Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

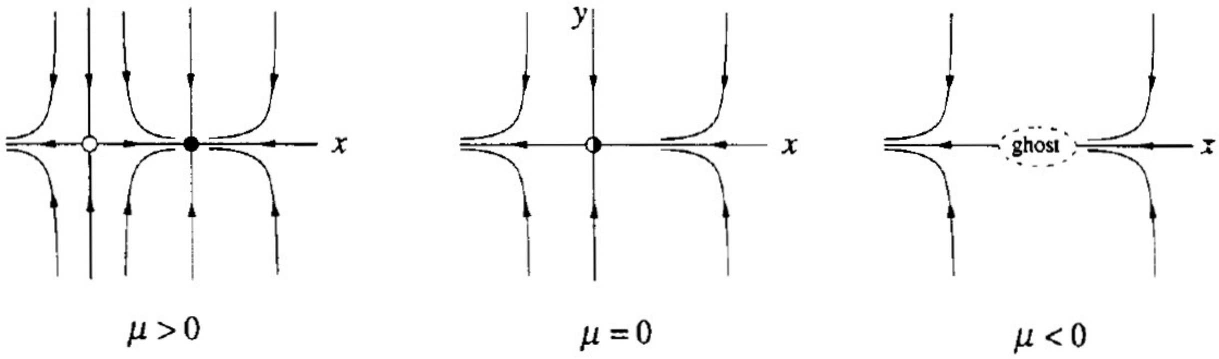
5 Bifurcations

5.1 Local bifurcations

Saddle-node

The saddle-node bifurcation is the basic mechanism for the creation and destruction of fixed points.

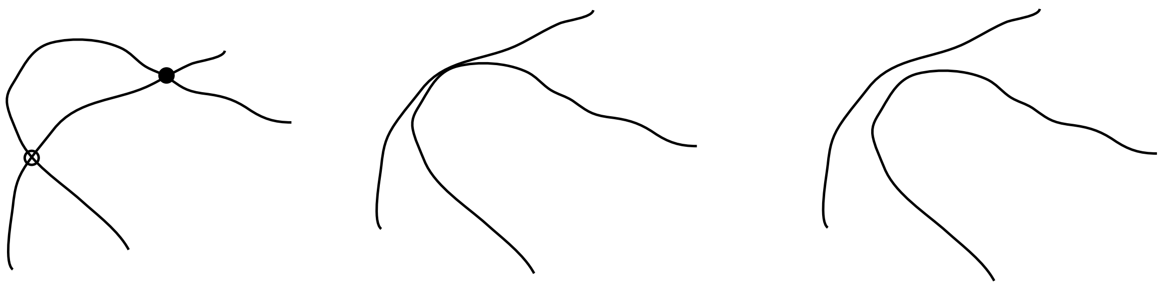
$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$



Consider a two-dimensional system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

that depends on a parameter μ. Then the nullclines can intersect for some value of μ. As μ varies the nullclines can pull away from each other (the fixed points approach each other and collide when $\mu = \mu_c$).



$\mu = \mu_c$

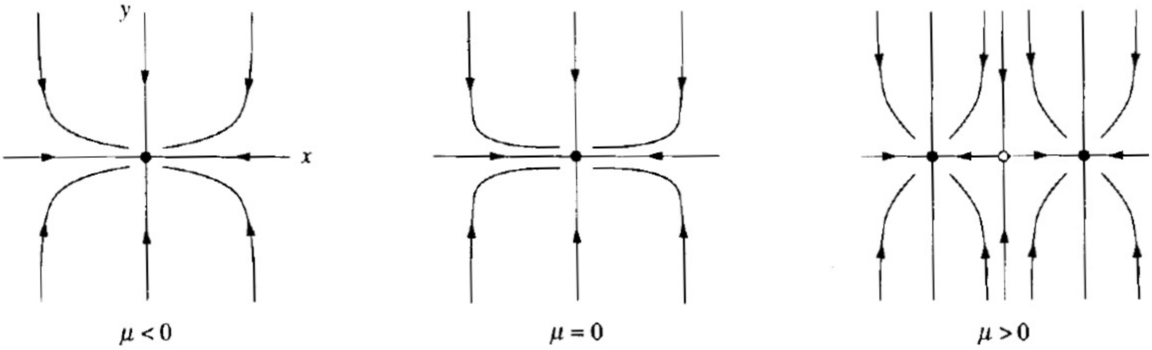
Transcritical and Pitchfork

We can construct examples of transcritical and pitchfork bifurcation:

$$\dot{x} = \mu x - x^2, \qquad \dot{y} = -y \qquad \text{transcritical}$$

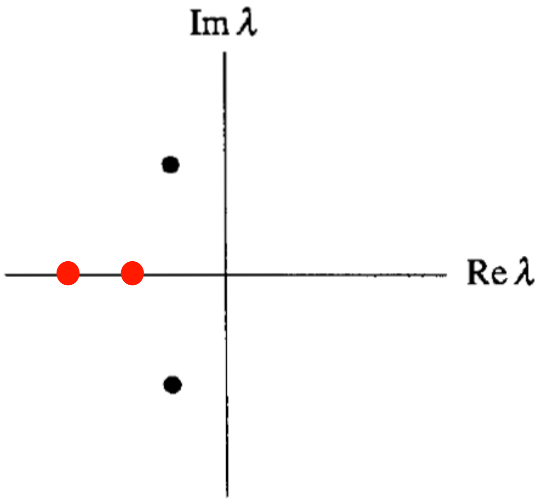
$$\dot{x} = \mu x - x^3, \qquad \dot{y} = -y \qquad \text{supercritical}$$
$$\dot{x} = \mu x + x^3, \qquad \dot{y} = -y \qquad \text{subcritical}$$

Example 1. The supercritical pitchfork system for $\mu < 0$, $\mu = 0$ and $\mu > 0$.



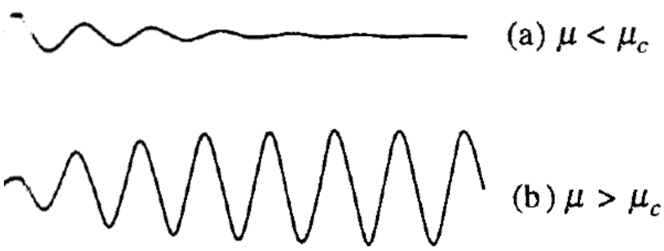
Hopf bifurcations

If the fixed point of a two-dimensional system is stable (for the fixed parameter μ), the eigenvalues λ_1, λ_2 must both lie in the left-half plane $\mathrm{Re}\lambda < 0$. To destabilize the fixed point, we need one or both of the eigenvalues to cross into the right-half plane as μ varies.



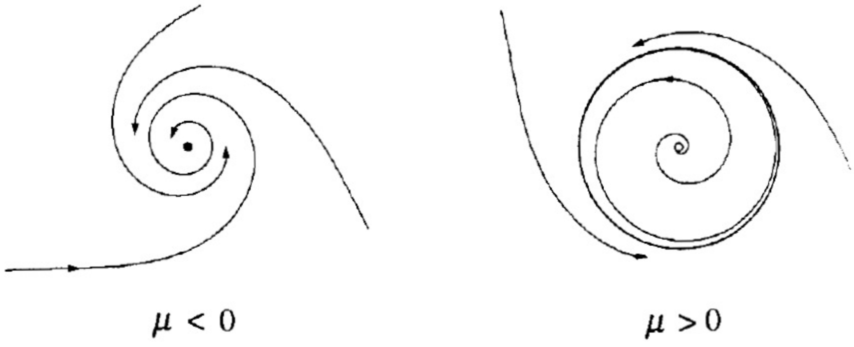
Supercritical HB

If the decay becomes slower and slower and finally changes to growth at a critical value μ_c, the equilibrium state will lose stability.The resulting motion is a small-amplitude, sinusoidal, limit cycle oscillation about the former steady state.

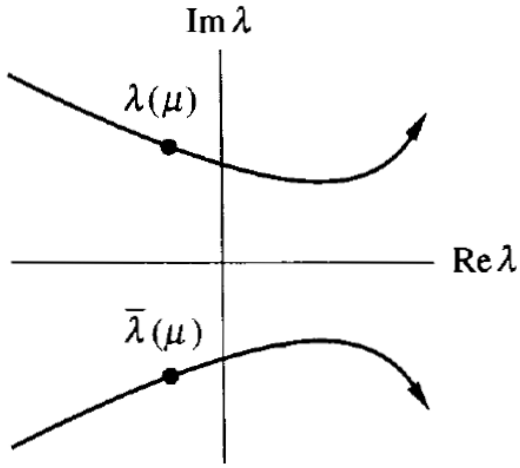


Example 2.

$$\dot{r} = \mu r - r^3, \qquad \dot{\theta} = \omega + br^2$$



Normally, the eigenvalues would follow a curvy path and cross the imaginary axis with nonzero slope:



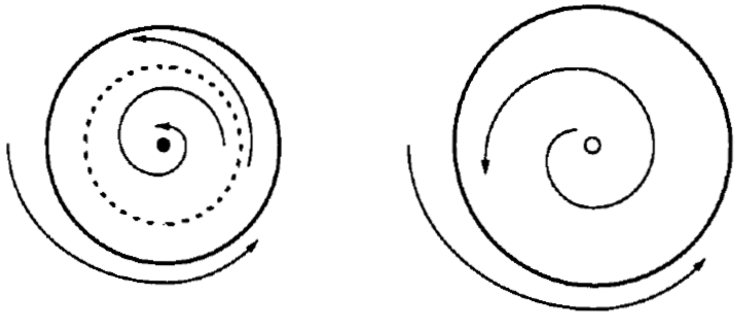
Subcritical HB

The subcritical case is always much more dramatic, and potentially dangerous in engineering applications. After the bifurcation the trajectories must jump to a distant attractor, which may be a fixed point, another limit cycle, infinity, or a chaotic attractor (in three and higher dimensions).

Example 3.

$$\dot{r} = \mu r + r^3 - r^5 \qquad \theta = \omega + br^2$$

A subcritical HB occurs at $\mu = 0$, where the unstable cycle shrinks to zero amplitude. For $\mu > 0$ the large amplitude limit cycle is suddenly the only attractor.



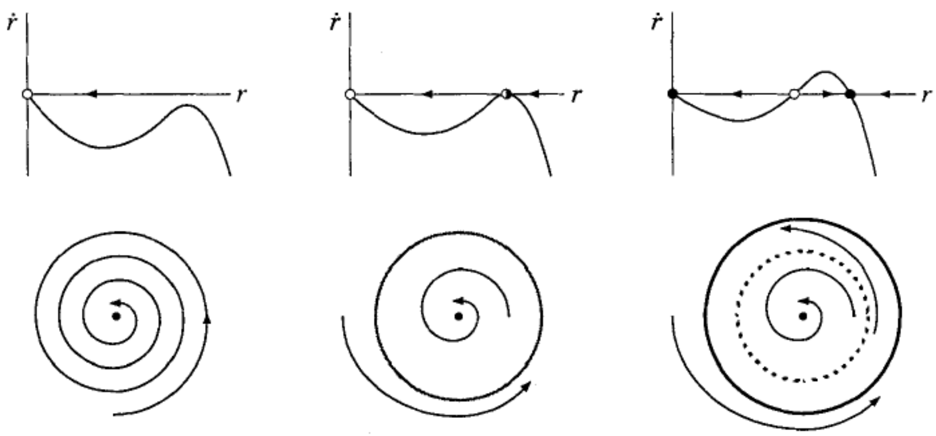
$\mu < 0$ $\qquad\qquad$ $\mu > 0$

5.2 Global bifurcations of cycles

Saddle-node bifurcation of cycles

A bifurcation in which two limit cycles coalesce and annihilate. An example occurs in the system:

$$\dot{r} = \mu r + r^3 - r^5 \qquad \theta = \omega + br^2$$

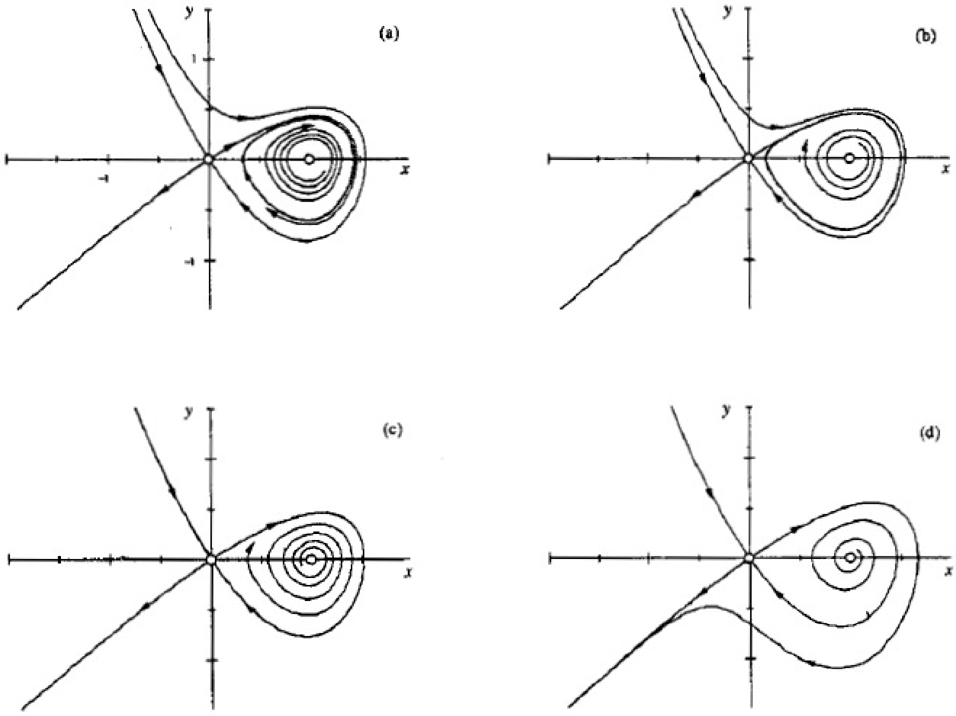


$\mu < \mu_c$ $\qquad\qquad$ $\mu = \mu_c$ $\qquad\qquad$ $0 > \mu > \mu_c$

Homoclinic bifurcation

In this bifurcation part of a limit cycle moves closer and closer to a saddle point. At the bifurcation the cycle touches the saddle point and becomes a homoclinic orbit. For example, consider the system

$$\dot{x} = y, \qquad \dot{y} = \mu y + x - x^2 + xy$$



The period of oscillations tends to infinity at the homoclinic bifurcation.

5.3 Heteroclinic bifurcation

In the phase portrait of a dynamical system, a *heteroclinic orbit* (sometimes called a heteroclinic connection) is a path in phase space which joins two different equilibrium points. (Note: if the equilibrium points at the start and end of the orbit are the same, the orbit is a homoclinic orbit). Consider the continuous dynamical system:

$$\dot{x} = f(x).$$

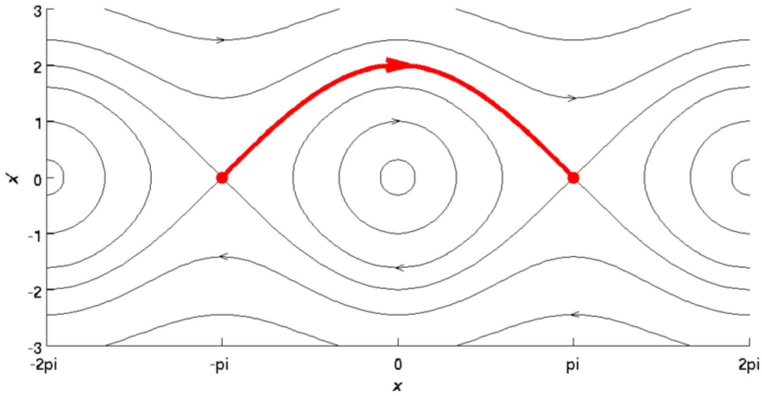
Suppose there are equilibria at $x = x_0$ and $x = x_1$. Then a solution $\phi(t)$ is a heteroclinic orbit from x_0 to x_1 if

$$\phi(t) \to x_0 \qquad \text{as} \qquad t \to -\infty$$

and

$$\phi(t) \to x_1 \qquad \text{as} \qquad t \to +\infty$$

This implies that the orbit is contained in the stable manifold of x_1 and the unstable manifold of x_0.
For example, the phase portrait of the pendulum equation $\ddot{x} + \sin x = 0$:



The highlighted curve shows the heteroclinc orbit from $(x, \dot{x}) = (-\pi, 0)$ to $(x, \dot{x}) = (\pi, 0)$. This orbit corresponds with the pendulum starting upright, making one revolution through its lowest position, and ending upright again.

A heteroclinic cycle is an invariant set in the phase space of a dynamical system. It is a topological circle of equilibrium points and connecting heteroclinic orbits.

A heteroclinic bifurcation is a global bifurcation involving a heteroclinic cycle.

State space reconstruction

If the dynamical system can be modelled by a system of ODEs, then one can make an unambiguous identification between the state variables and the phase-space coord of the dynamical system. When the equations of motion are unknown the attractor has to be reconstructed from some measured time series $z(t)$. From Takens embedding theorem we have the following: For almost every observable $z(t)$ and time delay τ, an m-dimensional portrait constructed from the vectors

$$\{z(t_0), z(t_0 + \tau), \ldots, z(t_0 + (m - 1)\tau)\}$$

can have the same properties (same Lyapunov exponents) as the original attractor. Strictly speaking, the phase portrait obtained by this procedure gives an embedding of the original manifold. The choice of delay τ is pretty arbitrary but the choice of the embedding dimension m is not. This presents us with some difficulties!