of money. One envelope contains twice as much as the other. You may pick one envelope and keep whatever amount it contains. You pick one envelope at random but before you open it you are offered the possibility to take the other envelope instead.

Would you switch?