Complexity Science Doctoral Training Centre

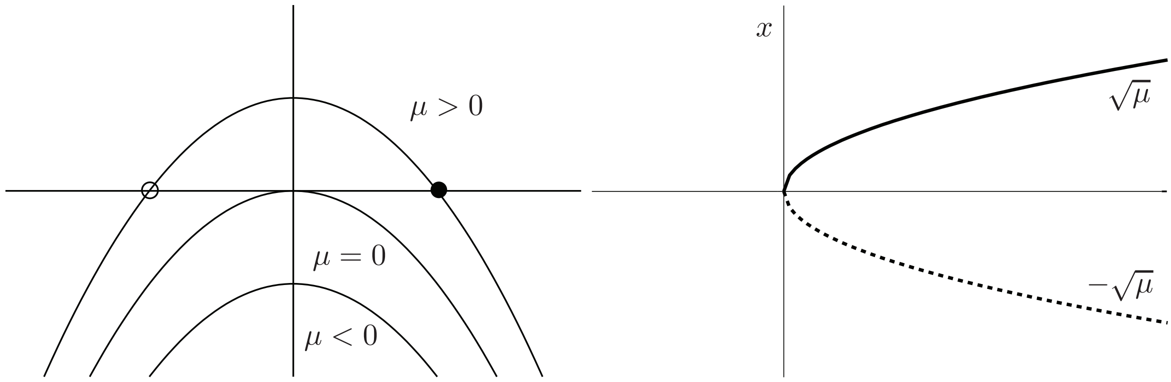
CO903 Complexity and Chaos in Dynamical Systems

Bifurcations

The qualitative structure of a flow can change as a parameter is varied. These qualitative changes are called *bifurcations* and the parameter values at which they occur are called *bifurcation* points.

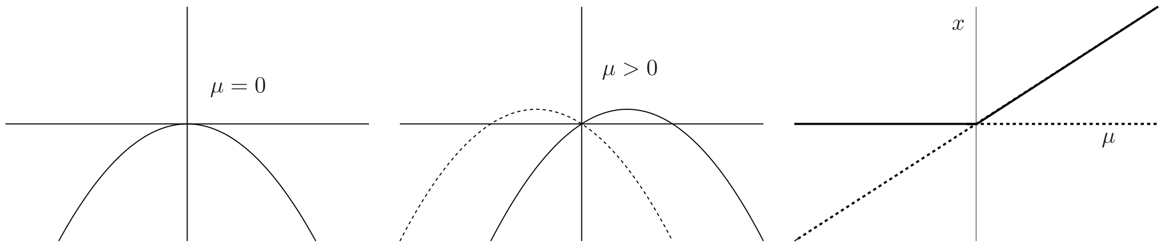
Saddle-Node bifurcation

$$\dot{x} = \mu - x^2$$



Transcritical bifurcation

$$\dot{x} = \mu x - x^2$$



Example 1.

$$\dot{x} = r \ln x + x - 1$$

Fixed point at $x = 1$. Let $u = x - 1$, then

$$\begin{aligned}\dot{u} &= r \ln(1 + u) + u \approx r\left(u - \frac{u^2}{2} + \ldots\right) + u \\ &= (r+1)u - \frac{1}{2} r u^2\end{aligned}$$

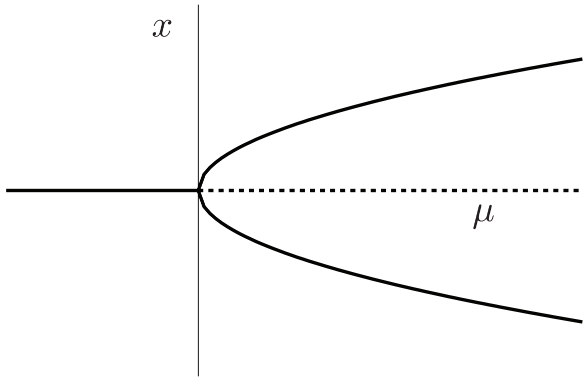
Rescale ($v = (r/2)u$):

$$\dot{v} = (r+1)v - v^2$$

By a near identity change of co-ords we have found the *normal form* for the dynamics (valid close to the bifurcation point).

Pitchfork bifurcation: supercritical

$$\dot{x} = \mu x - x^3$$



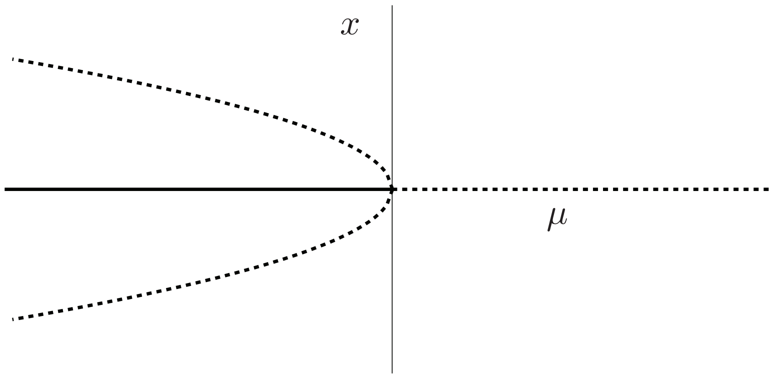
Shows critical slowing down at $\mu = 0$:

$$\int \frac{\mathrm{d}x}{x^3} = -\int \mathrm{d}t \Rightarrow x = \sqrt{\frac{1}{2(t+C)}}, \quad C = \frac{1}{2x_0^2}(x_0 \neq 0)$$

For large t, $x \sim t^{-1/2}$: power law decay rather than exponential $\mathrm{e}^{\mu t}$.

Pitchfork bifurcation: subcritical

$$\dot{x} = \mu x + x^3$$



Example 2.

$$\dot{x} = \mu x + x^3 - x^5$$

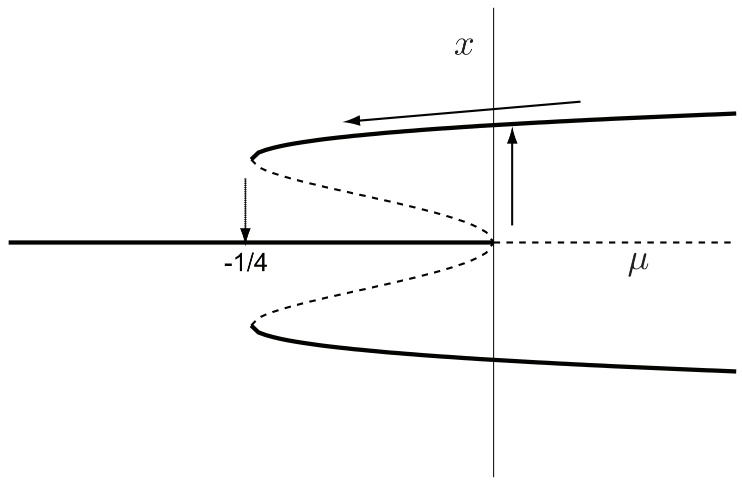
Fixed points

$$-(\mu + x^2) + x^4 = 0 \qquad \text{and} \qquad x = 0$$

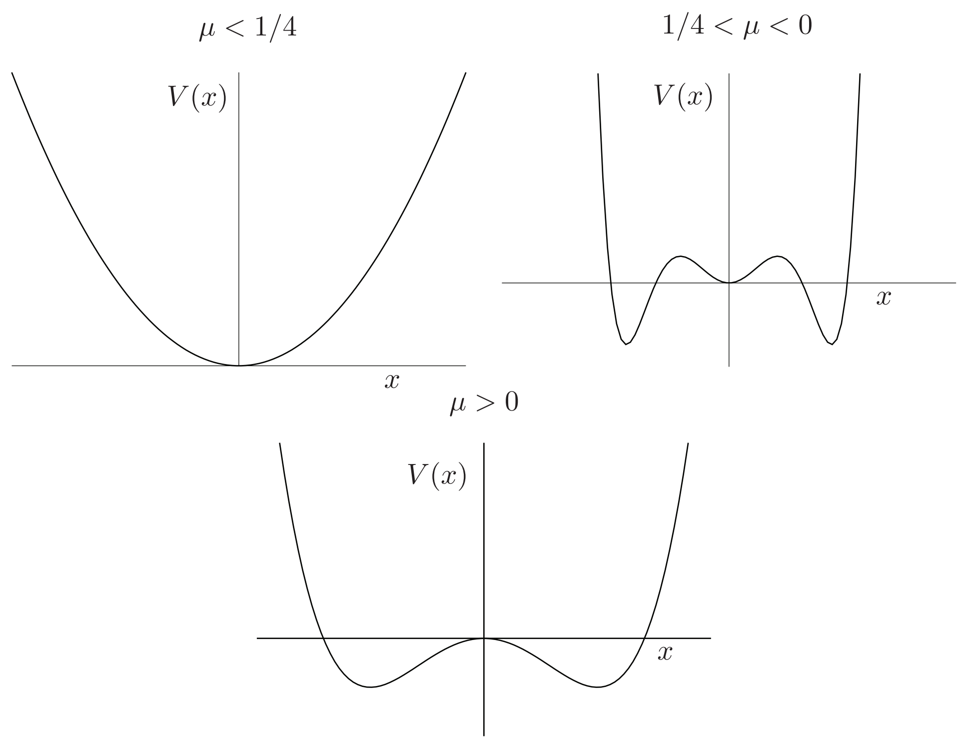
roots:

$$x^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

If $\mu > 0$ then $x^2 = (1 + \sqrt{1+4\mu})/2$: total of three fixed points. If $-1/4 < \mu < 0$, $x^2 = (1 \pm \sqrt{1+4\mu})/2$: total of five fixed points. Define $\mu_c = -1/4$.



1. In range $\mu_c < \mu < 0$ there co-exist 3 stable fixed points (and 2 unstable). There is *multi-stability*. (Local not global stability). Initial conditions determine the final state.
2. Bifurcation at μ_c is a saddle-node bifurcation.
3. System exhibits hysteresis and jump phenomenon.
4. If x^5 term was absent then blow up could occur.

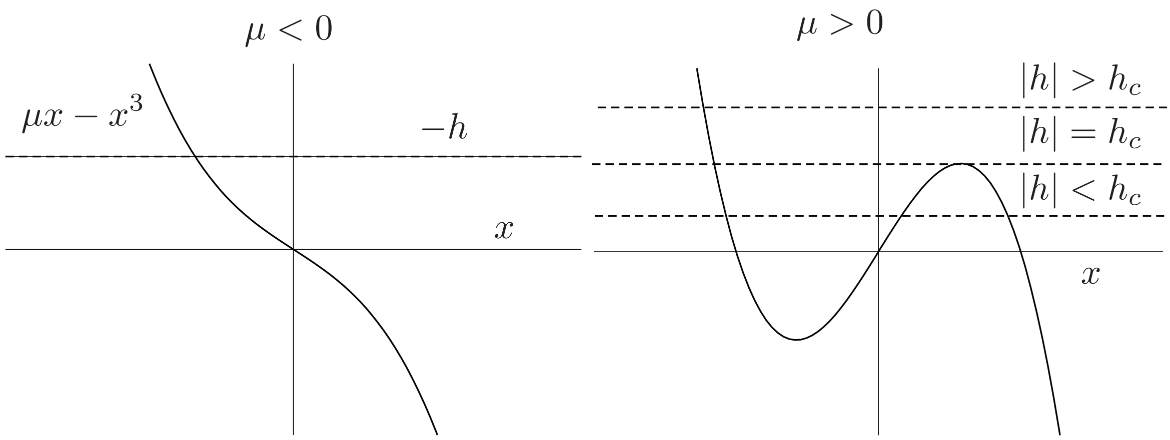


Cusp singularity

The pitchfork bifurcation is common in problems with reflection symmetry. Imperfections break this symmetry.

$$\dot{x} = h + \mu x - x^3$$

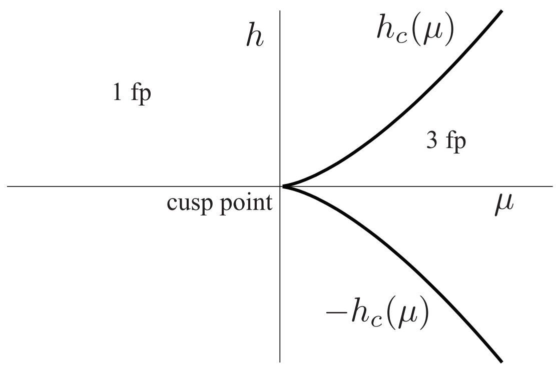
Co-dimension 2 rather than co-dimension 1.

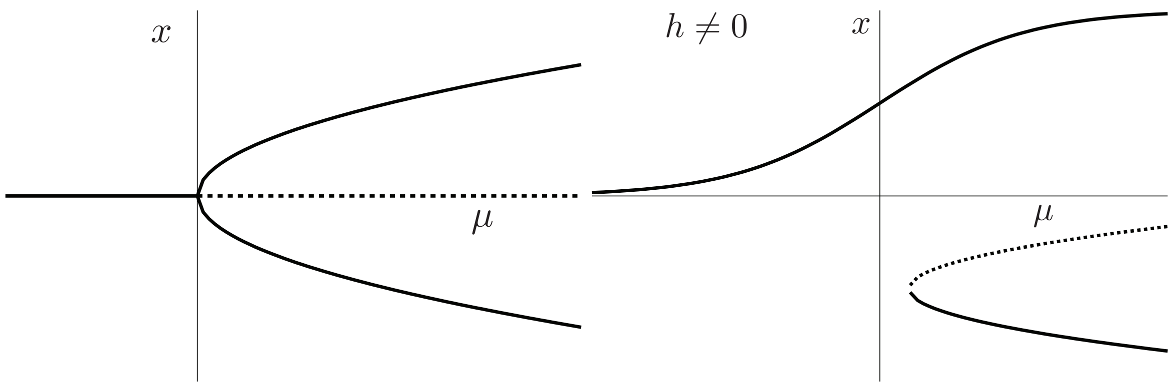


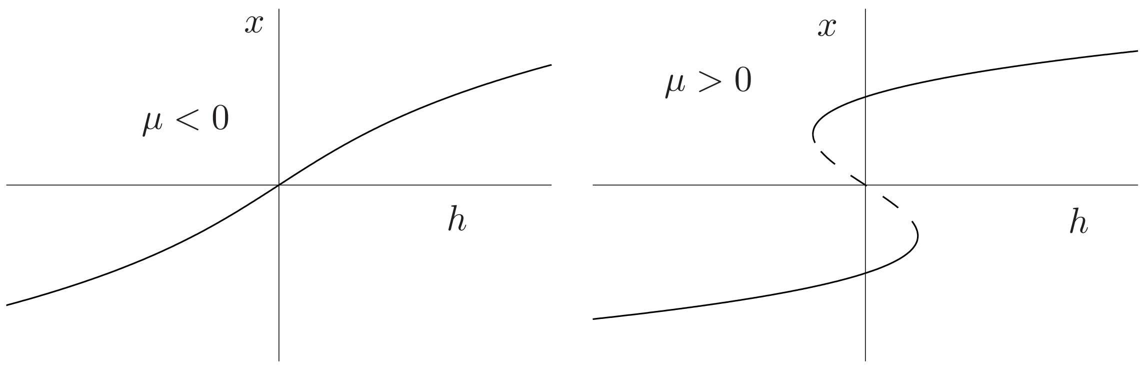
Critical case: horizontal line is tangent to min or max of $f(x) = \mu x - x^3$. Local max/min at $x = \pm\sqrt{\mu/3}$.

$$h_c(\mu) = \frac{2\mu}{3}\sqrt{\frac{\mu}{3}}$$

At $h = \pm h_c(\mu)$ there is a saddle-node bifurcation. There are two bifurcation curves $\pm h_c(\mu)$.







Jump phenomenon and catastrophe theory.

Example 3. Budworm population dynamics:

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}, \qquad A, B, R > 0$$

The budworm population $N(t)$ grows logistically (first term) in the absence of predators. The second term describes mortality due to predation (mainly by birds).

Non-dimensionalise: $x = N/A$.

$$\frac{A}{B}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{R}{B}Ax\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1 + x^2} \equiv f(x)$$

Introduce

$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}$$

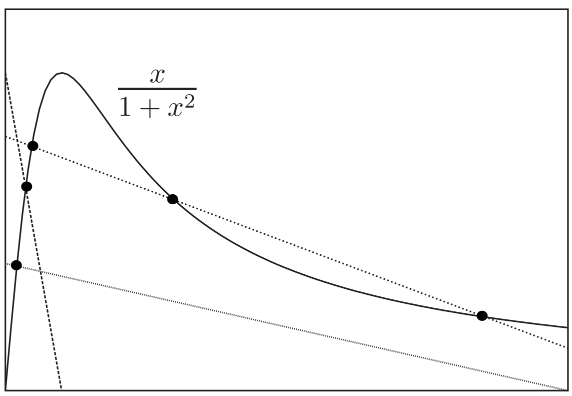
Fixed point at

$$\overline{x} = 0, \qquad r\left(1 - \frac{\overline{x}}{k}\right) - \frac{\overline{x}}{1 + \overline{x}^2} = 0$$

Linearisation:

$$f'(x) = r - \frac{2rx}{k} - \frac{2x}{(1 + x^2)^2}$$

so $f'(0) = r > 0$, so $\overline{x} = 0$ is unstable. Other roots may be found graphically by finding the intercepts of $x/(1 + x^2)$ and $r(1 - x/k)$:



Hence there can be either 1, 2 or 3 interceptions depending upon the choice of (k, r). For example when there are three fixed points $c > b > a > 0$, then since $x = 0$ is unstable a is stable, b unstable and c stable. We compute the details of the bifurcation in the following manner: Saddle-node occurs when $r(1-x/k)$ intersects $x/(1+x^2)$ tangentially. Thus we require $\overline{x}$ (given by $f(\overline{x}) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[r\left(1-\frac{x}{k}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{1+x^2}\right]$$

or that

$$-\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2}, \qquad x = \overline{x} \tag{1}$$

Substitution of r/k into the fixed point equation gives

$$r = \frac{2\overline{x}^3}{(1+\overline{x}^2)^2}$$

Substitution into (1) gives

$$k = \frac{2\overline{x}^3}{\overline{x}^2 - 1}$$

Since $k > 0$, we require $x > 1$. The bifurcation curve is defined by $(k(\overline{x}), r(\overline{x}))$ Challenge: plot the bifurcation curve ($r = r(k)$). In MATLAB you could try

```
ezplot3('2*x.^3./(x.^2-1)','2*x.^3./(1+x.^2)^2','0',[1,15]);view(0,90);
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In MATHEMATICA you could try

```
ParametricPlot[{2 x x x /(x x -1), 2 x x x/(1+x x)^2},{x,1,40}]
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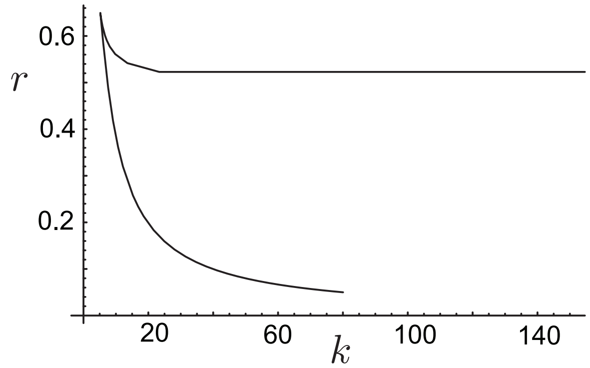
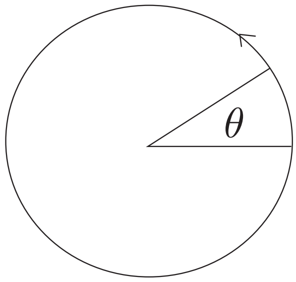


Flows on the circle



Basic model of an oscillator:

$$\dot{\theta} = f(\theta), \qquad \theta \in [0, 2\pi)$$

where $f(\theta) = f(\theta + 2\pi)$.

Uniform oscillator

$$\dot{\theta} = \omega, \qquad \theta = \theta_0 + \omega t$$

Period $T = 2\pi/\omega$.

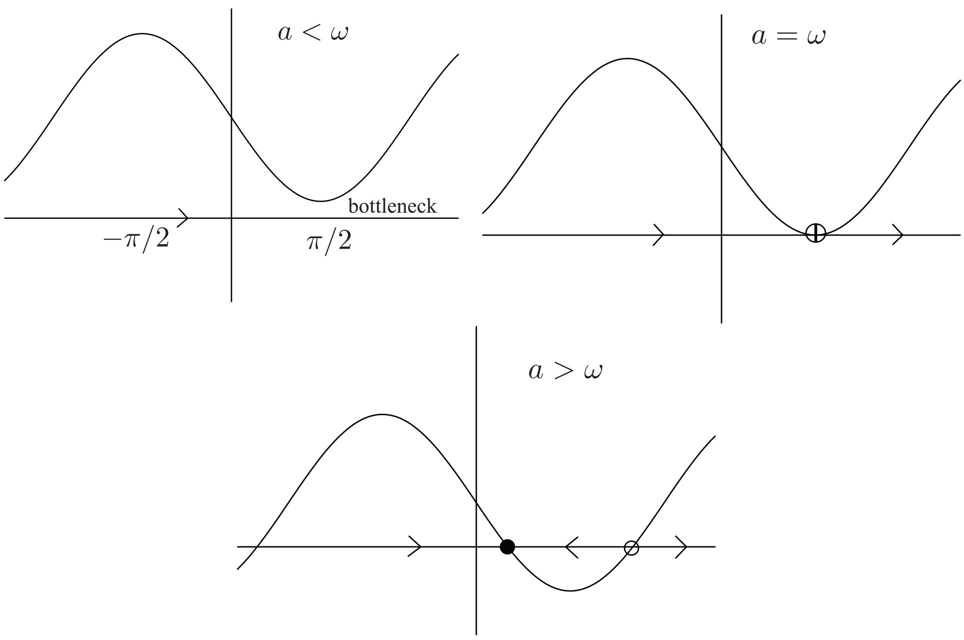
Non-uniform oscillator

$$\dot{\theta} = \omega - a \sin\theta$$

Consider here $\omega > 0$, $a \geq 0$ (similar results for negative ω and a).

$a < \omega$: Nonuniform flow which is fastest at $\theta = -\pi/2$ and slowest at $\pi/2$. When a is only slightly less than ω the system takes a long time to pass through the bottleneck at $\theta = \pi/2$ after which it quickly traverses the rest of the circle.

$a > \omega$: There exists a stable-unstable pair of fixed points at $\sin^{-1}[\omega/a]$ born via a saddle-node bifurcation. Oscillations do not exist.



Period:

$$T = \int \mathrm{d}t = \int_0^{2\pi} \frac{\mathrm{d}t}{\mathrm{d}\theta} \mathrm{d}\theta = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\omega - a \sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

[Hint: use the substitution $u = \tan\theta/2$].

Close to $a = \omega$

$$T = \frac{2\pi}{\sqrt{\omega + a}} \frac{1}{\sqrt{\omega - a}} \approx \frac{\sqrt{2}\pi}{\sqrt{\omega}} \frac{1}{\sqrt{\omega - a}}$$

so that we have a square root scaling law.