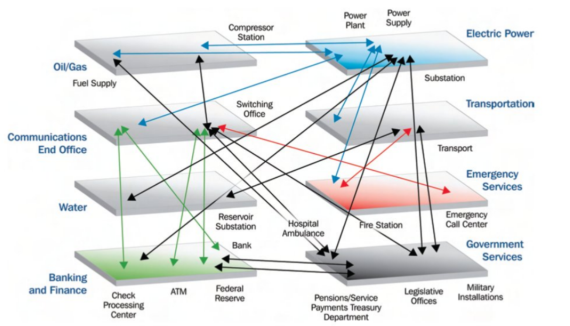
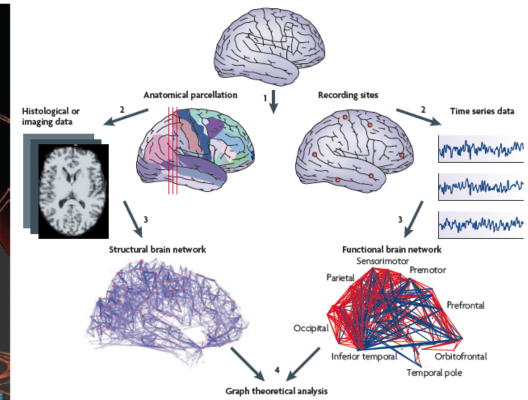


FIGURE 3.1 Connections and interdependencies across the economy. Schematic showing the interconnected infrastructures and their qualitative dependencies and interdependencies. SOURCE: Department of Homeland Security, National Infrastructure Protection Plan, available at http://www.dhs.gov/xprevprot/programs/editorial_0827.shtm.



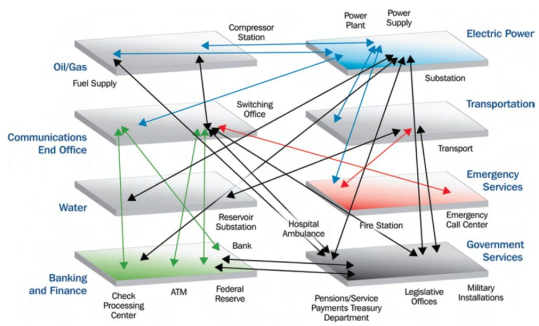
Complexity Summer School

Warwick, 29 April-1 May 2013

Interdependent and Multiplex Networks

Ginestra Bianconi

School of Mathematical Sciences, Queen Mary University of London, London, UK



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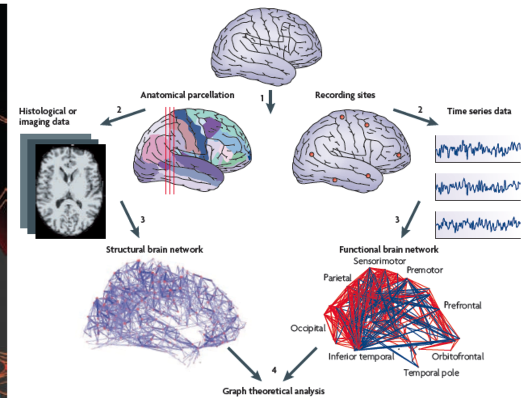


The function of many complex

technological social and biological

systems

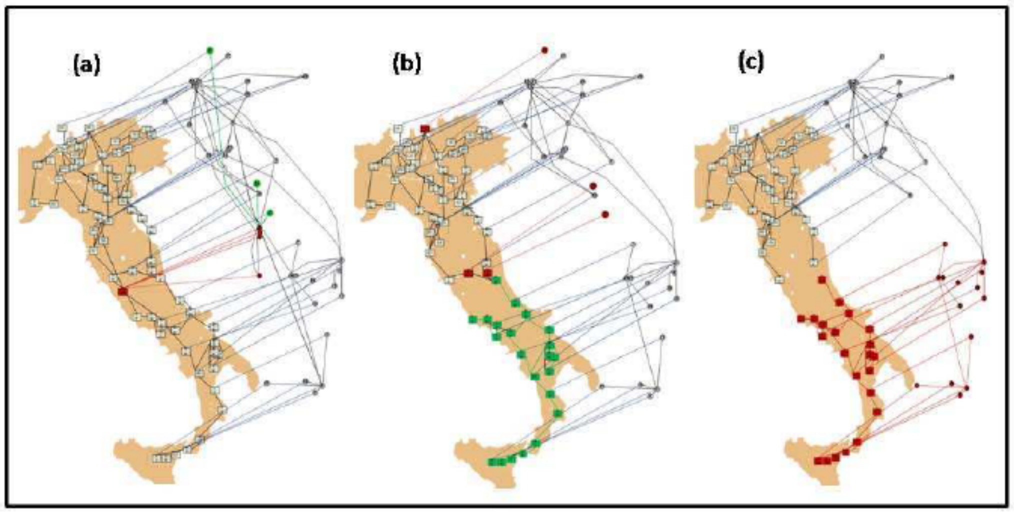
depends on the non-trivial interactions

between

interacting networks

Interacting infrastructure networks

Complex infrastructures are interdependent and a failure in one network can generate a cascade of failures in the Interdependent networks



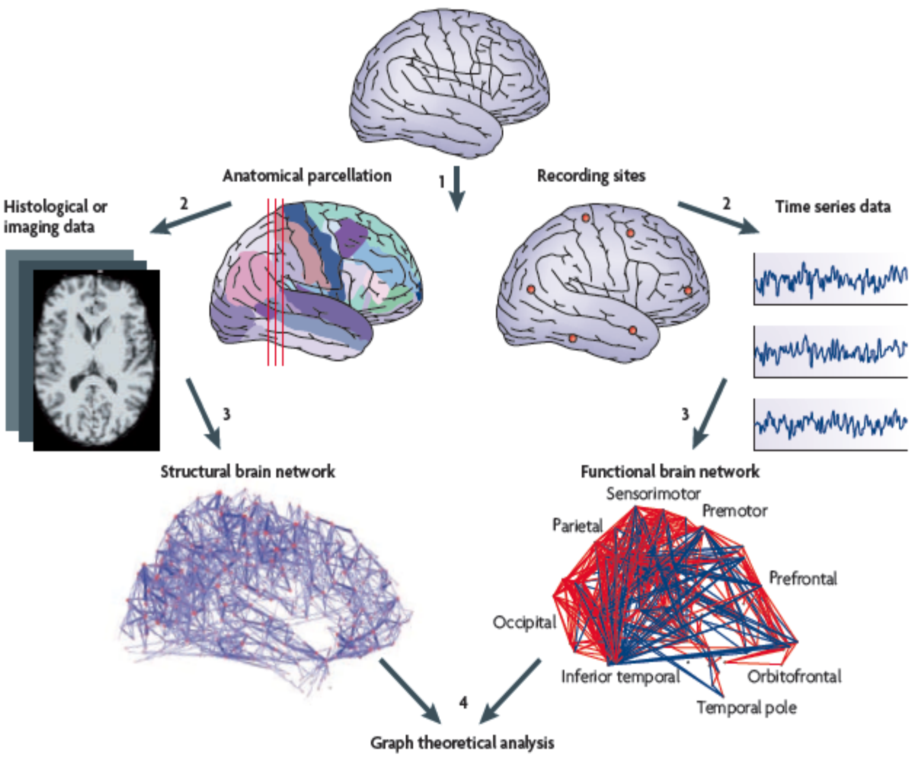
Buldyrev et al. Nature 2010

Interacting Transportation networks

Transportation networks are another major example of interacting networks. Here blue lines represent short-range commuting flow by car or train the red lines indicate airline flow for few selected cities

Vespignani Nature 2010

Interacting and multiplex Brain networks



The brain function is determined at the same time by the structural brain network and the functional brain network, in turn depending on the circulatory system

Bullmore Sporns 2009

Interacting
Social networks

Social networks are interacting and overlapping with profound implications for community detection algorithms

Y.Y. Ahn et al. Nature 2010

Interacting and multiplex networks

In order to

model, predict and control

complex networks

we need to understand the effect of interdependencies between networks and

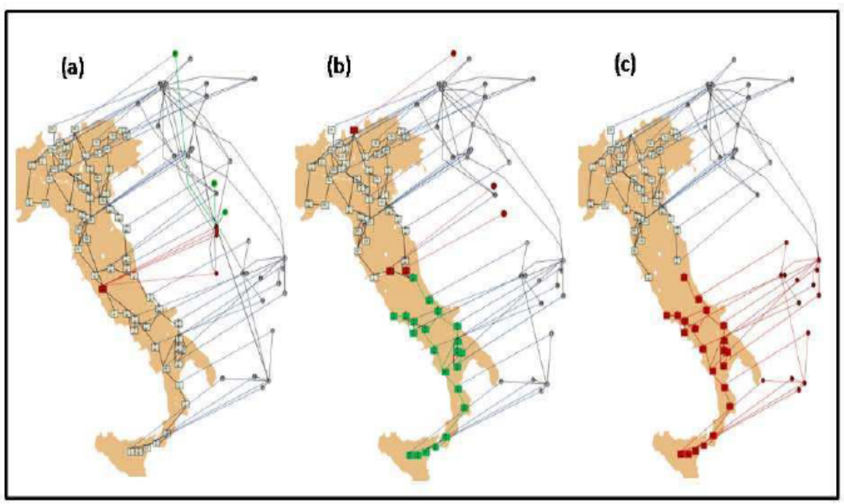
we need to fully characterize the evolution and dynamics of

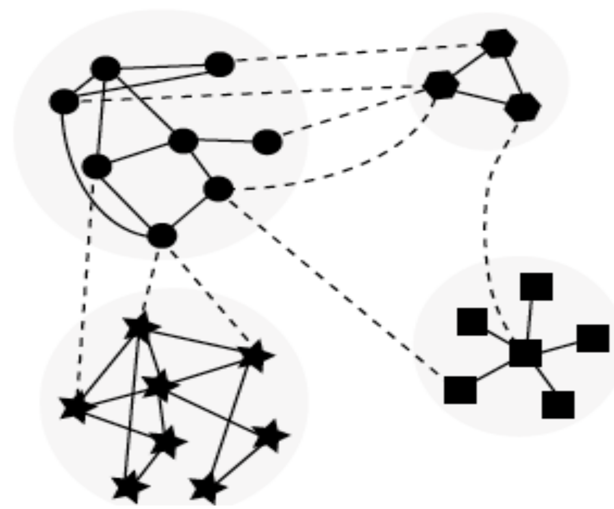
the

networks of networks

Interacting networks

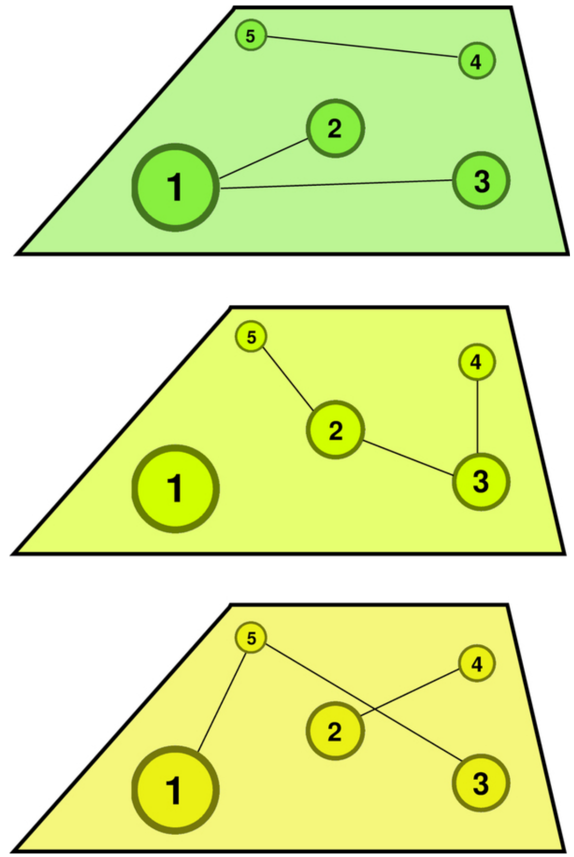
- Two or more interacting networks are formed by different nodes (ex. Power-grid network and Internet) but there might be complex interactions and interdependencies between the nodes



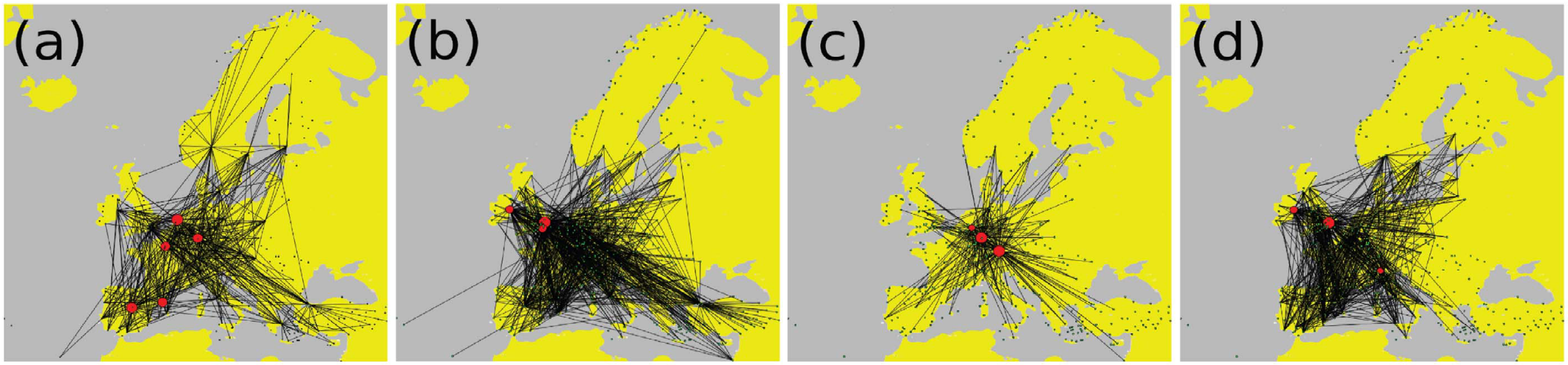


Multiplex

- A multiplex is formed by a set of nodes that are present at the same time on different networks,
- A multiplex is formed by M layers (in the figure M=3)
- Each layer is formed by a network



The airport network is a multiplex

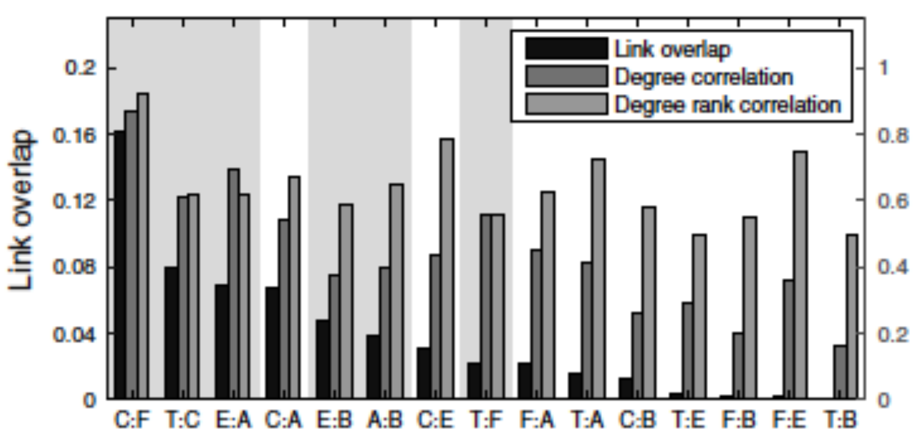


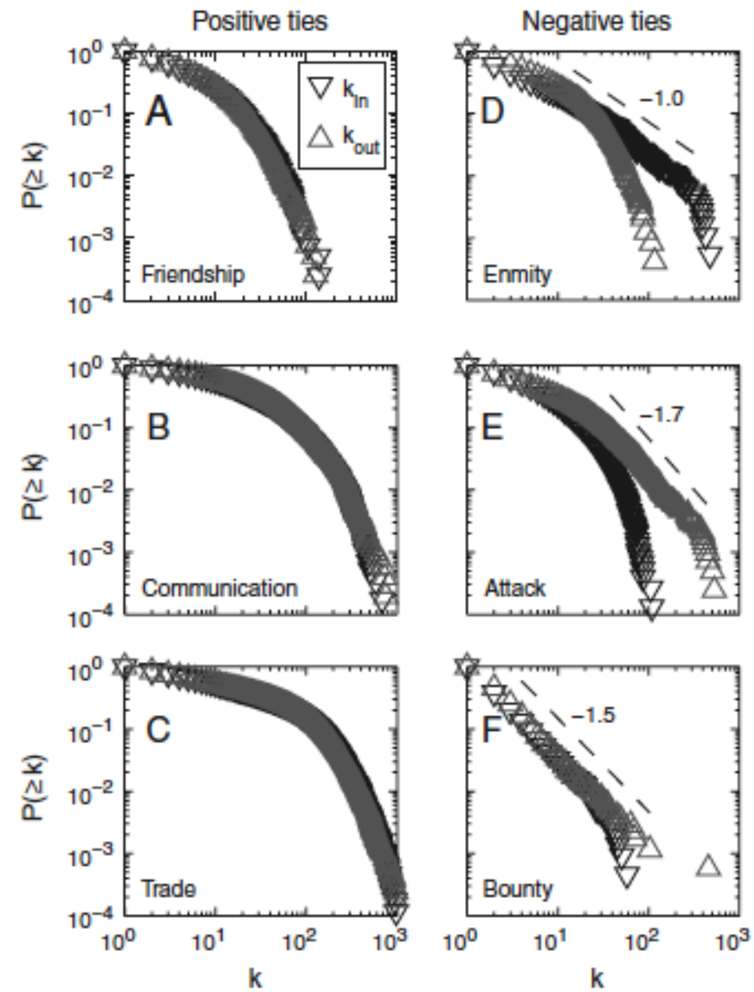
- (a) Only links belonging to all airline companies are plotted
- (b) The combined network where only nodes of degree k>75 have been plotted
- (c) A major airline network
- (d) Low cost airline network

Cardillo et al. Scientific Reports (2013).

The in silico multiplex social social network of an online game

- In this online game agents can belong to different networks Friendship, Communication, Trade, Enmity, Attack and Bounty networks





Szell et al. PNAS 2010

Representation of a multiplex

The straightforward representation a multiplex of N nodes formed by M layers is by means of the set of M adjacency matrices

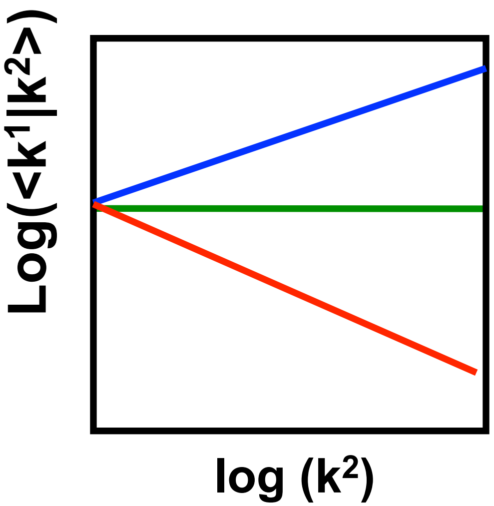
$$\boxed{a^{\alpha}}$$

with α=1,2,…M and matrix elements

$$\boxed{a_{ij}^{\alpha} = \begin{cases} 1 & if \;\; node \; i \; and \; node \;\; j \; are \; linked \; in \; layer \; \alpha \\ 0 & otherwise \end{cases}}$$

Multiplex Models

Conditional average degree in one layer (case of a duplex, i.e. two layers)



Positive degree correlations

No degree correlations

Negative degree correlations

$$\langle k^1 | k^2 \rangle = \sum_{k^1} k^1 P(k^1, k^2)$$

k^1 degree in network 1, k^2 degree in network 2
$P(k^1,k^2)$ probability that a node has degree k^1 in one layer and k^2 in the other layer

Growing multiplex (duplex)

- GROWTH

At each time a new node is added to the multiplex. Every new node has a copy in each layer and has m links in each layer.

- PREFERENTIAL ATTACHMENT

The probability that the new link is added to node i in layer α is given by Π^{α} with

$$\Pi_i^1 \propto a k_i^1 + (1-a) k_i^2$$

$$\Pi_i^2 \propto (1-b) k_i^1 + b k_i^2$$

and a,b $\leq$ 1.

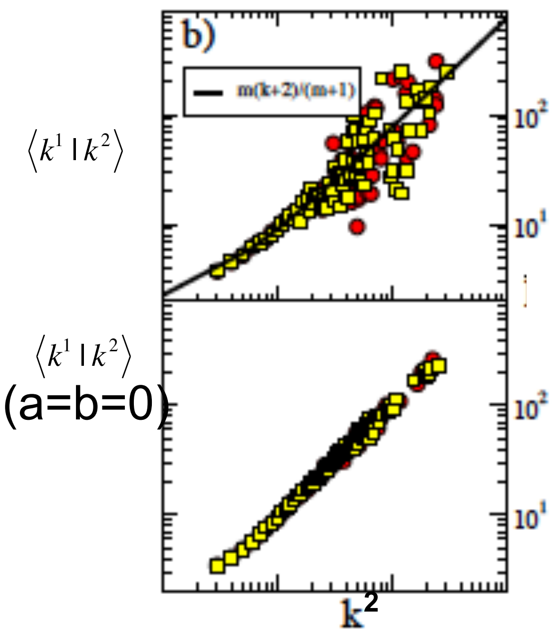
Degree correlations

- Case a=b=1 Exact solution

Nicosia et al arxiv:1302.7126

$$P(k^1,k^2)=\frac{2\Gamma(2+2m)\Gamma(k^1)\Gamma(k^2)\Gamma(k^1+k^2-2m+1)}{\Gamma(m)\Gamma(m)\Gamma(k^1-m+1)\Gamma(k^2-m+1)}$$

$$\left\langle k^1 \mid k^2 \right\rangle=\frac{m}{1+m}(k^2+2)$$



- For general a,b solving in the mean-field approximation it can be obtained

$$\left\langle k^1 \mid k^2 \right\rangle \propto k^2$$

- From the simulation results it is possible to conclude that the degree correlations are minimal in the a=b=1 case

Network measures: Overlap

- For two layers α and α' of the multiplex we can define the

  total overlap $O^{\alpha\alpha'}$ as

$$\boxed{O^{\alpha,\alpha'} = \sum_{i<j} a_{ij}^{\alpha} a_{ij}^{\alpha'}}$$

- For a node i of the multiplex, we can define the

  local overlap $o_i^{\alpha,\alpha'}$

$$\boxed{o_i^{\alpha,\alpha'} = \sum_{j} a_{ij}^{\alpha} a_{ij}^{\alpha'}}$$

Uncorrelated and correlated network ensembles

- A multiplex $\vec{G}$ can be seen as a set of graphs G_α in each layer a of the multiplex, i.e. $\vec{G} = (G_1, G_{2,} \ldots G_\alpha, \ldots . G_M)$
- A **uncorrelated multiplex ensemble** assign to every multiplex a probability given by

$$P(\vec{G}) = \prod_{\alpha=1,M} P_\alpha(G_\alpha)$$

- If instead

$$P(\vec{G}) \neq \prod_{\alpha=1,M} P_\alpha(G_\alpha)$$

the **multiplex ensemble is correlated**

Uncorrelated random multiplex

- **Microcanonical uncorrelated random multiplex**

Multiplex where we fix the total number of links L^{α} in every layer α

The probability that a node i is linked to a node j in layer α is given by

$$\boxed{p_{ij}^{\alpha} = \frac{L^{\alpha}}{N(N-1)/2}}$$

- **Canonical uncorrelated random multiplex**

Multiplex in which we fix the average total number of links $<L^{\alpha}>$ in every layer α

The probability that a node i is linked to a node j in layer α is given by

$$\boxed{p_{ij}^{\alpha} = \frac{<L^{\alpha}>}{N(N-1)/2}}$$

Average overlap in an uncorrelated random multiplex

- We can evaluate the average global overlap in the uncorrelated microcanonical random multiplex getting

$$< O^{\alpha,\alpha'} >= \sum_{i<j} < a_{ij}^{\alpha} a_{ij}^{\alpha'} >= \sum_{i<j} < a_{ij}^{\alpha} >< a_{ij}^{\alpha'} >$$

$$< O^{\alpha,\alpha'} >= \sum_{i<j} p_{ij}^{\alpha} p_{ij}^{\alpha'} = \frac{L^{\alpha} L^{\alpha'}}{N(N-1)/2}$$

- For sparse networks in which $L^{\alpha} \propto N$ **the global overlap is negligible**

$$\boxed{\frac{\langle O^{\alpha,\alpha'} \rangle}{L^{\alpha}} \to 0, \qquad \frac{\langle O^{\alpha,\alpha'} \rangle}{L^{\alpha'}} \to 0}$$

- **We can generalize this result and state that for every sparse uncorrelated network the global and local overlap are negligible!!**

Multilinks and Multiadjacency matrices

- Consider a vector $\vec{m} = (m_1, m_2, \ldots m_\alpha, \ldots m_M)\ with \quad m_\alpha = 0,1$

- A **multilink** $\vec{m}$ is the set of links connecting a given pair of nodes in the different layers of the multiplex and connecting them in a generic layer α only if $m_\alpha=1$.

- The **multiadjacency matrices** have elements $A_{ij}^{\vec{m}} = 1$ only if there is a multilink $\vec{m}$ between node i and node j and zero otherwise, i.e.

$$\boxed{A_{ij}^{\vec{m}} = \prod_{\alpha=1,\ldots M} [m_\alpha a_{ij}^\alpha + (1-m_\alpha)(1-a_{ij}^\alpha)]}$$

Case of two layers

Multiadjacency matrices

$$A_{ij}^{10} = \begin{cases} 1 & \textit{if node } i \textit{ and node } j \textit{ are linked in layer } 1 \textit{ and not linked in layer } 2 \\ 0 & \textit{otherwise} \end{cases}$$

$$A_{ij}^{01} = \begin{cases} 1 & \textit{if node } i \textit{ and node } j \textit{ are linked in layer } 2 \textit{ and not linked in layer } 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$A_{ij}^{11} = \begin{cases} 1 & \textit{if node } i \textit{ and node } j \textit{ are linked in layer } 1 \textit{ and in layer } 2 \\ 0 & \textit{otherwise} \end{cases}$$

Constraints on the multiadjacency matrices

$$\boxed{A_{ij}^{10} + A_{ij}^{01} + A_{ij}^{11} + A_{ij}^{00} = 1}$$

Multidegree

- The multidegree $\vec{m}$ is defined as

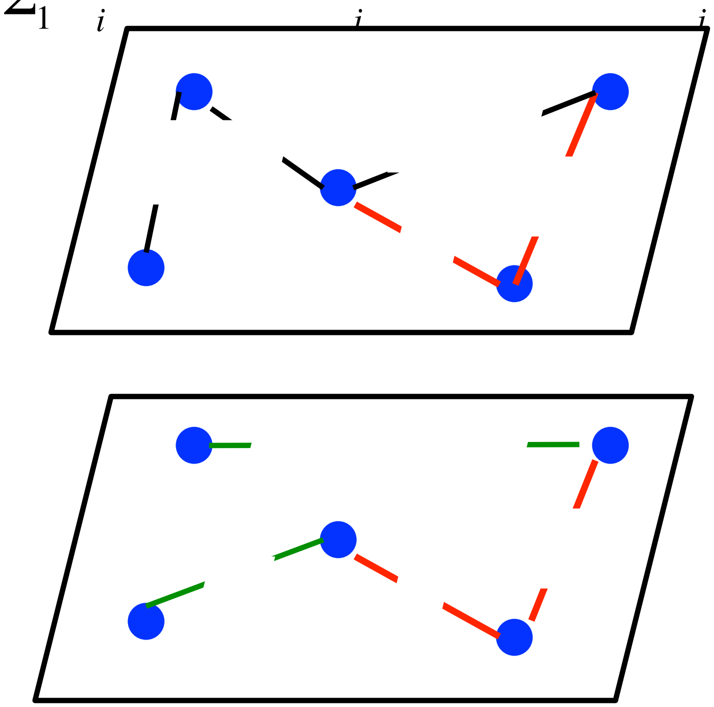
$$\boxed{k_i^{\vec{m}} = \sum_j A_{ij}^{\vec{m}}}$$

- In the case of two layers we have

$$\boxed{\begin{aligned} k_i^{10} &= \sum_j a_{ij}^1 (1 - a_{ij}^0) \\ k_i^{01} &= \sum_j (1 - a_{ij}^1) a_{ij}^0 \\ k_i^{11} &= \sum_j a_{ij}^1 a_{ij}^2 = o_i \end{aligned}}$$

Configuration model for the correlated multiplex(microcanonical ensemble)

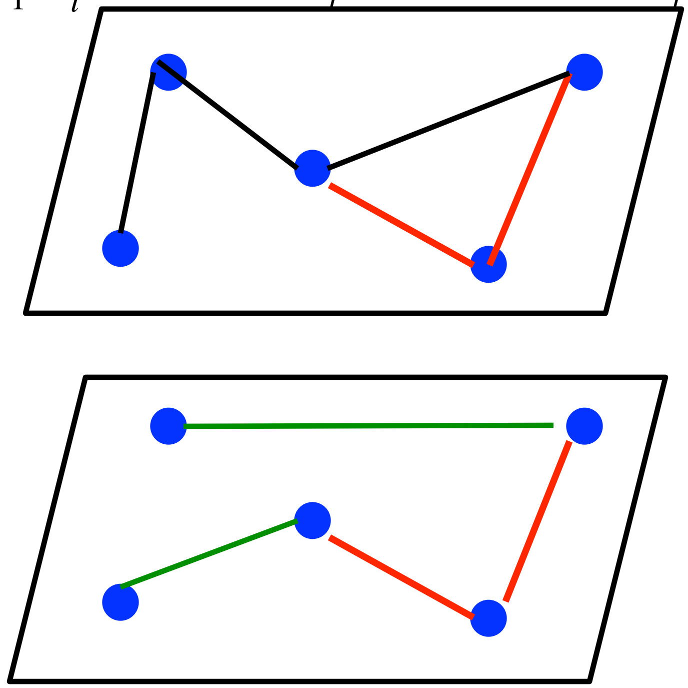
$$P(\vec{G}) = \frac{1}{\Sigma_1}\prod_i \delta(k^{10}{}_i - \sum_j A^{10}_{ij})\delta(k^{01}{}_i - \sum_j A^{01}_{ij})\delta(k^{11}{}_i - \sum_j A^{11}_{ij})$$



Ensemble of multiplex with given multidegree sequence

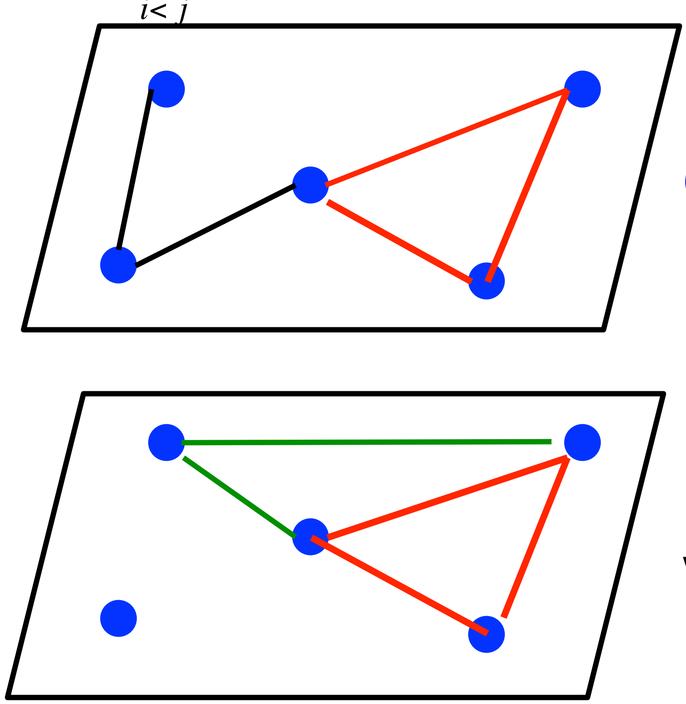
Configuration model for the correlated multiplex (microcanonical ensemble)

$$P(\vec{G}) = \frac{1}{\Sigma_1}\prod_i \delta\Big(k^{10}{}_i - \sum_j A^{10}_{ij}\Big)\delta\Big(k^{01}{}_i - \sum_j A^{01}_{ij}\Big)\delta\Big(k^{11}{}_i - \sum_j A^{11}_{ij}\Big)$$



Canonical network model for the correlated multiplex

$$P(\vec{G}) = \prod_{i<j} (p_{ij}^{10} A_{ij}^{10} + p_{ij}^{01} A_{ij}^{01} + p_{ij}^{11} A_{ij}^{11} + p_{ij}^{00} A_{ij}^{00})$$



Constructive algorithm

For every pair of nodes (i,j)

Draw a multilink $\vec{m}$

with probability $p_{ij}^{\vec{m}}$,

i.e. put a link in every layer

where m_α=1.

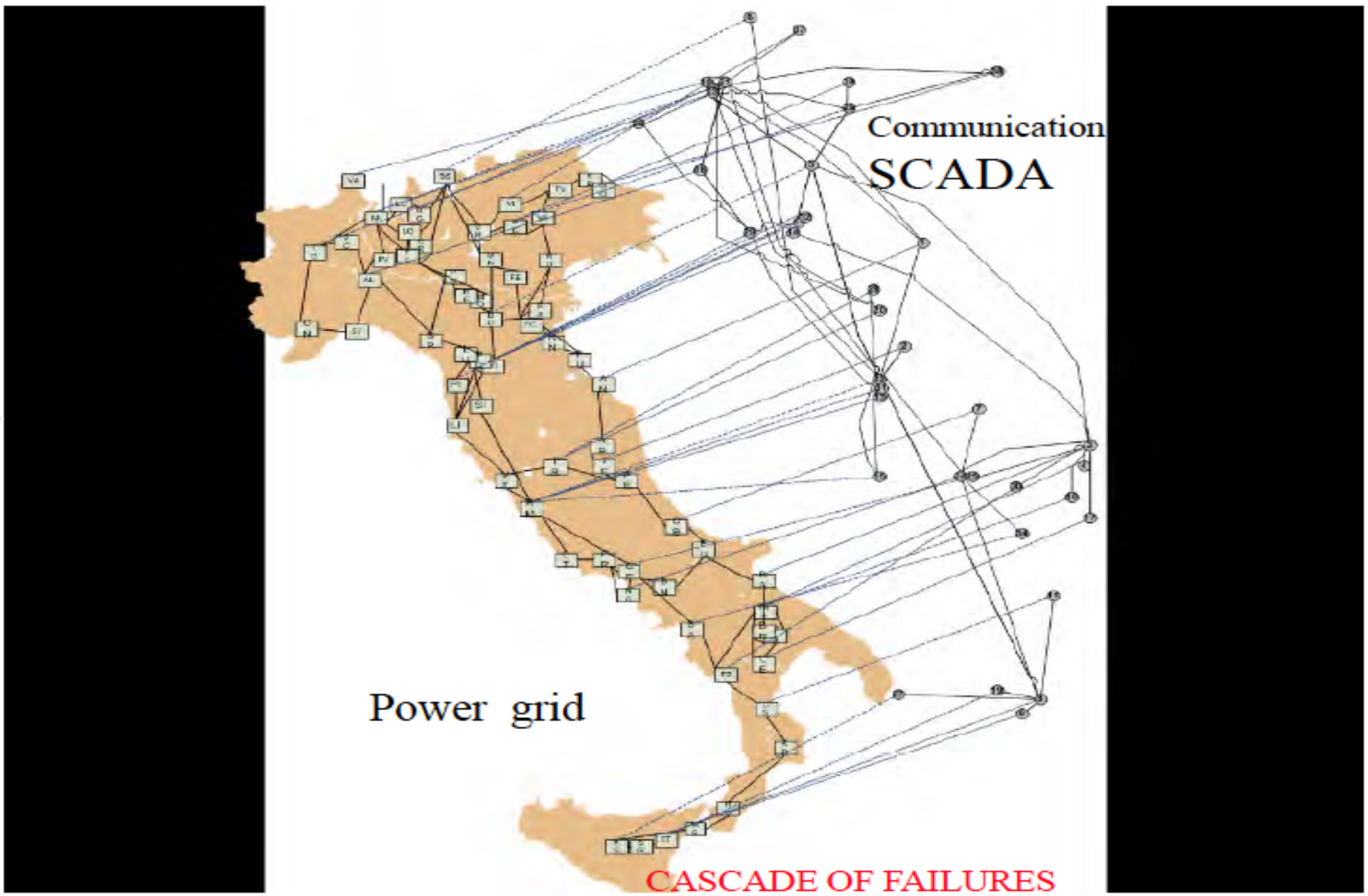
Percolation phenomena in interdependent networks

Cascade of failures: Blackout in Italy (28 September 2003)

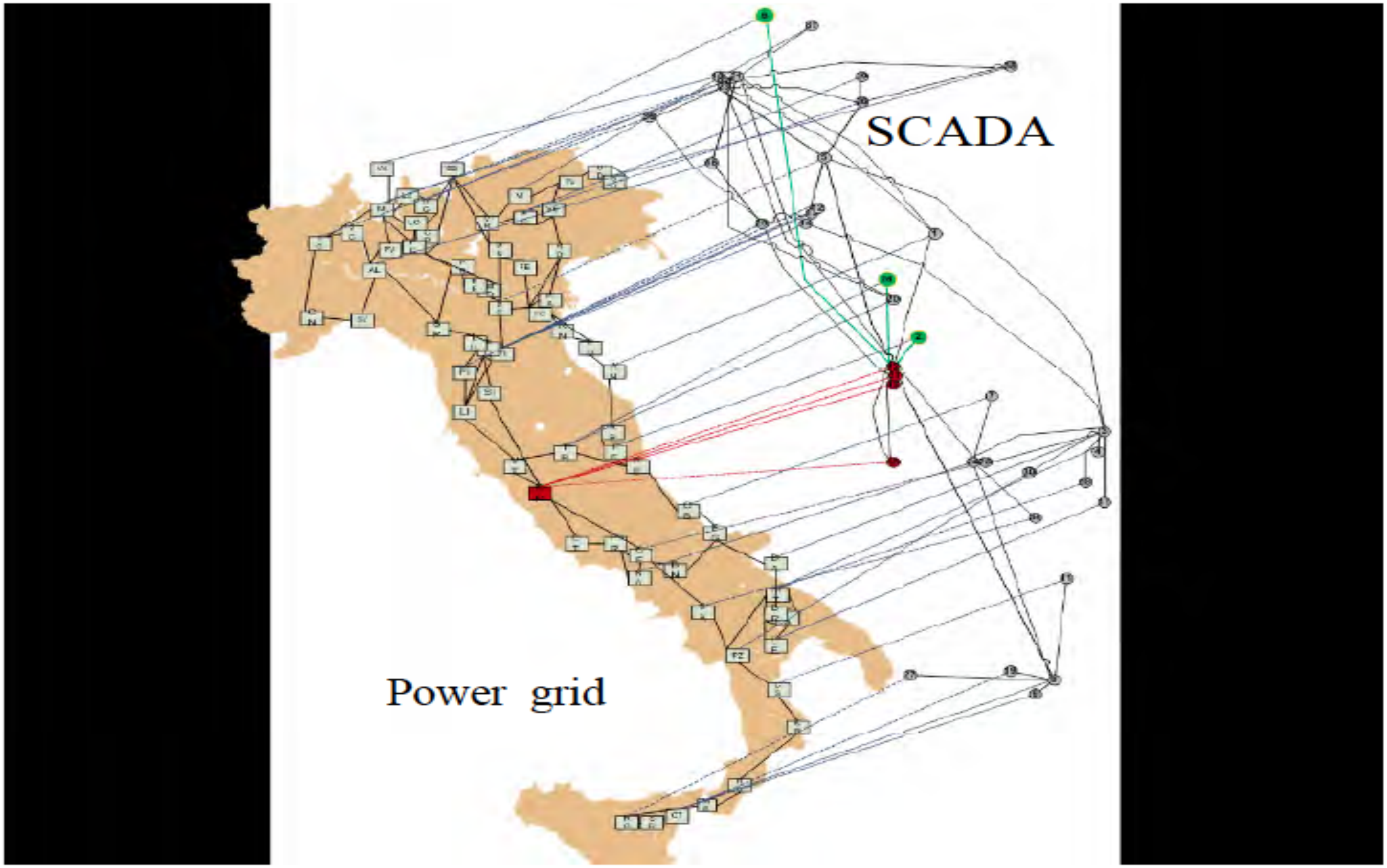
Cyber Attacks-CNN Simulation (2010)

Rosato et al Int. J. of Crit. Infrastruct. 4, 63 (2008)

From S. Havlin slides

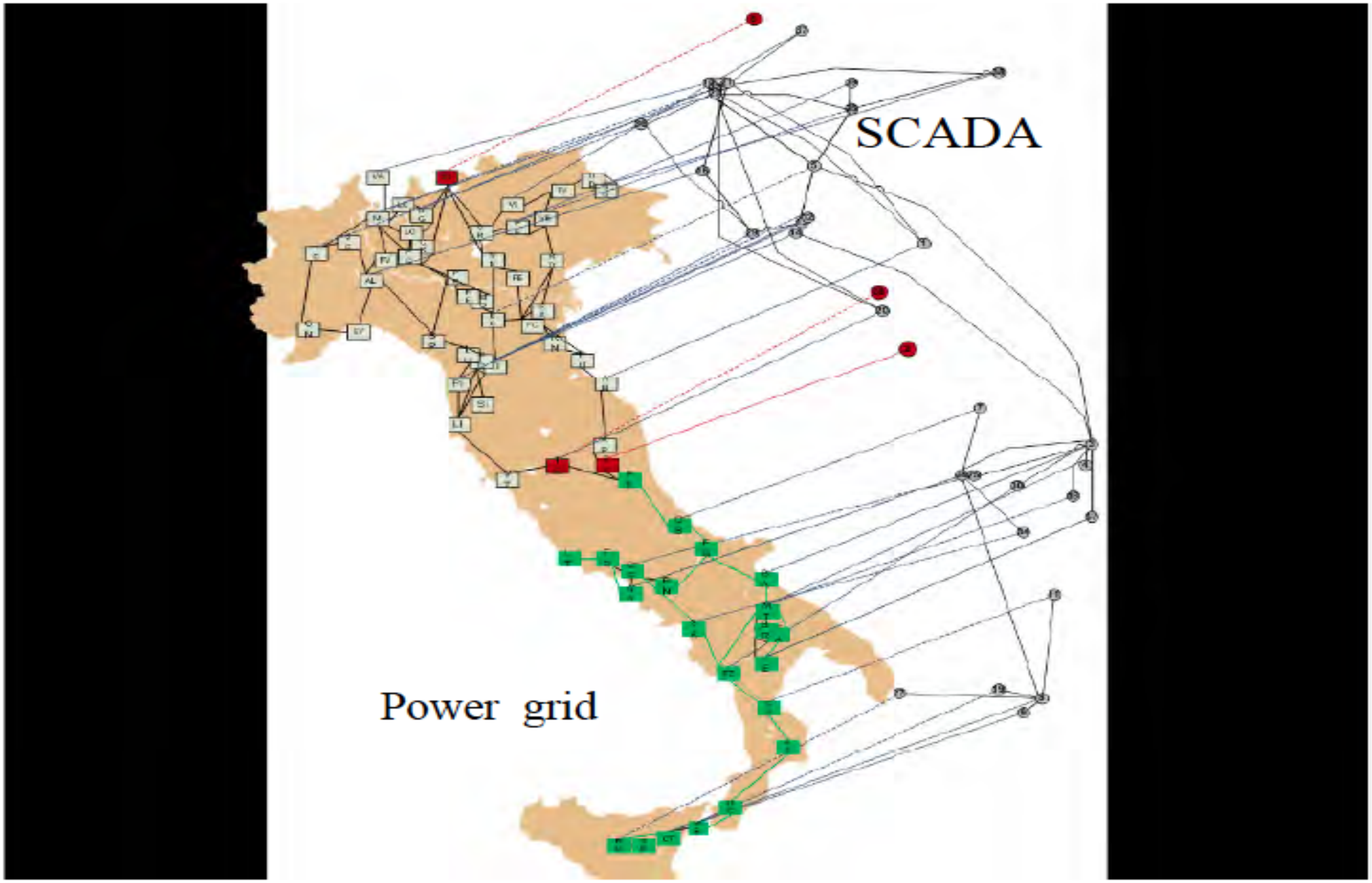


Cascade of failures: Blackout in Italy (28 September 2003)

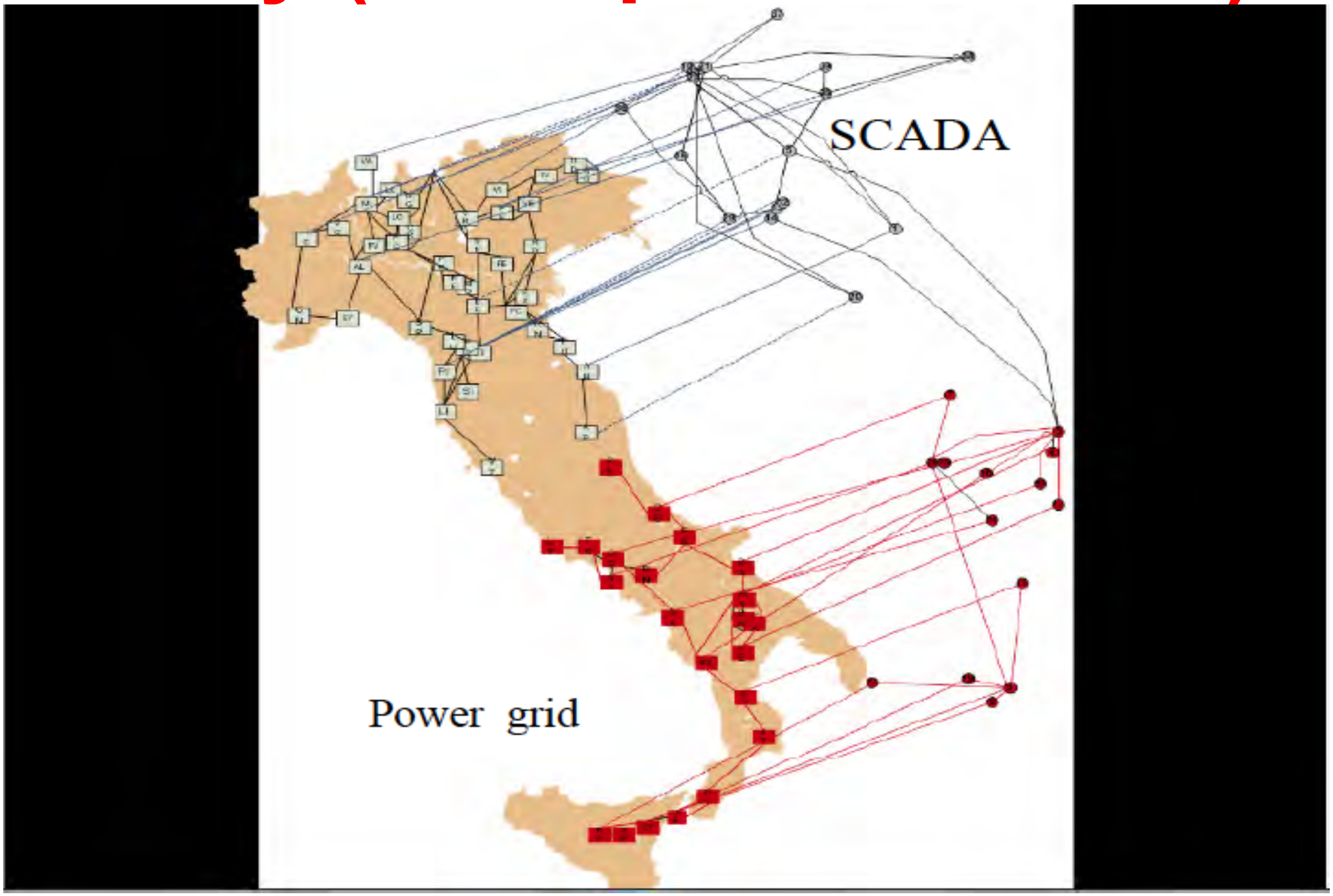


SCADA=Supervisory Control And Data Acquisition

Cascade of failures: Blackout in Italy (28 September 2003)

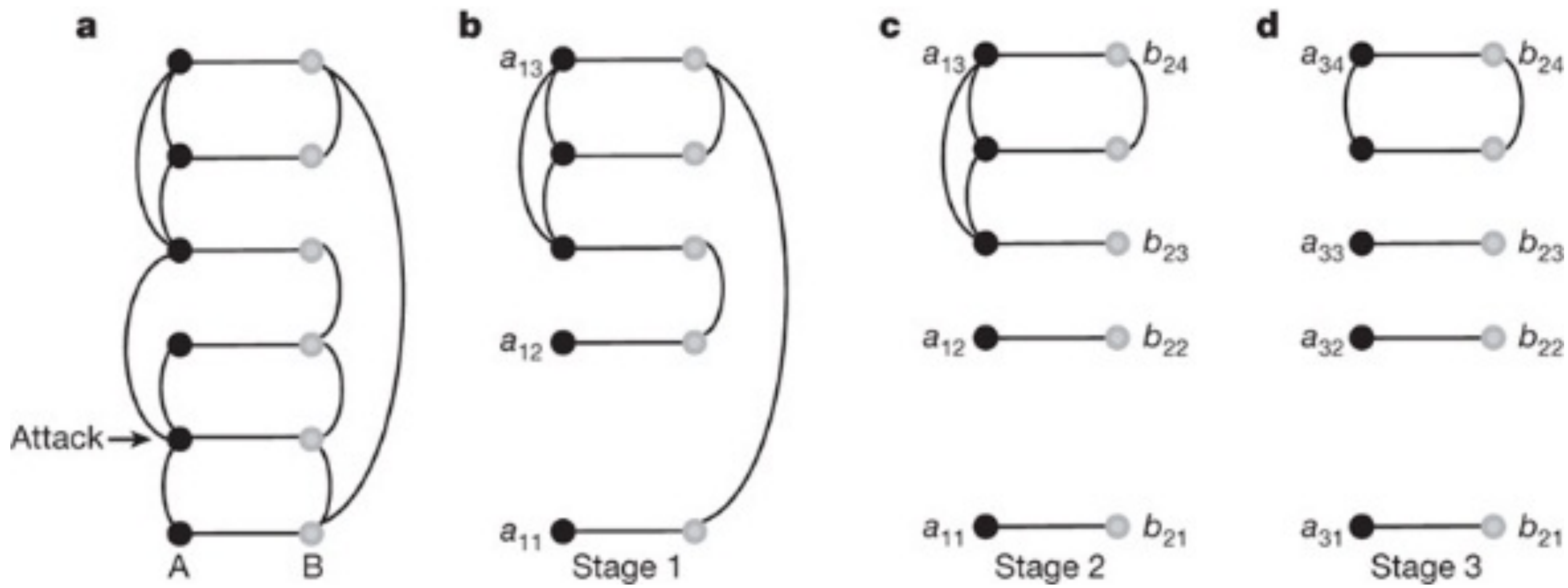


Cascade of failures: Blackout in Italy (28 September 2003)



Percolation on interdependent networks

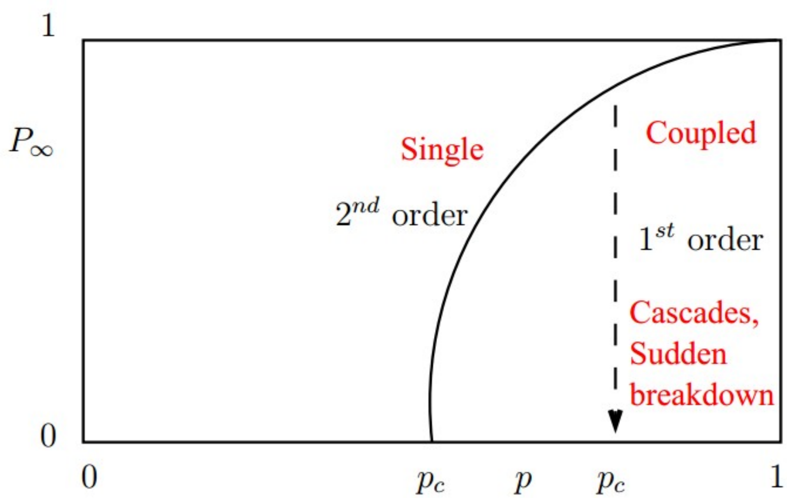
The model proposed by Buldyrev et al, Nature **(2010).**

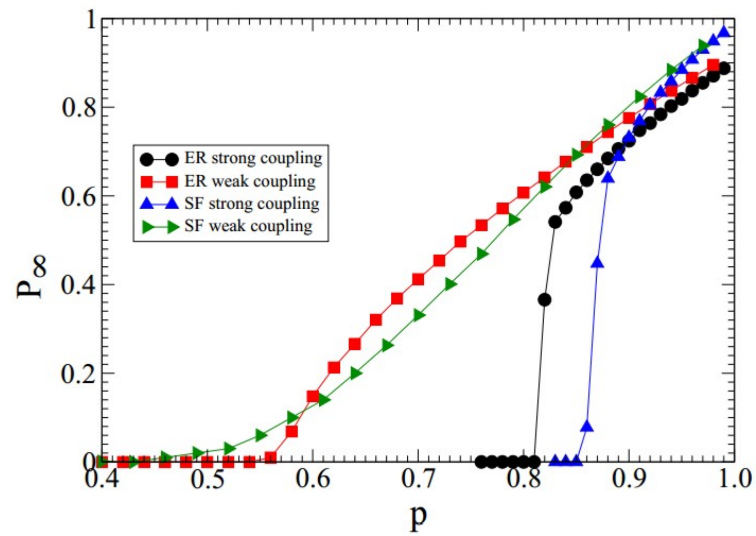


Iterative process of cascading failures:

•We start by randomly removing a fraction 1 − p of network A nodes and all the A-links that are connected to them;
•We remove the nodes in network B that depend on removed A-nodes together with the B-links that are connected to them.
•We continue the iterative process until the networks break into different independent connected components (or clusters).

The percolation transition of interdependent networks can be first-order!





The fraction of nodes of the giant vs. the probability p that a node is not randomly removed (Havlin et al. 2010)

In the next slides we will show that the emergence of the mutually connected giant component can be first order

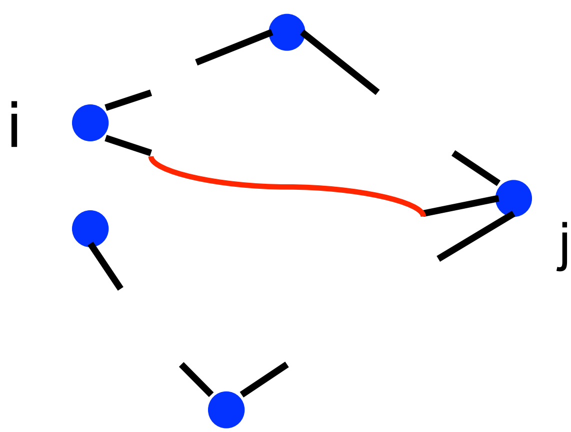
(case $p=1$)

To this end first we derive the Molloy Reed criterion

Preliminaries: probability that following a link we reach a node of degree k

In uncorrelated networks *the probability that following a link we reach a node of degree k is given by*

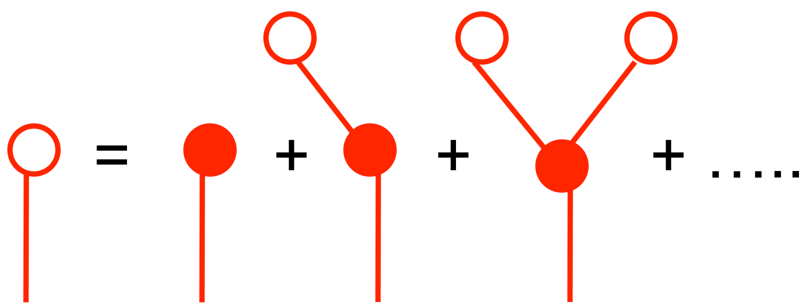
$$q_k = \frac{k}{\langle k \rangle} p_k$$



$$q_k = \frac{k}{\langle k \rangle N} N p_k = \frac{k}{\langle k \rangle} p_k$$

Emergence of the Giant Component in a network with p_k degree distribution

- S probability that a node is in the giant component
- S' probability that following a link we reach a node that is in the giant component
- In a locally tree like network S' satisfies

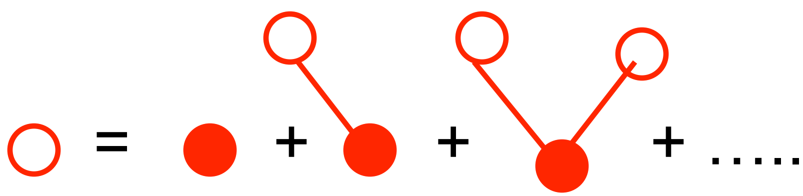


$$\boxed{1 - S' = \sum_k \frac{k}{\langle k \rangle} p_k (1 - S')^{k-1}}$$

Emergence of the Giant Component in a network with p_k degree distribution

- S probability that a node is in the giant component
- S' probability that following a link we reach a node that is in the giant component

- In a locally tree like network S satisfies



$$\boxed{1 - S = \sum_k p_k (1 - S')^k}$$

Using the generating functions

The equation for S and S' can be written in terms of the generating functions

$$\boxed{\begin{aligned} S' &= 1 - G_1(1 - S') \\ S &= 1 - G_0(1 - S') \end{aligned}}$$

$$\boxed{\begin{aligned} G_0(x) &= \sum_k p_k x^k \\ G_1(x) &= \sum_k \frac{k}{\langle k \rangle} p_k x^{k-1} \end{aligned}}$$

Molloy Reed condition

- The equation for S’ has always a S’=0 solution

$$\boxed{S' = 1 - \sum_k \frac{k}{\langle k \rangle} p_k (1 - S')^{k-1}}$$

- The non trivial solution S’>0 emerges for

$$\boxed{\frac{\langle k^2 \rangle}{\langle k \rangle} \geq 2}$$

Poisson network of average degree z

Generating functions for a Poisson network

$$G_0(x) = G_1(x) == \sum_k \frac{z^k}{k!} e^{-c} x^k = e^{-z(1-x)}$$

Simplification of the equations for S and S'

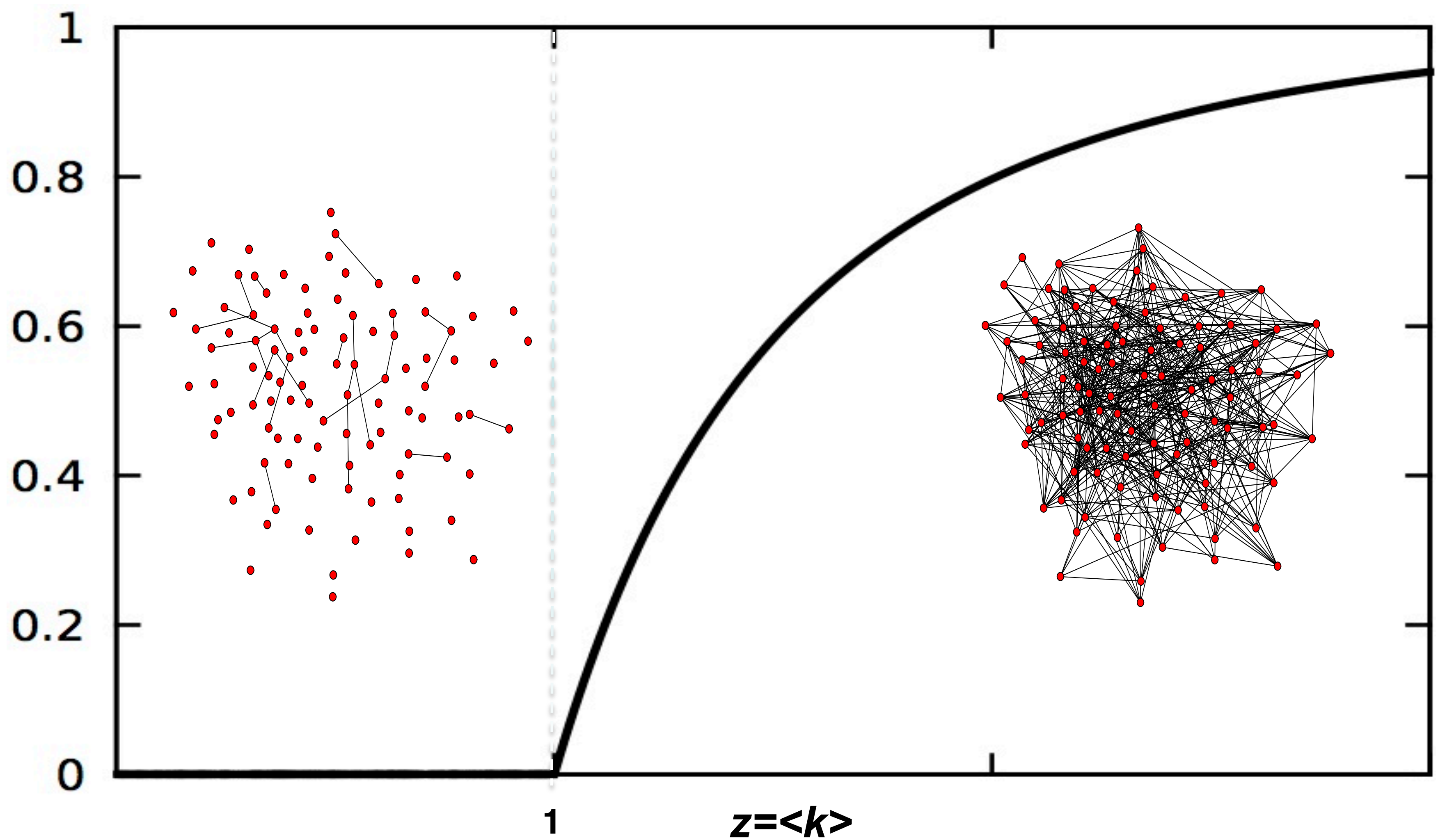
$$\boxed{\begin{aligned} S' &= 1 - G_1(1 - S') \\ S &= 1 - G_0(1 - S') \end{aligned}}$$

$$\Rightarrow$$

$$\boxed{S = S' = (1 - e^{-zS})}$$

Evolution of a random graph

disconnected nodes ➔ **NETWORK.**



Mutually connected giant component of interdependent networks

In the system of interdependent networks, the function or activity of a node depends on the function or activity of the linked nodes in the others networks.

We consider a duplex formed by two networks: network A and network B

A node of the mutually connected giant component must satisfy the following conditions:

1) *at least one of its neighbors in network A should belong to the mutually connected giant component;*

2) *at least one of its neighbors in network B should belong to the mutually connected giant component.*

Emergence of the mutually connected giant component

- S probability that a node is in the mutually connected giant component
- $S'_{A/B}$ probability that following a link in network A/B we reach a node that is in the mutually connected giant component

- On a locally tree-like multiplex the equations for S and S'_A, S'_B are given by

$$\boxed{S=\left[1-G_0^A(1-S'_A)\right]\left[1-G_0^B(1-S'_B)\right]}$$

$$\boxed{\begin{aligned} S'_A &= \left[1-G_1^A(1-S'_A)\right]\left[1-G_0^B(1-S'_B)\right] \\ S'_B &= \left[1-G_0^A(1-S'_A)\right]\left[1-G_1^B(1-S'_B)\right] \end{aligned}}$$

Two Poisson networks of average degree z

Generating functions for the two Poisson networks

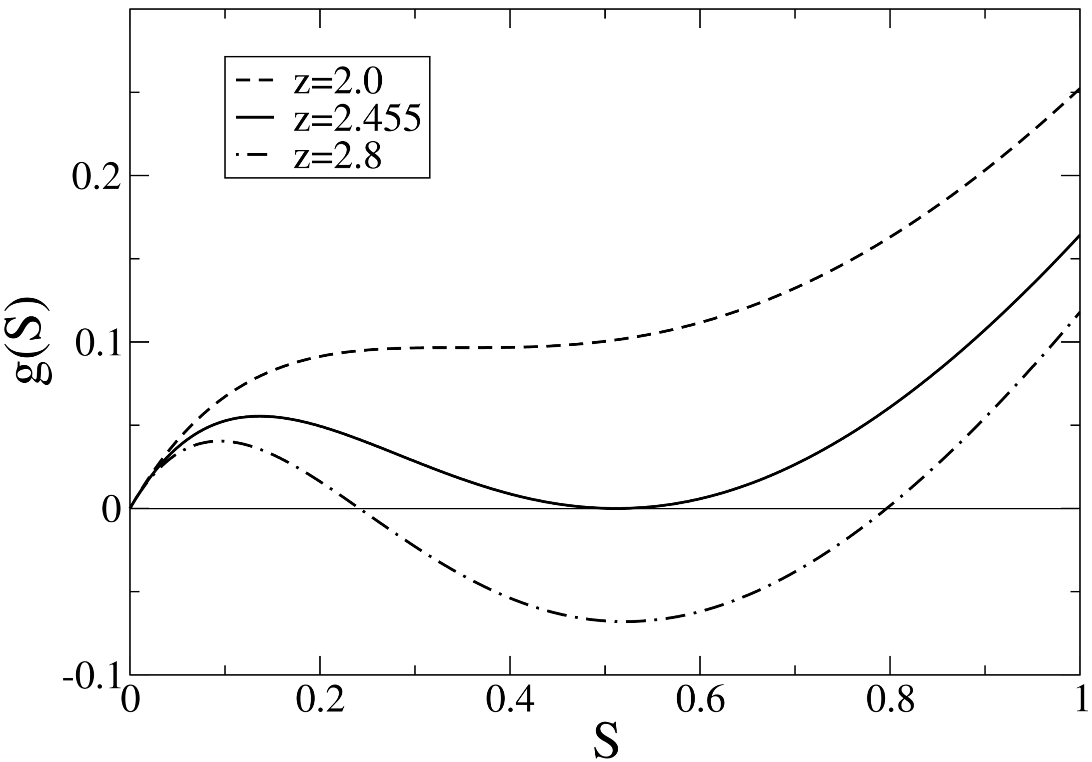
$$G_0^A(x) = G_0^B(x) = G_1^A(x) = G_1^B(x) = \sum_k \frac{z^k}{k!} e^{-c} x^k = e^{-z(1-x)}$$

Simplification of the equations for S and S'$_A$ and S'$_B$

$$\boxed{\begin{aligned} S &= \left[1 - G_0^A(1 - S_A^{'})\right]\left[1 - G_0^B(1 - S_B^{'})\right] \\ S_A^{'} &= \left[1 - G_1^A(1 - S_A^{'})\right]\left[1 - G_0^B(1 - S_B^{'})\right] \\ S_B^{'} &= \left[1 - G_0^A(1 - S_A^{'})\right]\left[1 - G_1^B(1 - S_B^{'})\right] \end{aligned}} \Rightarrow \boxed{S = S_A^{'} = S_B^{'} = (1 - e^{-zS})^2}$$

Percolation on two interdependent Poisson networks with average degree z

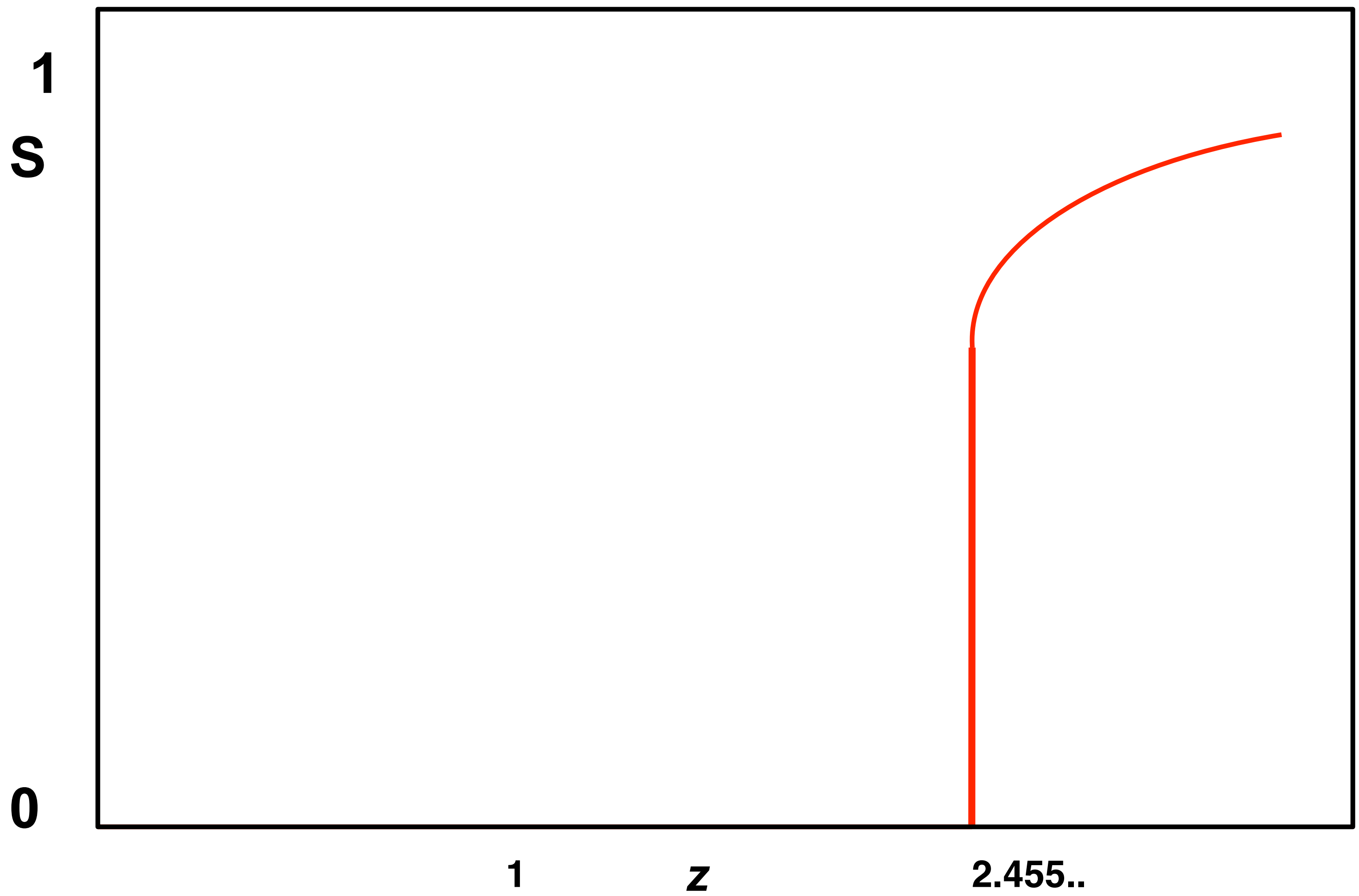
$$g(S) = S - \left(1 - e^{-zS}\right)^2 = 0$$



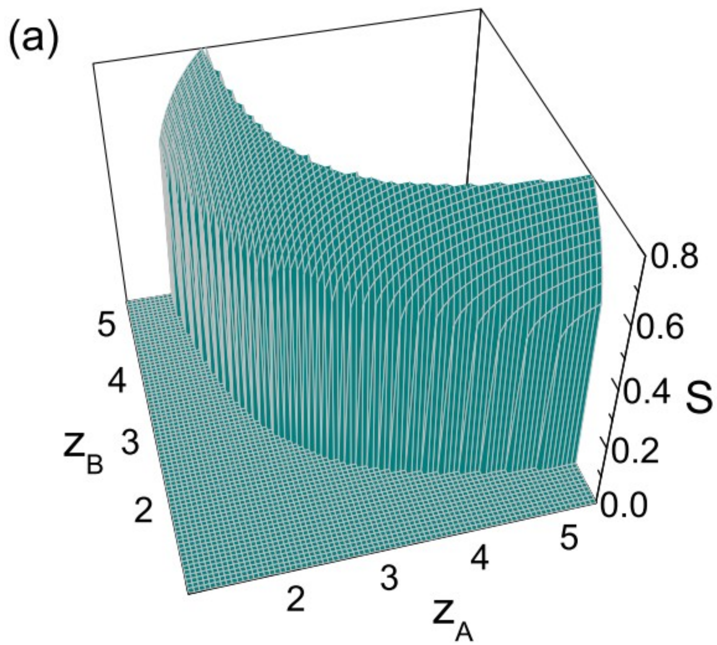
The percolation transition at z=2.455… is **first-order!**

Son S.-W., et al. EPL(2012)

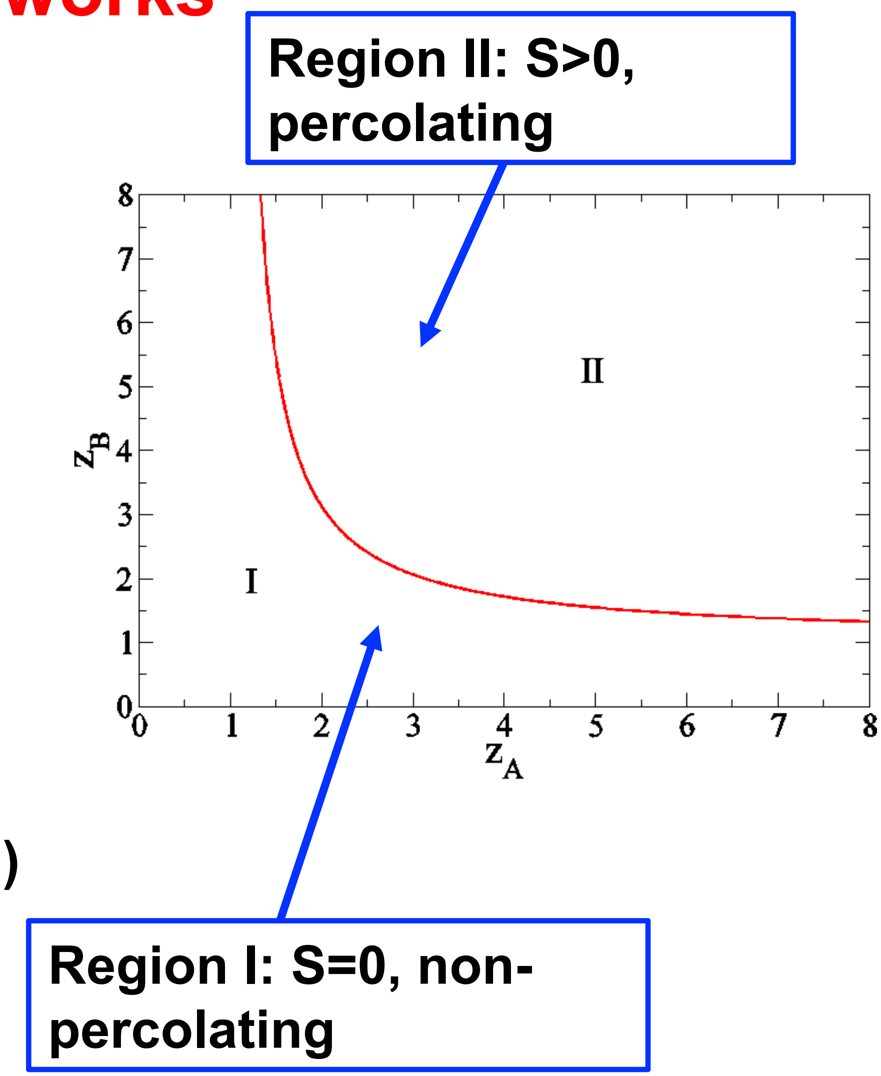
Emergence of the mutually connected giant component



Phase diagram of ER-ER interdepedent networks



Son S.-W., et al. EPL(2012)



Conclusions

- Many networks interact, coexist and coevolve with other networks.
- Many networks are also multiplex indicating the fact that two nodes might interact on different layers at the same time
- Modeling interacting and multiplex networks is only in its infancy and we need to develop a new series of non-equilibrium and equilibrium models and to compare their outcome to real data.
- Critical phenomena on multiplex and interacting networks show new surprising physics as the percolation first-order phase transitions.